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Euler's Enumerations

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Abstract: Many topics in enumerative combinatorics trace back to work of Leonhard Euler. We survey six of his research topics, ranging from narrow to broad, with attention to mathematical predecessors and subsequent work. The topics are derangements, Catalan numbers, Latin squares, two disjoint contributions to what eventually became graph theory, and integer partitions along with the associated sum of divisors function. Throughout, we highlight tools that Euler developed which are still important today, especially generating functions. In addition, we see his propensity for calculation, his extensive interactions with correspondents, and his habit of returning to topics years later. We conclude with a sampling of rich internet resources that enable further historical investigations.

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1. Introduction

From its beginnings, the history of mathematics includes many precursors to the field we now know as enumerative combinatorics. Hexagrams of the *I Ching*, meters of Vedic chants, and regular polyhedra studied by the ancient Greeks all contribute to our modern pursuits of counting discrete objects. But without doubt the most prominent predecessor to our contemporary work is Leonhard Euler. Within his prodigious output, still being published in the *Opera Omnia* [11] (the 80th volume is due in 2021), are many topics and tools used to this day. Here we survey several of his results in combinatorics and enumeration.

We begin with two fairly complete analyses, with both combinatorics and enumeration, on derangements and Catalan numbers (using modern terminology). Euler also made initial counts concerning Graeco-Latin squares and polyhedra, the latter in one of his two disconnected studies that led to the rich field of graph theory. Each of the topics mentioned so far appears in just a few publications, but several articles and a prominent book chapter were dedicated to integer partitions and the closely related sum of divisors function.

As demonstrated in his treatment of Catalan numbers and integer partitions, Euler greatly expanded the use of generating functions. We conclude with other integer sequences and expressions that are important tools for enumeration.

In this overview of Euler's foundational work in enumerative combinatorics, we will not go into detailed historical analysis. However, we do mention some related correspondence to show the collaborative nature of 18th century mathematics and provide some outlines of Euler's work to provide a sense of his style. Also, there are ample references to both more thorough accounts of Euler's methods and wide-ranging histories of the various topics. Rather than cite more than a dozen 18th century articles, we reference Euler's publications by Gustav Eneström's index when possible; these are available in the *Opera Omnia* [11] and online resources such as *The Euler Archive* [13].

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4
2	~~2~~	~~2~~	~~3~~	~~3~~	~~4~~	~~4~~	3	3	4	4	1	1	4	4	1	1	2	2	1	1	2	2	3	3
3	~~3~~	~~4~~	~~4~~	~~2~~	~~2~~	~~3~~	4	1	1	3	4	3	1	2	2	4	~~1~~	~~4~~	2	3	~~3~~	~~1~~	1	2
4	~~4~~	~~3~~	~~2~~	~~4~~	~~3~~	~~2~~	1	4	3	~~1~~	3	~~4~~	2	3	4	2	~~4~~	~~1~~	3	~~2~~	~~1~~	~~3~~	2	1

Table 1: Euler's analysis of four card Rencontre.

2. Derangements

Usually at the prompting of his patrons, Euler analyzed several games of chance to find probabilities and expected values. Perhaps with that experience of looking at gambling mathematically, in 1751 he considered the card game Rencontre, meaning meet or coincidence, in E201. This two-player game, with A versus B, reduces to the following: Suppose there are m cards labeled $1, \ldots, m$, shuffled, and turned over one at a time. If any card k appears at round k (i.e., if a coincidence occurs), then A wins. If no such meetings occur, then B wins. What are the probabilities of winning for each player?

Table 1 from Euler's article demonstrates all possibilities for the game with $m = 4$. In the first six columns, the 1 card appears on the first draw and A wins; the other cards in those orders are no longer relevant, thus they are crossed out. In the orders of the columns labeled 17, 18, 21, and 22, A wins in the second round as 1 was not the first card and 2 was the second card. Columns 10, 12, and 20 show A winning in the third round. Finally, columns 8 and 15 show A winning in the fourth round. Altogether, A wins for $6 + 4 + 3 + 2 = 15$ of 24 possible orders. That leaves B winning for the remaining 9 orders, now known as derangements, where no number is drawn on its corresponding turn.

Euler developed a table including the $6, 4, 3, 2$ data (the triangle can now be found in the On-Line Encyclopedia of Integer Sequences [36, A068016]) and a simple recursive relationship similar to the rule for Pascal's triangle. Then, with a bit of algebra and what we might now call informal induction, he arrived at the following expression for the probability that A wins:

$$1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots \tag{1}$$

(without the contemporary factorial symbol and "&c." rather than an ellipsis). Another table shows that the probabilities for A and B winning stabilize to nine decimal places already at $m = 12$. More surprising yet, he recognized that the probability for B to win with an infinite number of cards is the infinite series for $1/e$.

As happened with several topics, Euler returned to derangements much later in his career. He presented E738 in 1779—he died in 1783 and there was such a backlog of his work that this was not published until 1811. There is no longer motivation from a card game, just mathematical interest in counting ways to change an order where no item returns to its original position. For n items, call this number D_n. We saw above that $D_4 = 9$. The sequence is [36, A000166] with a low identification number, as expected.

