Generating Functions for Monomial Characters of Wreath Products $\mathbb{Z}/d\mathbb{Z} \wr S_n$

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Abstract: Let $\mathbb{Z}/d\mathbb{Z} \wr S_n$ denote the wreath product of the cyclic group $\mathbb{Z}/d\mathbb{Z}$ with the symmetric group $S_n$. We define generating functions for monomial (induced one-dimensional) characters of $\mathbb{Z}/d\mathbb{Z} \wr S_n$ and express these in terms of determinants and permanents. This extends work of Littlewood (The Theory of Group Characters and Representations of Groups, 1940) and Merris and Watkins (Linear Algebra Appl., 64, 1985) on generating functions for the monomial characters of $S_n$.

Keywords: Character; Generating function; Wreath product

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1. Introduction

Let $z = (z_{i,j})$ be an $n \times n$ matrix of indeterminates and let $S_n$ be the symmetric group. For each linear functional $\theta : \mathbb{C}[S_n] \to \mathbb{C}$, define the generating function

$$\text{Imm}_\theta(z) := \sum_{w \in S_n} \theta(w)z_{1,w_1} \cdots z_{n,w_n} \in \mathbb{C}[z]$$

for $\theta$, and call this the $\theta$-immanant. Such functions appeared originally in [7, p. 81] for $\theta$ equal to irreducible $S_n$-characters $\chi^\lambda$, and were extended in [14, §3] to general $\theta$. As is the case with many functions, a simple formula for a generating function for $\theta$ can be as useful as a simple formula for the numbers $\{\theta(w) | w \in S_n\}$ themselves.

Particularly simple generating functions for the monomial (induced one-dimensional) characters of $S_n$ are expressed in terms of integer partitions, ordered set partitions, and submatrices of $z$. Call a nonnegative integer sequence $\lambda = (\lambda_1, \ldots, \lambda_r)$ satisfying $\lambda_1 + \cdots + \lambda_r = n$ a weak composition of $n$ and write $|\lambda| = n$, $\ell(\lambda) = r$. If the components of $\lambda$ are weakly decreasing and positive, call it an (integer) partition of $n$ and write $\lambda \vdash n$. For any weak composition $\lambda$ of $n$, call a sequence $(I_1, \ldots, I_r)$ of pairwise disjoint subsets of $[n] := \{1, \ldots, n\}$ an ordered set partition of $[n]$ of type $\lambda$ if $|I_j| = \lambda_j$ for $j = 1, \ldots, r$. (Note that this nonstandard terminology allows empty sets in ordered set partitions, whereas standard terminology [13, pp. 39, 73] does not.) Given subsets $I$, $J$ of $[n]$, define the $(I, J)$-submatrix of $z$ to be

$$z_{I,J} = (z_{i,j})_{i \in I, j \in J}. \quad (2)$$

The class function space of $S_n$ has two standard bases consisting of monomial characters: the induced trivial character basis $\{\eta^\lambda = \text{triv} \uparrow_{S_n}^\lambda | \lambda \vdash n\}$ and the induced sign character basis $\{\epsilon^\lambda = \text{sgn} \uparrow_{S_n}^\lambda | \lambda \vdash n\}$, where $S_\lambda$ is the Young subgroup of $S_n$ indexed by $\lambda$. (See, e.g., [9].) Littlewood [7, §6.5] and Merris and Watkins [8] came close to expressing the $\eta^\lambda$- and $\epsilon^\lambda$-immanants as

$$\text{Imm}_{\eta^\lambda}(z) = \sum_{(J_1, \ldots, J_r)} \det(z_{J_1,J_1}) \cdots \det(z_{J_r,J_r}),$$

$$\text{Imm}_{\epsilon^\lambda}(z) = \sum_{(J_1, \ldots, J_r)} \text{per}(z_{J_1,J_1}) \cdots \text{per}(z_{J_r,J_r}). \quad (3)$$
where the sums are over all ordered set partitions \((J_1, \ldots, J_\ell)\) of \([n]\) of type \(\lambda = (\lambda_1, \ldots, \lambda_\ell)\). For example, we have
\[
\text{Imm}_{21}(z) = \det \begin{bmatrix} z_{1,1} & z_{1,2} & z_{3,3} \\ z_{2,1} & z_{2,2} \\ z_{3,1} & z_{3,2} & z_{3,3} \end{bmatrix} = 3z_{1,1}z_{2,2}z_{3,3} - z_{1,2}z_{2,1}z_{3,3} - z_{1,3}z_{2,2}z_{3,1} - z_{1,1}z_{2,3}z_{3,2},
\]
and \(\epsilon^{21}(123) = 3\), \(\epsilon^{21}(213) = \epsilon^{21}(321) = \epsilon^{21}(132) = -1\), \(\epsilon^{21}(312) = \epsilon^{21}(231) = 0\). While Littlewood, Merris, and Watkins may not have written the equations (3) explicitly, we call them the *Littlewood–Merris–Watkins identities*. These identities have played an important role in the evaluation of (type-A) Hecke algebra characters at Kazhdan–Lusztig basis elements \([3], [4], [5]\), the formulation of a generating function for irreducible Hecke identities \([1]\) and \([2]\), and the interpretation of characters of chromatic symmetric functions \([3], [10]\). The identity in our main result (Theorem 3.1) plays an important role in the evaluation of hyperoctahedral group characters at elements of the type-BC Kazhdan-Lusztig basis \([11]\).