Euler found two equivalent recurrence relations for D_n. First, he gave a careful combinatorial argument for

$$D_n = (n-1)(D_{n-1} + D_{n-2}). \tag{2}$$

Second, after computing values through D_{10}, he observed a more succinct recursion, which we can write as

$$D_n = nD_{n-1} + (-1)^n. \tag{3}$$

He showed convincingly how (3) implies (2). Euler recognized that the other implication is more challenging; he worked from (2) with small values of n to (3) and declared the derivation "quite clear" (satis patet).

Using (3), one can derive (1) very directly [9, pp. 167–169]. However, Euler did not do this.

This is the one result we consider where Euler's contributions had almost all been made previously. Knobloch [23] details earlier work of Pierre Rémond de Montmort, Nicolaus (I) Bernoulli, and Abraham de Moivre, along with additional results from Euler's notebooks and a paper by Johann Lambert written between Euler's two presentations. Despite correspondence between Euler and some of these colleagues, it is agreed that he was unaware of their work as he was "invariably generous in bestowing praise and sharing credit" [9, p. 155].

3. Catalan numbers

Our second topic is barely included in Euler's official publications, which number around 850. Fortunately, some 3100 letters exchanged between Euler and almost 300 correspondents have been preserved. Of these, the 196 communications between Euler and Christian Goldbach, from 1729 to 1764 (the year of Goldbach's death), hold the most interest for us. (See Kleinert [20] for recent details on the status of publishing Euler's writings, especially the correspondence series IVA, for which he is the general editor.)

Euler's letter of September 1751 introduced the "curious" (merkwürdig) question of counting triangulations of polygons: "as a polygon of n sides is cut up into $n-2$ triangles by $n-3$ diagonals, in how many different manners can this be done?" [12, p. 1039]. The German transcription of the letter includes reproductions of Euler's drawings of a quadrilateral and pentagon with labeled vertices [12, p. 490]; he denoted the triangulations but did not draw them out. Let C_{n-2} be the number of triangulations for a convex n-gon. He offered these values for $n = 3, \ldots, 10$, the now-familiar sequence $1, 2, 5, 14, 42, 132, 429, 1430$ ([36, A000108], one of the longest entries in that encyclopedia), a conjectured formula

$$C_{n-2} = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)}, \tag{4}$$

and the associated generating function

$$1 + 2a + 5a^2 + 14a^3 + 42a^4 + 132a^5 + \cdots = \frac{1 - 2a - \sqrt{1-4a}}{2a^2} \tag{5}$$

with the observation that $a = 1/4$ leads to an infinite series that sums to 4. Goldbach replied the next month, explaining that he understands (5) but would have had difficulty determining the polynomial coefficients given just the right-hand side of the equation. Calling the generating function A, Goldbach noted that

$$1 + aA = A^{1/2} \tag{6}$$

and worked out the first few coefficients. In December 1751, Euler wrote how (5) follows from the usual expansion of $\sqrt{1-4a}$ and generalized to the series for $\sqrt[n]{1-n^2a}$ before turning to their ongoing discussion on sums of squares.

Apparently Euler shared some aspects of the problem with another frequent correspondent, Johann Andreas von Segner. (While Euler archived Segner's 159 letters to him, unfortunately Euler's letters to Segner seem to have been lost.) Segner found a new relation and Euler arranged for its publication [34]. Segner's main result, in our notation, is

$$C_n = C_0C_{n-1} + C_1C_{n-2} + C_2C_{n-3} + \cdots + C_{n-3}C_2 + C_{n-2}C_1 + C_{n-1}C_0 \tag{7}$$

which he proved combinatorially in terms of triangulations. He ended with a table of values up to C_{18}, the number of triangulations of a 20-gon.

We know from Euler's correspondence with Gerhard Friedrich Müller of the Imperial Academy of Sciences in Saint Petersburg that Euler regularly offered unsigned "abstracts" of articles in the academy's journals which often, in fact, added material. In this case, Euler mentioned the recurrence that leads to (4) and corrected the values of C_{13} to C_{18}, explaining that "some errors had crept in" to Segner's computations (eingeschlichene Fehler). For good measure, Euler gave (correct) values through C_{22} [10]. (The summaries for this journal volume are sometimes designated E265*, a notation that Eneström used for additions to his original list.)

Segner's recurrence (7) can be used to establish Goldbach's equation (6) and prove Euler's formula (4), but there is no record that any of these three thought more about the topic. The first published proofs came in two noteworthy volumes of Joseph Liouville's *Journal de Mathématique Pures et Appliquées.* Orly Terquem asked Liouville about establishing (4) from (7) and Liouville disseminated the problem widely. The 1838 and 1839 journal issues include six articles on the topic by Jacques Binet, Eugène Charles Catalan, Gabriel Lamé, and Olinde Rodrigues. For more details, see Igor Pak's thorough history [38, pp. 177–189] which includes slightly earlier work of the Mongolian scientist Sharabiin Myangat, also known as Ming Antu. Pak's essay includes his conclusion that the name Catalan numbers traces to John Riordan and was solidified by Martin Gardner.