Let \(G = G(n)\) be the wreath product \(\mathbb{Z}/d\mathbb{Z} \rtimes S_n\). (We will suppress \(d\) and sometimes \(n\) from the notation.) Its class function space has \(2^d\) standard bases consisting of monomial characters, and it is possible to use a matrix of \(dn^2\) indeterminates to construct generating functions analogous to (3) for the elements of these bases. In Section 2 we review \(G\) and its monomial characters; in Section 3 we present our generating functions for these.

## 2. \(G\) and its monomial characters

The group \(G\) is generated by \(n\) elements \(s_1, \ldots, s_{n-1}, t\) subject to the relations
\[
s_i^2 = e \quad \text{for } i = 1, \ldots, n-1,
\]
\[
t^d = e,
\]
\[
ts_1ts_1 = s_1ts_1t,
\]
\[
s_is_j = s_js_i \quad \text{for } |i - j| \geq 2,
\]
\[
ts_j = st \quad \text{for } j \geq 2,
\]
\[
s_is_js_i = s_js_is_j \quad \text{for } |i - j| = 1.
\]

A one-line notation for elements of \(G\), analogous to that for elements of \(S_n\), uses sequences of integer multiples of complex \(d\)th roots of unity. Let \(\zeta\) be a primitive \(d\)th root of unity, and let \(S\) be the set of sequences
\[
\{(\zeta^{\gamma_1}w_1, \ldots, \zeta^{\gamma_n}w_n) | w_1 \cdots w_n \in S_n, (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n\}.
\]
We define an action of \(G\) on \(S\) by letting the generators act on a sequence \((a_1, \ldots, a_n)\) as follows.

1. \(s_i \circ (a_1, \ldots, a_n) = (a_1, \ldots, a_{i-1}, a_{i+1}, a_i, a_{i+2}, \ldots, a_n)\),
2. \(t \circ (a_1, \ldots, a_n) = (\zeta a_1, a_2, \ldots, a_n)\).

Letting each element \(g \in G\) act on the sequence \((1, \ldots, n)\), we obtain a bijection between \(G\) and \(S\). If \(g \circ (1, \ldots, n) = (\zeta^{\gamma_1}w_1, \ldots, \zeta^{\gamma_n}w_n)\), we define this second sequence to be the *one-line notation* of \(g\), and we write \(g = (\gamma, w)\), where \(\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}/d\mathbb{Z}^n, w \in S_n\). In particular, the identity element \(e\) has one-line notation \(1 \cdots n\).

Since \(G\) is a finite group, Brauer’s Induced Character Theorem implies that the set of monomial characters of \(G\) spans the *trace space* \(\mathcal{T}(G)\) of \(G\), the set of all linear functionals \(\theta : \mathbb{C}[G] \to \mathbb{C}\) satisfying \(\theta(gh) = \theta(hg)\) for all \(g, h \in G\). (See, e.g., [12].) This includes all \(G\)-characters. \(\mathcal{T}(G)\) has dimension equal to the number of conjugacy classes of \(G\), equivalently, to the number of sequences \(\lambda = (\lambda^0, \ldots, \lambda^{d-1})\) of \(d\) (possibly empty) integer partitions, with
\[
|\lambda^0| + \cdots + |\lambda^{d-1}| = n.
\]
We call such a sequence a *d-partition* of \([n]\) and write \(\lambda \vdash n\).

In order to describe natural bases of \(\mathcal{T}(G)\), we introduce certain subgroups of \(G\) which are analogous to Young subgroups of \(S_n\). Fix *d-partition* \(\lambda = (\lambda^0, \ldots, \lambda^{d-1}) \vdash n\), and define \(r_k = \ell(\lambda^k)\) for \(k = 0, \ldots, d-1\). We will say that an ordered set partition of \([n]\) of type
\[
(\lambda^0_0, \ldots, \lambda^0_{r_0}, \lambda^1_0, \ldots, \lambda^1_{r_1}, \ldots, \lambda^{d-1}_0, \ldots, \lambda^{d-1}_{r_{d-1}}),
\]

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has type \( \lambda \). In particular, let \( \mathbf{K}(\lambda) = (K^0, \ldots, K^{r_d-1}) \) be the ordered set partition of \([n]\) of type \( \lambda \) whose blocks are the \( r_0 + \cdots + r_{d-1} \) subintervals

\[
K^0 = [1, \lambda_0], \quad K^d = [\lambda_0 + 1, \lambda_1 + \lambda_2], \quad \ldots, \quad K^{d-1} = [n - \lambda_d + 1, n]
\]

(5) of \([n]\). For \( 1 \leq i \leq j \leq n \), define the element \( t_i = s_{i-1} \cdots s_is_{i+1} \in \mathcal{G} \), and let

\[
\mathcal{G}([i, j]) \cong \mathbb{Z}/d\mathbb{Z} \rtimes \mathcal{S}_{j-i+1}
\]

be the subgroup of \( \mathcal{G} \) generated by \( \{t_i, s_i, \ldots, s_{j-1}\} \). For \( k = 0, \ldots, d-1 \), use (5) to define the subgroup

\[
\mathcal{G}(\lambda, k) := \mathcal{G}(K^0) \cdots \mathcal{G}(K^{d-1}) \cong \mathcal{G}(\lambda^0) \times \cdots \times \mathcal{G}(\lambda^{d-1})
\]

of \( \mathcal{G} \), and finally define the Young subgroup

\[
\mathcal{G}(\lambda) := \mathcal{G}(\lambda, 0) \cdots \mathcal{G}(\lambda, d-1) \cong \prod_{k=0}^{d-1} (\mathcal{G}(\lambda^0) \times \cdots \times \mathcal{G}(\lambda^0))
\]

of \( \mathcal{G} \). Each element \( y \in \mathcal{G}(\lambda) \) factors uniquely as \( y_0 \cdots y_{d-1} \) with \( y_k \in \mathcal{G}(\lambda, k) \).

Several natural representations of \( \mathcal{G} \) are defined by using symmetric group representations and induction from \( \mathcal{G}(\lambda) \). First, observe that the subgroup of \( \mathcal{G} \) generated by \( s_1, \ldots, s_{n-1} \) is isomorphic to \( \mathcal{S}_n \), and that each \( r \)-dimensional \( \mathcal{S}_n \)-representation \( \rho \) can trivially be extended to an \( r \)-dimensional \( \mathcal{G} \)-representation in at least \( d \) ways: by defining \( \rho(t) = \zeta^k I \) for \( k = 0, \ldots, d-1 \). If the character of the \( \mathcal{S}_n \)-representation is \( \chi \), call its extension \( \delta_k \chi \). Thus the two one-dimensional \( \mathcal{S}_n \)-representations

\[
1: s_i \mapsto 1 \quad (w \mapsto 1 \text{ for all } w \in \mathcal{S}_n),
\]

\[
\epsilon: s_i \mapsto -1 \quad (w \mapsto (-1)^{\text{inv}(w)} \text{ for all } w \in \mathcal{S}_n)
\]

yield \( 2d \) one-dimensional \( \mathcal{G} \)-representations:

\[
\delta_k : (s_i, t) \mapsto (1, \zeta^k), \quad (g = (\gamma, w) \mapsto \zeta^{k(\gamma_1 + \cdots + \gamma_n)} \text{ for all } g \in \mathcal{G}),
\]

\[
\delta_k \epsilon : (s_i, t) \mapsto (-1, \zeta^k), \quad (g = (\gamma, w) \mapsto (-1)^{\text{inv}(w)} \zeta^{k(\gamma_1 + \cdots + \gamma_n)} \text{ for all } g \in \mathcal{G}),
\]

for \( k = 0, \ldots, d-1 \). Here, \( \text{inv}(w) \) denotes the Coxeter length of \( w \). (See, e.g., [2, p. 15].) Next, observe that for any \( d \)-tuple \( (H_0, \ldots, H_{d-1}) \) of subgroups of a group \( G \) which satisfy

\[
H := H_0 \cdots H_{d-1} \cong H_0 \times \cdots \times H_{d-1},
\]

(6) and any \( d \)-tuple \( (\theta_0, \ldots, \theta_{d-1}) \) of characters of these, we have that the function \( \theta = \theta_0 \otimes \cdots \otimes \theta_{d-1} \) defined by \( \theta(\theta_0(\cdots(\theta_{d-1}(h_{d-1}) \cdots \theta_0(h_0))) = \theta_0(h_0) \cdots \theta_{d-1}(h_{d-1}) \) is a character of \( H \), and \( \theta^G_H \) is a character of \( G \). In particular, the Young subgroup \( \mathcal{G}(\lambda) \) has the form (6) with \( H_k = \mathcal{G}(\lambda, k) \). For every \( d \)-tuple \( \beta = (\beta_0, \ldots, \beta_{d-1}) \in \{1, \epsilon\}^d \) of one-dimensional symmetric group characters we have the one-dimensional \( \mathcal{G}(\lambda) \)-character

\[
\delta_0 \beta_0 \otimes \cdots \otimes \delta_d \beta_{d-1},
\]