Additional combinatorial interpretations of the Catalan numbers started with the French publications of 1838/1839 and continue to this day. Richard Stanley included 66 in the second volume of his *Enumerative Combinatorics* [37, Exercise 6.19] and kept an expanding list on his website. Finally, he stopped at 214 combinatorial interpretations which are the core of his 2015 monograph [38].

4. Arrays of integers

The first posthumous edition of Jacques Ozanam's famous book on recreational mathematics includes an example of an important concept. In the last of four volumes, in the chapter on games, puzzle number 39 of the "various amusing tricks" section explains that 16 playing cards—the jacks, queens, kings, and aces of all four suits—can be positioned such that there is exactly one of each rank and of each suit in every row, column, and corner-to-corner diagonal [26, p. 434]. Can you make the cards so well arranged (si bien disposées)? The original solution is shown in Figure 3 which is postponed to a later page.

An $n \times n$ Latin square consists of n copies of n symbols such that each symbol appears exactly once in each row and in each column. A completed Sudoku puzzle is a 9×9 Latin square with additional structure. The card ranks in Ozanam's puzzle make a 4×4 Latin square, as do the card suits. When two Latin squares overlap such that no ordered pair of symbols is repeated, as with the ranks and suits above, they are said to make a Graeco-Latin square. Less creatively, two such Latin squares are called orthogonal. Ozanam is believed to be the first to feature this construction. (Some give credit to Claude Bachet in the 17th century, but Richardson [30] challenges that claim.)

Euler considered these combinatorial objects in 1776 in service of another recreational mathematics topic, magic squares, in E795 (not published until 1848). The connection can be seen in the 3×3 examples from the article shown in Table 2. The terminology Latin and Graeco-Latin follows from Euler's choices of alphabets. To make the magic square, add the values assigned to the Latin letters (multiples of 3) and Greek letters (1, 2, 3). Notice that while the Graeco-Latin square has some repetition on the diagonals, the resulting magic square has distinct entries and the same sum for each row, column, and diagonal. Euler developed methods for building magic squares from Latin and Greek squares with some exceptions to the orthogonality condition, working up to 6×6 magic squares.

$a\gamma$	$b\beta$	$c\alpha$
$b\alpha$	$c\gamma$	$a\beta$
$c\beta$	$a\alpha$	$b\gamma$

2	9	4
7	5	3
6	1	8

Table 2: A 3×3 Graeco-Latin square and a magic square derived from it using $a = 0$, $b = 6$, $c = 3$; $\alpha = 1$, $\beta = 3$, $\gamma = 2$.

Unhappy with not finding a 6×6 Graeco-Latin square, Euler returned to the topic in 1779 with one of his longest articles, E530, which filled 155 pages in its original published form. Within the multitude of examples are two items that resonated powerfully for many years. First, he defined the directrix of an $n \times n$ Latin square, now called a transversal: a collection of n squares, one per row and one per column, with no two of the squares having the same symbol. For example, in the Latin square of Table 2 (with Latin letters), the diagonal is a transversal. This is a foundational concept related to constructing orthogonal Latin squares.

Second, after developing transformations of Latin squares and much searching, Euler transitioned from saying that a 6×6 Graeco-Latin square would need to be "completely irregular" to claiming that any $(4k+2) \times (4k+2)$ Graeco-Latin square is impossible. (For all other dimensions, he could construct them.) He was right about two (trivial) and six, which he posed as a problem involving 36 officers of six regiments having six different ranks, as confirmed by Gaston Tarry in 1900 (or should credit to go Thomas Clausen in the 1840s?). But it turns out these are the only impossibilities; Ernest Parker, Raj Bose, and Sharadchandra Shrikhande showed by 1960 that the rest of Euler's claim is false. See [21] for details of this history (and a case for giving Clausen priority).

Towards the theme of this survey, Euler did enumeration work with Latin squares. He began counting reduced Latin squares, where the first row and column are in a standardized order; see Table 3. For $n = 2, 3, 4, 5$, Euler found that the number of reduced $n \times n$ reduced Latin squares to be 1, 1, 4, 56, respectively [36, A000315]. The magnitude of the $n = 6$ value led to Euler's extreme efforts; the count was shown to be 9408 by Michel Frolov in 1890. See Dénes and Keedwell [7, pp. 138–149] for more of this history and extensive information on Latin squares and their applications.

1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	4	3	2	1	4	3	2	3	4	1	2	4	1	3
3	4	1	2	3	4	2	1	3	4	1	2	3	1	4	2
4	3	2	1	4	3	1	2	4	1	2	3	4	3	2	1

Table 3: Euler's list of the four 4×4 reduced Latin squares.

Euler's other enumeration result here is a count of possible second rows for a reduced Latin square. The first entry must be 2 and, for the rest, k cannot be the kth entry. From Table 3 we see there are three possible second rows for a 4×4 reduced Latin square. This is a special case of the derangements discussed above. The sequence begins 1, 1, 3, 11, 53, 309 (Euler computed through $n = 10$; [36, A000255]) and he derived recurrences similar to (2) and (3).

5. Graph theory

Two famous results of Euler, the Königsberg bridges problem and the polyhedral formula, are widely considered to be the beginnings of graph theory. In this section, we will clarify what Euler did and did not do and highlight a lesser-known enumeration result.