(7) the corresponding monomial \( \mathcal{G} \)-character

\[
\beta^\lambda := (\delta_0 \beta_0 \otimes \cdots \otimes \delta_d \beta_{d-1}) \bigg|_{\mathcal{G}(\lambda)},
\]

and the basis \( \{\beta^\lambda | \lambda \vdash n\} \) of \( \mathcal{T}(\mathcal{G}) \). The irreducible character basis \( \{\chi^\lambda | \lambda \vdash n\} \) of \( \mathcal{T}(\mathcal{G}) \) can be defined somewhat similarly. Given \( \lambda = (\lambda^0, \ldots, \lambda^{d-1}) \vdash n \), define the \( d \)-partition \( \lambda^* = (|\lambda^0|, \ldots, |\lambda^{d-1}|) \), and the \( \mathcal{G}(\lambda^*) \)-character

\[
\delta_0 \chi^{\lambda^0} \otimes \cdots \otimes \delta_d \chi^{\lambda^{d-1}},
\]

where \( \chi^\lambda \) is the irreducible \( \mathcal{G}(|\lambda^0|) \)-character indexed by the partition \( \lambda^k \). The corresponding induced characters

\[
\chi^\lambda = (\delta_0 \chi^{\lambda^0} \otimes \cdots \otimes \delta_d \chi^{\lambda^{d-1}}) \bigg|_{\mathcal{G}(\lambda^*)}
\]

are the irreducible characters of \( \mathcal{G} \). (See, e.g., [1, p. 219].)
For the purpose of creating generating functions for characters $\beta^\lambda$, it will be convenient to realize each as the character of a submodule of the group algebra $\mathbb{C}[G]$, with $G$ acting by left multiplication. To do this, we consider an arbitrary finite group $G$, a subgroup $H$, an $H$-character $\theta$, and the element
\[ T_H^\theta := \sum_{h \in H} \theta(h^{-1}) h \in \mathbb{C}[G]. \]

**Proposition 2.1.** Let $H$ be a subgroup of a finite group $G$ and let $\rho$ be a one-dimensional complex representation of $H$ with character $\theta$ ($= \rho$). Let $U = \langle u_1, \ldots, u_r \rangle$ be a transversal of representatives of cosets of $H$ in $G$. Let $G$ act by left multiplication on the submodule
\[ V := \text{span}_\mathbb{C}\{u_i T_H^\theta \mid 1 \leq i \leq r\} \tag{8} \]
of $\mathbb{C}[G]$. Then $V$ is a $G$-module with character $\theta^G_H$.

**Proof.** To see that $V$ is a $G$-module, consider the action of $g \in G$ on the $j$th element of the defining basis of $V$. Let $u_i H$ be the unique coset satisfying $gu_i H = u_i H$, i.e., $u_i^{-1} gu_j \in H$. Then we have
\[
gu_j T_H^\theta = gu_j \sum_{h \in H} \theta(h^{-1}) h = u_i \sum_{h \in H} \theta(h^{-1}) u_i^{-1} gu_j h = u_i \sum_{h' \in H} \theta((h')^{-1} u_i^{-1} gu_j) h'
\]
for $\theta = \rho$ is a homomorphism. It follows that in the $j$th column of the matrix representing $g$, all components are 0 except for the $i$th, which is $\theta(u_i^{-1} gu_j)$. But this is precisely the formula for entries of the matrix $\rho^G_H(g)$.

(See, e.g., [9, Defn. 1.12.2].)

For $\chi = \theta^G_H$, Proposition 2.1 allows us to express $T_H^\chi$ as a sum of conjugates of $T_H^\theta$.

**Lemma 2.2.** Let groups $G$, $H$, transversal $U = \langle u_1, \ldots, u_r \rangle$, $H$-character $\theta$, and $G$-module $V$ be as in Proposition 2.1, and let $A = (a_{i,j})$ be the matrix of $g \in G$ with respect to the defining basis (8) of $V$. Then $a_{i,j}$ equals the coefficient of $g^{-1}$ in $u_j T_H^\theta u_i^{-1}$. In particular if $\chi$ is the character of $V$, then we have the identity
\[
\sum_{i=1}^{r} u_i T_H^\theta u_i^{-1} = \sum_{g \in G} \chi(g) g^{-1}.
\]
in $\mathbb{C}[G]$.

**Proof.** By the proof of Proposition 2.1, we have $a_{i,j} = \theta(u_i^{-1} gu_j)$ if some $h \in H$ satisfies $g = u_h u_j^{-1}$, and is 0 otherwise. On the other hand, we have
\[
u_j T_H^\theta u_i^{-1} = \sum_{h \in H} \theta(h^{-1}) u_j h u_i^{-1}.
\]
If there is no $h \in H$ satisfying $g^{-1} = u_j h u_i^{-1}$, then the coefficient of $g^{-1}$ in (9) is 0. Otherwise, the coefficient of $g^{-1}$ is
\[
\theta(h^{-1}) = \theta(u_i^{-1} gu_j).
\]
It follows that $a_{i,j}$ is equal to the coefficient of $g^{-1}$ in $u_j T_H^\theta u_i^{-1}$. Thus $\chi(g) = \sum_i a_{i,i}$ is equal to the coefficient of $g^{-1}$ in $\sum_i u_i T_H^\theta u_i^{-1}$. \qed

For $G = G$, $H = G(\lambda)$, and $\theta$ as in (7), the module $V$ (8) has a particularly nice form. The element $T_H^\theta$ factors as $T_{G(\lambda,0)}^{\beta_0} \cdots T_{G(\lambda,d-1)}^{\beta_{d-1}}$, and each coset $u G(\lambda)$ of $G(\lambda)$ has a unique representative $g = (\gamma, w)$ satisfying $\gamma_1 = \cdots = \gamma_n = 0$ and $w_i < w_{i+1}$ for $i, i + 1$ belonging to the same block of $K(\lambda)$, i.e.,
\[
w_1 < \cdots < w_{\lambda_1^1}, \quad w_{\lambda_1^1+1} < \cdots < w_{\lambda_1^1+\lambda_2^2}, \quad \ldots, \quad w_{n-\lambda_{d-1}^d} < \cdots < w_n.
\]