Correspondence suggests that Karl Ehler, the mayor of Danzig (now Gdansk) and an amateur mathematician, introduced Euler to the situation in Königsberg (Kaliningrad) with seven bridges connecting three land masses and an island. Is there a path that crosses each bridge exactly once? Euler shared a solution with the Austrian astronomer Giovanni Jacopo Marinoni, explaining that the problem was "banal" (vulgaris) but might be an example of the "geometry of position" (geometriam situs) mentioned by Gottfried Leibniz. To Ehler, who requested a solution, Euler sent an amusing reply:

> . . . this type of solution bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle. Because of this, I do not know why even questions which bear so little relationship to mathematics are solved more quickly by mathematicians than by others.

In 1735, Euler proved that the requested path is impossible, in E53. Shifting to modern notation, write $\mathcal{V}$ for the set of vertices of a graph, E for the number of edges, and $\deg(v)$ for the degree of the vertex v, the number of edges incident to v. A lasting lemma that Euler proved is

$$\sum_{v \in \mathcal{V}} \deg(v) = 2E.$$

He eventually characterized which graphs have what are now known as Eulerian paths or circuits based on the degrees of their vertices.

It is worth pointing out a few things that Euler did not do. He did not provide a rigorous method for how to build a desired path when one exists; such an algorithm was first published by Carl Hierholzer in 1873 with another given by Pierre-Henry Fleury in 1883. In no way did Euler abstract this problem to a more general connected system; Figure 1 shows the generalizations he did make—more complicated systems of rivers with more bridges. On a related note, Euler did not draw any sort of dot and line illustration now used in graph theory; the first such figure we found for this Königsberg graph was by W.W. Rouse Ball in 1892. See Hopkins and Wilson [18] for more details.

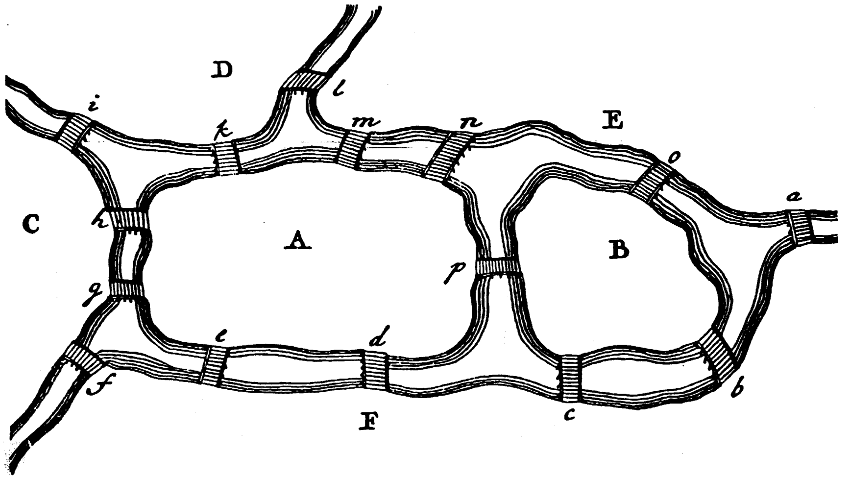

Figure 1: Euler's generalization of the Königsberg bridges problem. This system admits Eulerian paths with endpoints in areas D and E. Image from [13].

Concerning paths in graphs, it is interesting to note that Euler's treatment of knight's paths on a chessboard, E309 presented in 1759, includes what are now called Hamiltonian circuits. Ozanam's 1723 collection also includes such a puzzle, but this appears in the chess literature by 1141 at the latest [35].

In 1750, Euler shared with Goldbach "general properties of the bodies enclosed by plane surfaces," a mix of what we would now call combinatorial and geometric results on polyhedra. The novelty of the topic required linguistic creativity in addition to mathematical insight:

> In order to present his new insights into the combinatorics of polyhedral bodies, Euler lacks well-established terms, which shows how new this mode of thinking about geometrical objects was at the time. There is a Latin term for the faces (*hedrae*, from Greek $\varepsilon\delta\rho\alpha\iota$) which Euler retains, but since there is no separate word for the vertices, he uses "solid angles" (*anguli solidi*), and for the edges he has to coin a new use for the Latin word *acies* (literally "sharpness", "blade"). [12, p. 1030]

Switching to planar graphs, write V for the number of vertices, E for the number of edges, and F for the number of faces. Here are some of Euler's results from the subsequent publications, E230 and E231, starting with the celebrated polyhedral formula:

$$V + F = E + 2,\ 2E \geq 3F,\ 2E \geq 3V,\ 2V \geq F + 4,\ 2F \geq V + 4,\ 3V \geq E + 6,\ 3F \geq E + 6.$$

There has been criticism of Euler's imprecise notion of polyhedron and method of proof; see [31] for a careful analysis.

As for enumeration, Euler included several tables in E230. One gives the possible range of V and E values for polyhedra having 4 to 25 faces, another the range of V and F values for polyhedra having 6 to 60 edges. He noted that there cannot be a polyhedron with 7 edges. The article builds to a tabulation of the 37 types (genera) of polyhedra having up to 10 vertices (see [36, A001651]).

A related manuscript of René Descartes, from perhaps 1630, survives only because Leibniz transcribed it in 1676 (Descartes died in 1650), and those notes were not found until 1860, by Louis-Alexandre Foucher de Careil. Descartes determined the relations

$$2V \geq F + 4,\ 2F \geq V + 4$$

which allowed him to prove algebraically that there can be only five regular solids. (Euler considered each of the Platonic solids as examples, but did not argue with these tools that they are the only possibilities.) Descartes concluded with a relation involving V, F, and P, the number of "plane angles" which, with the relation $P = 2E$

to Euler's terminology, can easily be converted to the polyhedral formula. See Federico for a well-informed and circumspect consideration of priority for the polyhedral formula [14].

There are more geometric results in these articles of Descartes and Euler that we will not consider. Also, as with the Königsberg bridges problem, it is important to highlight that Euler did not generalize this research beyond polyhedra. Furthermore, he made no connections between the two topics discussed in this section.

For more on the history of graph theory, Biggs, Lloyd, and Wilson [3] cover the two centuries from the Königsberg article to Dénes König's 1936 book on the subject. The importance of Euler's work can be seen elsewhere in this inaugural issue of *Enumerative Combinatorics and Applications*, where a survey by Gross and Tucker involves both Eulerian circuits and the Euler genus, related to extensions of the polyhedral formula [17].

6. Integer partitions and the sum of divisors function

Integer partitions are a basic mathematical object, but their first significant study came with Euler, who greatly expanded the machinery of generating functions towards these ends. (Generating functions were used as early as 1708 by Montmort, but Euler may have developed the concept independently.) The partitions of a positive integer n are the ways to write n as a sum of positive integers without regard to order of the summands. For instance, the partitions of 4 are $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. Write $p(4) = 5$ for the count [36, A000041]. Records show Euler worked on partitions and related ideas in letters to several correspondents and some dozen publications from 1740 to 1775.

The key insight (again, using notation adopted since Euler) is the generating function

$$\sum_{n=0}^{\infty} p(n)x^n = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots$$
$$= \prod_{k=1}^{\infty} \frac{1}{1-x^k}. \tag{8}$$

Expanding a few terms of the geometric series product gives $5x^4$ corresponding to $p(4) = 5$.

Euler's first publication on partitions answers questions posed to him by Philipp Naudé in 1740: How many ways can 50 be written as the sum of 7 positive integers, either distinct or allowing repetition? Euler modified (8) with another variable to track the number of parts. Writing $p(n, m)$ for the number of partitions of n with exactly m parts,

$$\sum_{n=0}^{\infty} p(n,m)x^n z^m = \prod_{k=1}^{\infty} \frac{1}{1-zx^k}. \tag{9}$$

To allow at most one of each summand, he truncated the geometric series of (8). Writing $q(n)$ for the number of partitions of n with distinct summands [36, A000009],

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{k=1}^{\infty}(1+x^k) \tag{10}$$

in which z can also be included as in (9) to find the generating function for the analogous $q(n, m)$. To compute the answers to Naudé's questions, Euler developed recurrence relations for $p(n, m)$ and $q(n, m)$.

Euler later found a seminal result in the theory of partitions: $q(n)$ equals the number of partitions of n into odd parts. Using generating functions, the proof is just a few lines of algebra:

$$\begin{aligned}\sum_{n=0}^{\infty} q(n)x^n &= \prod_{k=1}^{\infty}(1+x^k)\\ &= \prod_{k=1}^{\infty}\frac{(1+x^k)(1-x^k)}{1-x^k}\\ &= \prod_{k=1}^{\infty}\frac{1-x^{2k}}{1-x^k}\\ &= \prod_{\ell=1}^{\infty}\frac{1}{1-x^{2\ell-1}},\end{aligned}$$

the modification of (8) for only odd summands. This is the prime example of results showing that two types of partitions (or their generalizations) are equinumerous. There are a vast and growing number of such bijections now known, proven algebraically, combinatorially, or both.

The product $\prod(1-x^k)$, the denominator of (8), arises so often that, in the contemporary work of q-series, its analogue is simply denoted by the Pochhammer symbol $(q)_\infty$. Already in 1740, Euler realized the pattern

$$\prod_{n=1}^{\infty}(1-x^n) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \cdots$$
$$= \sum_{k=-\infty}^{\infty} (-1)^k x^{\frac{k(3k-1)}{2}} \tag{11}$$

where the exponents are generalized pentagonal numbers (for positive k, these are the normal pentagonal figurate numbers), but had significant difficulty finding a proof. Ten years later, in 1750, Euler shared with Goldbach the lemma

$$(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\cdots = 1 - \alpha - \beta(1-\alpha) - \gamma(1-\alpha)(1-\beta) - \delta(1-\alpha)(1-\beta)(1-\gamma) - \cdots$$

which allowed him to prove what we now call the pentagonal number theorem. He returned to the topic in 1775 with additional proofs. Bell provides a detailed account of Euler's long history with this result [2], drawing from the notebooks and correspondence in addition to several publications.