Letting $G(\lambda)^-$ be the set of such coset representatives, we have
\[
V = V(\lambda, \beta) = \text{span}_\mathbb{C}\{u T_{\bar{G}(\lambda,0)}^{\beta_0} \cdots T_{\bar{G}(\lambda,d-1)}^{\beta_{d-1}} \mid u \in G(\lambda)^-\},
\]
and the following special case of Lemma 2.2.
Corollary 2.3. Fix a d-partition $\lambda \vdash n$. For each one-dimensional $G(\lambda)$-character $\theta$ of the form (7), the monomial $G$-character $\beta^\lambda = \theta \big|_{G(\lambda)}$ satisfies

$$\sum_{u \in G(\lambda)} uT^\theta_{G(\lambda)}u^{-1} = \sum_{g \in G} \beta^\lambda(g^{-1})g.$$  \hspace{1cm} (11)

For $d = 1, 2$, the group $G$ (equal to the symmetric group or the hyperoctahedral group) has real-valued irreducible characters. Therefore each group element is conjugate to its inverse, and the final sum of (11) may be expressed as $\sum_{g \in G} \beta^\lambda(g^{-1})g$.

3. Main result

To construct generating functions analogous to (3) for monomial characters of $G$, we first extend the ring $\mathbb{C}[z] = \mathbb{C}[z_1, z_2, \ldots, z_n]$ and its $n!$-dimensional subspace span\{\$z_1, w_1 \cdots z_n, w_n | w \in S_n\$\} which make the definition (1) possible. The one-line notation (4) for elements of $G$ suggests that we define a ring $\mathbb{C}[x]$ using the $dn^2$ indeterminates

$$x = \{x_{i,p} | i, p \in [n], k \in \mathbb{Z}/d\mathbb{Z}\}$$

and the $d^n n!$-dimensional subspace

$$\text{span}\{x_{1,g_1} \cdots x_{n,g_n} | g \in G\}$$

of $\mathbb{C}[x]$, where $g_1 \cdots g_n$ has the form (4). We call (12) the $G$-immanant space.

One can think of $x$ as an $n \times dn$ matrix of indeterminates, and of each monomial in the $G$-immanant space as a collection of $n$ entries of $x$, with one entry per row and one entry per column (mod $d$). For example let $(n, d) = (4, 3)$, so that $\zeta = e^{2\pi i/3}$. Let us economize notation by writing $\hat{m} := \zeta^m$, $\check{m} := \zeta^2 m$, e.g., $x_{2,3} = x_{2, \check{3}}$. Then the 48 indeterminates are

$$x = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,\check{1}} & x_{1,\check{2}} & x_{1,\check{3}} & x_{1,\check{4}} \\
 x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & x_{2,\check{1}} & x_{2,\check{2}} & x_{2,\check{3}} & x_{2,\check{4}} \\
x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} & x_{3,\check{1}} & x_{3,\check{2}} & x_{3,\check{3}} & x_{3,\check{4}} \\
x_{4,1} & x_{4,2} & x_{4,3} & x_{4,4} & x_{4,\check{1}} & x_{4,\check{2}} & x_{4,\check{3}} & x_{4,\check{4}} \end{bmatrix}. \hspace{1cm} (13)$$

Some monomials belonging to the $G$-immanant space are

$$x_{1,2}x_{2,4}x_{3,3}x_{4,1}, \quad x_{1,\check{1}}x_{2,\check{2}}x_{3,\check{3}}x_{4,\check{4}}, \quad x_{1,3}x_{2,\check{1}}x_{3,2}x_{4,\check{2}}, \quad x_{1,\check{1}}x_{2,\check{2}}x_{3,4}x_{4,\check{4}}.$$

For $u \in S_n$, $g \in G$, we will find it convenient to write monomials in the $G$-immanant space as $x^{u,g} := x_{u_1,g_1} \cdots x_{u_n,g_n}$.

By the action defined after (4) and commutativity, these monomials satisfy

$$x^{s_i u, s_i g} = x_{u_1,g_1} \cdots x_{u_{i-1},g_{i-1}} x_{u_{i+1},g_{i+1}} x_{u_i,g_i} x_{u_{i+2},g_{i+2}} \cdots x_{u_n,g_n} = x^{u,g}$$

and thus

$$x^{u,g} = x^{u^{-1}, u^{-1} g} = x^{e, u^{-1} g}$$

for all $u \in S_n$, $g \in G$. It follows that for any fixed $u \in S_n$, the $G$-immanant subspace of $\mathbb{C}[x]$ may also be expressed as span$_{\mathbb{C}}\{x^{u,g} | g \in G\}$. The left- and right-regular representations of $G$,

$$h_1 \circ g \circ h_2 = h_1 gh_2$$

for $g, h_1, h_2 \in G$, define left- and right-actions of $G$ on the $G$-immanant space,

$$h_1 \circ x^{e,g} \circ h_2 = x^{e,h_1 gh_2}.$$  \hspace{1cm} (15)