Combining (8) and (11) allowed Euler to give the following recurrence for $p(n)$:

$$p(n) = p(n-1) - p(n-2) + p(n-5) + p(n-7) - p(n-12) - p(n-15) + \cdots$$

where $p(0) = 1$ and $p(n) = 0$ for $n < 0$. George Andrews explained, "No one has ever found a more efficient algorithm for computing $p(N)$. It computes a full table of values of $p(n)$ for $n \le N$ in time $\mathcal{O}(N^{3/2})$." [1, p. 208]

The pentagonal number theorem leads to another surprising recurrence. Euler was interested in the sum of divisors function $\sigma(n)$ used in the definitions of perfect and amicable numbers. For example, $\sigma(4) = 1+2+4 = 7$ [36, A000203]. The logarithmic derivative of (11) leads to the recurrence

$$\sigma(n) = \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12) - \sigma(n-15) + \cdots$$

where $\sigma(0) = n$ and $\sigma(n) = 0$ for $n < 0$.

Euler also gets credit for the first application of integer partitions. One of King Frederick II's assignments was analysis of the Genoese lottery, where bettors choose multiple numbers. Euler eventually wrote five articles on the topic. One of these, E338 presented in 1765, considers the probabilities that various runs of numbers are chosen. For instance, for four numbers, the possibilities are

- a single run $a, a+1, a+2, a+3$ of length four, or
- a length three run and a singelton, $a, a+1, a+2, b$ where $b \neq a-1$ or $a+3$, or
- two length two runs, $a, a+1, b, b+1$, or
- a length two run and two singletons, $a, a+1, b, c$, or
- four singletons, a, b, c, d (each with difference at least 2).

These five types (especes) correspond to $p(4) = 5$. See Bradley [4] for more details.

These are just some of Euler's enumeration results on integer partitions, but he did not incorporate any combinatorial interpretations. Those eventually came from James Joseph Sylvester, his colleagues, and his students in the late 1800s. For instance, a Ferrers diagram, named for Norman Macleod Ferrers, represents a partition by a system of dots showing the summands. Figure 2 shows the Ferrers diagram for the partitions $3+1$ and $2+1+1$. Notice that these are conjugate, in that the rows of one match the columns of the other. This gives a visual explanation for a relation that Euler approached through a great deal of symbolic manipulation:

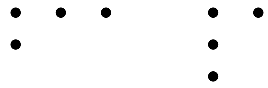

Figure 2: Ferrers diagrams of the conjugate partitions $3+1$ and $2+1+1$, respectively.

the number of partitions of n with exactly k parts equals the number of partitions of k with largest part k. Fabian Franklin later used Ferrers diagrams to give an ingenious combinatorial proof of the pentagonal number theorem (11).

For more on the development of partitions, Dickson gave encyclopedic coverage to 1919 [8, Ch. III] and Andrews provides a more expository and updated treatment [1]. Note, though, that Knobloch [22] documents a richer pre-Eulerian history of integer partitions than Dickson and Andrews acknowledge.

7. Additional topics

Finally, we consider some tools of contemporary enumerative combinatorics that were developed or created by Euler, bypassing his work on binomial coefficients and figurate numbers.

In 1778, Euler presented results on general multinomial coefficients that arise from $(1+x+x^2+x^3+\cdots)^n$. For example, writing $\binom{n}{k}_2$ for the coefficient of k in the expansion of $(1+x+x^2)^n = (1+x(1+x))^n$, he expressed these trinomial coefficients in terms of binomial coefficients:

$$\binom{n}{k}_2 = \binom{n}{k}\binom{k}{0} + \binom{n}{k-1}\binom{k-1}{1} + \binom{n}{k-2}\binom{k-2}{2} + \cdots .$$

Beware that terminology is not consistent across the literature; some authors parametrize trinomial coefficients by effectively considering $(x^{-1}+1+x)^n$, but this does not generalize to quadrinomial coefficients, etc. One can also find coefficients from $(x+y+z)^n$ called trinomial coefficients.

There is an interesting pedagogical example from Euler involving trinomial coefficients in E326 from 1763. He computed several terms $\binom{n}{n}_2$ (central trinomial coefficients, [36, A002426]) and, looking for a recurrence, noticed that

$$3\binom{n}{n}_2 - \binom{n+1}{n+1}_2 = F_{n-1}(F_{n-1}+1)$$

where $F_{-1}=1$, $F_0=0$, and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 1$. (Euler did not call these Fibonacci numbers but knew "Binet's formula" for them.) However, the magic disappears with the next term, where the weighted difference 464 does not match $21 \cdot 22 = 462$. The reader would have been suspect with the section heading "a notable example of fallacious induction." After revealing the stratagem, Euler explained, "Since the rule is obtained by examination of the first ten terms of the given sequence, who can ever doubt that the same rule holds for the whole series? Less certain inductions were often successful." He went on to show the correct relation

$$\binom{n+1}{n+1}_2 = \binom{n}{n}_2 + \frac{n}{n+1}\left(\binom{n}{n}_2 + 3\binom{n-1}{n-1}_2\right).$$

See Ferraro [15] for more on what Euler meant by induction and his use of examples and rigorous proofs.