Now we state a natural $G$-analog of the definition (1). For any function $\theta : G \to \mathbb{C}$, define the type-$G$ $\theta$-immanent to be the generating function

$$\text{Imm}_\theta^G(x) = \sum_{g \in G} \theta(g^{-1})x^{e,g}.$$  \hspace{1cm} (16)
for evaluations of $\theta$. Our counterintuitive use of $g^{-1}$ in place of $g$ is necessitated by Proposition 2.1 – Corollary 2.3. (See also [15, Eq. (1)].) By the comment following Corollary 2.3, symmetric group and hyperoctahedral group ($\mathcal{B}_n \cong \mathbb{Z}/2\mathbb{Z} \wr S_n$) immanants can be written without inverses:

$$\text{Imm}^\mathcal{B}_n(\theta) = \sum_{w \in \mathcal{B}_n} \theta(w)x^{\varepsilon(w)}, \quad \text{Imm}^\mathcal{G}_n(g) = \sum_{g \in \mathcal{G}_n} \theta(g)x^\mathcal{G}.$$  

For economy, we will suppress $\mathcal{G}_n$ from the notation of symmetric group immanants.

Immanants for the $2d$ one-dimensional characters $\delta_0, \ldots, \delta_{d-1}$, $\delta_{0\epsilon}, \ldots, \delta_{d-1\epsilon}$ of $\mathcal{G}$ may be expressed very simply in terms of $n \times n$ matrices whose entries are linear combinations of the indeterminates $x$. In particular, define the $d \times n$ matrices $Q_0(x), \ldots, Q_{d-1}(x)$ by $Q_k(x) = (q_{i,j,k}(x))_{i,j \in [n]}$, where

$$q_{i,j,k}(x) = x_{i,j} + \zeta^{-k}x_{i,-j} + \zeta^{-2k}x_{i,-j} + \cdots + \zeta^{-(d-1)k}x_{i,-j}.$$  

Then we have

$$\text{per}(Q_k(x)) = \sum_{g=(\gamma, \tau) \in \mathcal{G}} (-1)^{\text{inv}(w)}\zeta^{-(\gamma+\cdots+\tau)}x^\mathcal{G} = \text{Imm}^\mathcal{G}_n(\theta),$$  

$$\text{det}(Q_k(x)) = \sum_{g=(\gamma, \tau) \in \mathcal{G}} (-1)^{\text{inv}(w)}\zeta^{-(\gamma+\cdots+\tau)}x^\mathcal{G} = \text{Imm}^\mathcal{G}_n(\theta).$$  

Returning to our $(n,d) = (4,3)$ example, we have the matrices

$$Q_0(x) = \begin{bmatrix} x_{1,1} + x_{1,1} + x_{1,1} & x_{1,2} + x_{1,2} + x_{1,2} & x_{1,3} + x_{1,3} + x_{1,3} & x_{1,4} + x_{1,4} + x_{1,4} \\ \vdots & \vdots & \vdots & \vdots \\ x_{4,1} + x_{4,1} + x_{4,1} & x_{4,2} + x_{4,2} + x_{4,2} & x_{4,3} + x_{4,3} + x_{4,3} & x_{4,4} + x_{4,4} + x_{4,4} \end{bmatrix},$$  

$$Q_1(x) = \begin{bmatrix} x_{1,1} + \zeta^2x_{1,1} + \zeta x_{1,1} & x_{1,2} + \zeta^2x_{1,2} + \zeta x_{1,2} & x_{1,3} + \zeta^2x_{1,3} + \zeta x_{1,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ x_{4,1} + \zeta^2x_{4,1} + \zeta x_{4,1} & x_{4,2} + \zeta^2x_{4,2} + \zeta x_{4,2} & x_{4,3} + \zeta^2x_{4,3} + \zeta x_{4,3} & \cdots \end{bmatrix},$$  

$$Q_2(x) = \begin{bmatrix} x_{1,1} + \zeta^2x_{1,1} + \zeta^2 x_{1,1} & x_{1,2} + \zeta^2x_{1,2} + \zeta^2 x_{1,2} & x_{1,3} + \zeta^2x_{1,3} + \zeta^2 x_{1,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ x_{4,1} + \zeta^2x_{4,1} + \zeta^2 x_{4,1} & x_{4,2} + \zeta^2x_{4,2} + \zeta^2 x_{4,2} & x_{4,3} + \zeta^2x_{4,3} + \zeta^2 x_{4,3} & \cdots \end{bmatrix}.$$  

Immanants for the other monomial characters of $\mathcal{G}$ may be expressed as sums of products of $\mathcal{G}_p$-immanants of $p \times p$ submatrices of $Q_0(x), \ldots, Q_{d-1}(x)$, or of $\mathcal{G}(p)$-immanants of $p \times dp$ submatrices of $x$. In terms of the submatrix notation (2), these submatrices will have the forms $Q_k(x)_{M,M}$ and $x_{M,C,M}$ for some subset $M \subset [n]$, where

$$C_d = \{\zeta^k \mid k \in \mathbb{Z}/d\mathbb{Z}\}, \quad C_d M = \{\zeta^k m \mid k \in \mathbb{Z}/d\mathbb{Z}, m \in M\}.$$  

For example, with $(n,d) = (4,3)$ and $M = \{2,3\}$, we have

$$C_3 = \{1, \zeta, \zeta^2\}, \quad C_3 M = \{2, 3, 2, 3, 3, 2, 3\},$$  

and the matrices (18) and (13) have submatrices

$$Q_0(x)_{M,M} = \begin{bmatrix} x_{2,2} + x_{2,2} + x_{2,2} & x_{2,3} + x_{2,3} + x_{2,3} \\ x_{3,2} + x_{3,2} + x_{3,2} & x_{3,3} + x_{3,3} + x_{3,3} \end{bmatrix},$$  

$$x_{M,C,M} = \begin{bmatrix} x_{2,2} & x_{2,3} & x_{2,3} & x_{2,3} \\ x_{3,2} & x_{3,3} & x_{3,3} & x_{3,3} \end{bmatrix}.$$  

Now we may state $\mathcal{G}$-analogs of the Littlewood-Merris-Watkins generating functions (3) in terms of the monomial characters $\nu^\mathcal{G} = 1^\mathcal{G}$ and $e^\mathcal{G} = \zeta^\mathcal{G}$ of $\mathcal{G}_p$, for $p \leq n$.

**Theorem 3.1.** Fix $d$-partition $\lambda = (\lambda^0, \ldots, \lambda^{d-1}) \vdash n$, and let $a_\lambda = |\lambda^k|$. Fix symmetric group character sequence $\beta = (\beta_0, \ldots, \beta_{d-1}) \in \{1, e\}^d$ and define

$$\beta_k^\lambda = \beta_k^\lambda \zeta^\mathcal{G} \in \{e^\mathcal{G}, \zeta^\mathcal{G}\}, \quad k = 0, \ldots, d-1.$$
Then we have
\[
\text{Imm}_{\beta_{k-1}^6}(x) = \sum_{(I_0, \ldots, I_{d-1})} \text{Imm}_{\beta_{0}^6}(Q_0(x)_{I_0, I_0}) \cdots \text{Imm}_{\beta_{d-1}^6}(Q_{d-1}(x)_{I_{d-1}, I_{d-1}}),
\]
where the sum is over all ordered set partitions of \([n]\) of type \(\mathbf{x} = (a_0, \ldots, a_{d-1}).\)

**Proof.** Define the \(G(\lambda)\)-character \(\theta = \delta_0 \beta_0 \odot \cdots \odot \delta_{d-1} \beta_{d-1}\) and let \(\beta^\lambda = \theta \uparrow_{G(\lambda)}^G\). By (16), (15), and Corollary 2.3, we can express the left-hand side of (19) as
\[
\sum_{g \in \mathcal{G}} \beta^\lambda(g^{-1})x^{e,g} = \sum_{g \in \mathcal{G}} \beta^\lambda(g^{-1})(g \circ x^{e,c}) = \sum_{u \in \mathcal{G}(\lambda)^-} (uT_{G(\lambda)}^0)^{u^{-1} \circ x^{e,c}}.
\]

Now consider the right-hand side of (19), and define \(r_k = \ell(\lambda^k)\). By (3), we may rewrite this as a sum of products of permanents and determinants,
\[
\sum_{\mathcal{J}} \left( \prod_{i=0}^{r_0} \text{Imm}_{\beta_{0}^6}(Q_0(x)_{J_i^0, J_i^0}) \right) \cdots \left( \prod_{i=0}^{r_{d-1}} \text{Imm}_{\beta_{d-1}^6}(Q_{d-1}(x)_{J_{d-1}^{d-1}, J_{d-1}^{d-1}}) \right),
\]
where the sum is over all ordered set partitions \(\mathcal{J} = (J_0^0, \ldots, J_{r_0}^0, \ldots, J_1^{d-1}, \ldots, J_{r_{d-1}}^{d-1})\) of \([n]\) of type \(\lambda\), and where \(\text{Imm}_m = \text{det}\), \(\text{Imm}_m = \text{per}\). For all \(i, k\), the indeterminates that appear in \(Q_k(x)_{J_i^k, J_i^k}\) are \(x_{J_i^k, C_i J_i^k}\). By (17), we may again rewrite (21) as a sum
\[
\sum_{\mathcal{J}} \left( \prod_{i=1}^{r_0} \text{Imm}_{\beta_{0}^6}(x_{J_i^0, C_i J_i^0}) \right) \cdots \left( \prod_{i=1}^{r_{d-1}} \text{Imm}_{\beta_{d-1}^6}(x_{J_i^{d-1}, C_i J_i^{d-1}}) \right)
\]
in which each factor of each term has the form
\[
\text{Imm}_{\beta_{k}^6}(x_{J_i^k, C_i J_i^k}) = \begin{cases} \sum_{g = (n, w) \in \mathcal{G}(J_i^k)} \zeta^{-k(\gamma_1 + \cdots + \gamma_n)}(x_{J_i^k, C_i J_i^k})^{e,g} & \text{if } \beta_k = 1, \\ \sum_{g = (n, w) \in \mathcal{G}(J_i^k)} \zeta^{-k(\gamma_1 + \cdots + \gamma_n)(-1)^l(w)}(x_{J_i^k, C_i J_i^k})^{e,g} & \text{if } \beta_k = e. \end{cases}
\]