There are two integer sequences named for Euler which both arose in his analytic work but have come to have important combinatorial interpretations. There are the Euler numbers [36, A000111] and the Eulerian numbers [36, A008292] (the latter have two parameters, so the terms make a triangular array). Combinatorially, the Euler numbers, also known as secant numbers, count odd alternating permutations and are playfully called zig numbers by Conway and Guy [6, pp. 110–111] who also discuss zag numbers and zigzag arrangements. The Eulerian numbers count permutations with a specified number of descents; Petersen's monograph [28] makes the argument that "the array of Eulerian numbers is just as interesting as Pascal's triangle." Foata [16] explores Euler's motivation to consider the polynomials whose coefficients are Eulerian numbers.

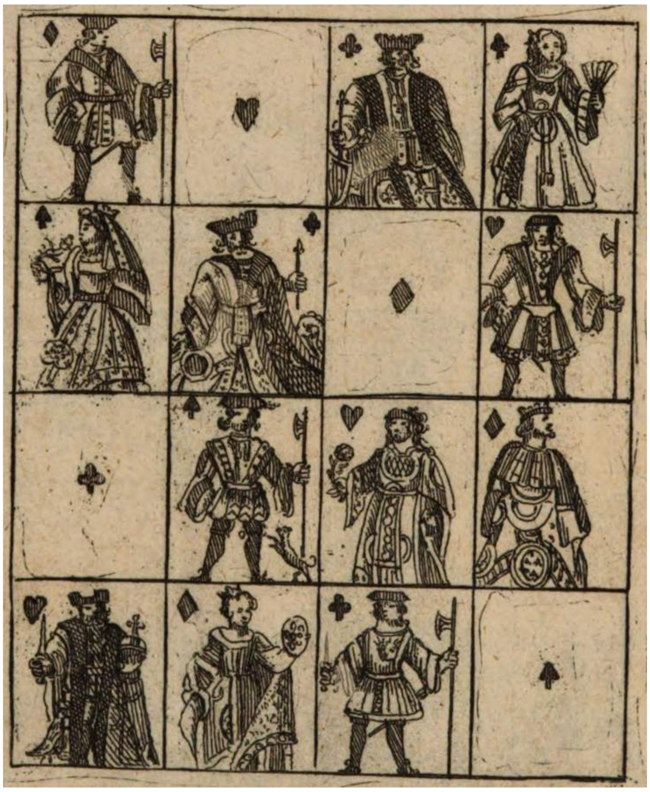

Figure 3: Ozanam's puzzle solution [26, Fig. 35]; the first row should be $J\diamondsuit, A\heartsuit, K\spadesuit, Q\clubsuit$. Image from the Bayerische Staatsbibliothek, http://mdz-nbn-resolving.de/urn:nbn:de:bvb:12-bsb10594196-7, Einband, Scan 38.

8. Further reading

Beyond his publications and correspondence, there are still Euler's notebooks. Knobloch reports that they include discussion of triangulations of polygons, some enumeration of magic squares and rectangles [24], and even musical topics such as counting chord inversions and certain rhythmic patterns [25]. Hopefully the notebooks will be made widely available soon.

For more on Euler and combinatorics, this survey is both a refinement and an expansion of Hopkins and Wilson [19]. Suggested general mathematical surveys of Euler's work include Dunham [9] and two volumes collecting Sandifer's *How Euler Did It* column [32,33]. Calinger recently completed the authoritative biography of Euler [5].

Focusing on historical sources, three websites in particular are highly recommended. Richard Pulskamp's *Sources in the History of Probability and Statistics* [29] is a remarkable resource with files of publications from before Blaise Pascal to around 1890 with ample translations. The Euler section includes 13 of his articles on games, financial mathematics, statistics, etc., and supplementary material, such as Montmort, de Moivre, Lambert, Laplace, and Michaelis's writings on Rencontre, all with translations. Another extensive web resource is Igor Pak's *Catalan Numbers Page* [27] with some 150 links to videos, historical sources, etc.

The greatest accolades go to *The Euler Archive* [13]. Now on its second set of directors and at least its third electronic home, this digital library includes files for almost every original Euler publication, translations and links to related articles for many of those, much of the correspondence, notes on Eneström's index, and more. With this amazing resource, you can follow Pierre-Simon Laplace's exhortation, "Read Euler, read Euler. He is the master of us all."

References

[1] G. Andrews, *Partitions*, pp. 205–229, in Combinatorics: Ancient and Modern, ed. R. Wilson and J.J. Watkins, Oxford University Press, Oxford (2013).

[2] J. Bell, *A summary of Euler's work on the pentagonal number theorem*, Arch. Hist. Exact Sci. 64 (2010) 301–373.

[3] N.L. Biggs, E.K. Lloyd, and R.J. Wilson, *Graph Theory: 1736–1936*, Clarendon Press, Oxford (1976).

[4] R.E. Bradley, *The Genoese lottery and the partition function*, pp. 203–215 in Euler at 300: An Appreciation, ed. R.E. Bradley, L.A. D'Antonio, and C.E. Sandifer, Mathematical Association of America, Washington, DC (2007).

[5] R.S. Calinger, *Leonhard Euler: Mathematical Genius in the Enlightenment*, Princeton University Press, Princeton, NJ (2016).

[6] J.H. Conway and R.K. Guy, *The Book of Numbers*, Springer, New York (1996).

[7] J. Dénes and A.D. Keedwell, *Latin Squares and Their Applications*, Academic, New York (1974).

[8] L.E. Dickson, *History of the Theory of Numbers. Vol. II: Diophantine Analysis*, Chelsea, New York (1966).

[9] W. Dunham, *Euler: The Master of Us All*, Mathematical Association of America, Washington, DC (1999).

[10] L. Euler, *Summarium dissertationum mathematica VI*, Novi. Comm. Petrop. 7 (1758/1759), 13–15.

[11] L. Euler, *Opera Omnia*, 79 volumes, Birkhäuser, Basel (1911–2019).

[12] L. Euler, *Correspondence of Leonhard Euler with Christian Goldbach*, ed. F. Lemmermeyer and M. Martmüller, Opera Omnia, Series IVA, Vol. 4, Birkhäuser, Basel (2015).

[13] L. Euler, *The Euler Archive*, 2020. Available at https://scholarlycommons.pacific.edu/euler/.

[14] P.J. Federico, *Descartes on Polyhedra*, Sources in the History of Mathematics and Physical Sciences, Vol. 4, Springer, New York (1982).

[15] G. Ferraro, *Proofs, arbitrary exemplifications, and inductive generalizations in Euler's mathematical practice*, in Handbook of the History and Philosophy of Mathematical Practice, ed. B. Sriraman, Springer, Cham, Switzerland (2020).

[16] D. Foata, *Eulerian polynomials: From Euler's time to the present*, pp. 254–273 in The Legacy of Alladi Ramakrishnan in the Mathematical Sciences, ed. K. Alladi, J.R. Klauder, and C.R. Rao, Springer, New York (2010).

[17] J.L. Gross and T.W. Tucker, *Enumerating graph embeddings and partial-duals by genus and Euler genus*, Enumer. Combin. Appl. 1 (2021), Article S2S1.

[18] B. Hopkins and R. Wilson, *The truth about Königsberg*, College Math. J. 35 (2004), 198–207.

[19] B. Hopkins and R. Wilson, *Euler's science of combinations*, pp. 395–408 in Leonhard Euler: Life, Work and Legacy, ed. R.E. Bradley and C.E. Sandifer, Studies in the History and Philosophy of Mathematics, Vol. 5, Elsevier, Amsterdam (2007).

[20] A. Kleinert, *Leonhardi Euleri Opera omnia: Editing the works and correspondence of Leonhard Euler*, Pr. Kom. Hist. Nauki PAU 14 (2015), 13–35.

[21] D. Klyve and L. Stemkoski, *Graeco-Latin squares and a mistaken conjecture of Euler*, College Math. J. 37 (2006), 2–15.

[22] E. Knobloch, *The mathematical studies of G.W. Leibniz on combinatorics*, Historia Math. 1 (1974), 409–430.

[23] E. Knobloch, *Euler and the history of a problem in probability theory*, Gaṇita Bhāratī 6 (1984), 1–12.

[24] E. Knobloch, *Leonhard Eulers Mathematische Notizbücher*, Ann. of Sci. 46 (1989), 277–302.

[25] E. Knobloch, *The sounding algebra: Relations between combinatorics and music from Mersenne to Euler*, pp. 27–48 in Mathematics and Music, eds. G. Assayag, H.G. Feichtinger, and J.F. Rodrigues, Springer, Berlin (2002).

[26] J. Ozanam, *Récréations mathématiques et physiques, Vol. IIII*, Jombert, Paris (1723).

[27] I. Pak, *Catalan Numbers Page*, 2020. Available at https://www.math.ucla.edu/∼pak/lectures/Cat/pakcat.htm.

[28] T.K. Petersen, *Eulerian Numbers*, Springer, New York (2015).

[29] R.J. Pulskamp, *Sources in the History of Probability and Statistics*, 2020. Available at https://www.cs.xu.edu/math/Sources.

[30] J.T.E. Richardson, *Who introduced Western mathematicians to Latin squares?*, British J. Hist. Math. 34 (2019), 95–103.

[31] D.S. Richeson, *Euler's Gem*, Princeton University Press, Princeton, NJ (2008).

[32] C.E. Sandifer, *How Euler Did It*, Mathematical Association of America, Washington, DC (2007).

[33] C.E. Sandifer, *How Euler Did Even More*, Mathematical Association of America, Washington, DC (2015).

[34] J.A. Segner, *Enumeratio modorum quibus figurae planae rectilineae per diagonales dividuntur in triangula*, Novi. Comm. Petrop. 7 (1758/1759), 203–210.

[35] D. Singmaster, *An historical tour of binary and tours*, pp. 207–240 in Games of No Chance 5, ed. U. Larsson, Mathematical Sciences Research Publications, Vol. 70, Cambridge University Press, Cambridge (2019).

[36] N.J.A. Sloane et al., *The On-Line Encyclopedia of Integer Sequences*, 2020. Available at https://oeis.org.

[37] R.P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, New York (1999).

[38] R.P. Stanley, *Catalan Numbers*, Cambridge University Press, New York (2015).