Define the set partition \(\mathbf{K} = (K_0^0, \ldots, K_{r_0}^0, \ldots, K_1^{d-1}, \ldots, K_{r_{d-1}}^{d-1})\) of type \(\lambda\) as in (5), and for each ordered set partition \(\mathcal{J}\) of type \(\lambda\) define \(u = u(\mathcal{J}) \in \mathcal{G}(\lambda)^-\) to be the element whose one-line notation has the \(\lambda^k\) consecutive letters \(K_i^k\) in positions \(J_i^k\), for \(k = 0, \ldots, d-1\) and \(i = 1, \ldots, r_k\). In particular, \(u^{-1}\) is the element in \(\mathfrak{S}_n \subset \mathcal{G}\) whose one-line notation contains the increasing rearrangement of \(J_i^k\) in the consecutive positions \(K_i^k\) for \(k = 0, \ldots, d-1\) and \(i = 1, \ldots, r_k\). By (10), the map \(\mathcal{J} \mapsto u(\mathcal{J})\) defines a bijective correspondence between ordered set partitions of type \(\lambda\) and \(\mathcal{G}(\lambda)^-\). Thus in the expansion of the product (22), the monomials which appear are precisely the set \(\{x^{u^{-1}}y^{-1} | y \in \mathcal{G}(\lambda)\}\). Factoring \(y = y_0 \cdots y_{d-1}\) with \(y_k \in \mathcal{G}(\lambda, k)\), we may express the coefficient of each such monomial as
\[
\delta_0 \beta_0(y_0^{-1}) \cdots \delta_{d-1} \beta_{d-1}(y_{d-1}^{-1}) = \theta(y^{-1}).
\]
Using these facts and (14), (15), we may rewrite (21) as
\[
\sum_{u \in \mathcal{G}(\lambda)^-} \theta(y^{-1})x^{u^{-1}, y^{-1}} = \sum_{u \in \mathcal{G}(\lambda)^-} \sum_{y \in \mathcal{G}(\lambda)} \theta(y^{-1})(yu^{-1} \circ x^{e,c}) = \sum_{u \in \mathcal{G}(\lambda)^-} (uT_{G(\lambda)}^0)^{u^{-1} \circ x^{e,c}}
\]
to see that it is equal to (20).\(\square\)

We illustrate with an example. Consider the group \(\mathcal{G} = \mathbb{Z}/3\mathbb{Z} \wr \mathfrak{S}_6\). Its trace space \(\mathcal{T} \mathcal{G}\) has dimension equal to the number of 3-partitions of 6, and its immanent space
\[
\text{span}_{\mathbb{C}} \{x_{i_1, g_1} \cdots x_{i_6, g_6} | (g_1, \ldots, g_6) \in \mathcal{G} \}
\]
requires the \(6^2 - 3 = 108\) indeterminates \(\{x_{i, m}, x_{i, \bar{m}}, x_{i, \bar{m}} | i, m \in [6]\}\) where we define \(\bar{m} := \zeta m\), \(\bar{m} := \zeta^2 m\), as in (13). The \(2^3 = 8\) monomial character bases correspond to the triples of one-dimensional symmetric group characters \((1, 1, 1), (1, 1, \epsilon), (1, \epsilon, 1), \ldots, (\epsilon, \epsilon, \epsilon), \) so that the basis corresponding to \((\epsilon, \epsilon, 1)\) is
\[
\{(\epsilon, \epsilon, 1)^\lambda = (\epsilon \otimes \delta_1 \epsilon \otimes \delta_2) \uparrow_{G(\lambda)}^G | \lambda \vdash 6\}.
\]
Consider the basis element \((\epsilon, \epsilon, 1)^{(21,1,2)}\). To evaluate \((\epsilon, \epsilon, 1)^{(21,1,2)}(g)\) for all \(g \in \mathcal{G}\), we write its immanant \(\text{Imm}^g_{(\epsilon, \epsilon, 1)^{(21,1,2)}}(x)\) as a sum of 60 terms

\[
\text{Imm}^g_{(\epsilon, \epsilon, 1)^{(21,1,2)}}(Q_0(x)_{123,123})\text{Imm}^{i_1}_{1}(Q_1(x)_{4,4})\text{Imm}^{i_2}_{2}(Q_2(x)_{56,56}) + \text{Imm}^g_{(\epsilon, \epsilon, 1)^{(21,1,2)}}(Q_0(x)_{123,123})\text{Imm}^{i_1}_{1}(Q_1(x)_{5,5})\text{Imm}^{i_2}_{2}(Q_2(x)_{46,46}) + \text{Imm}^g_{(\epsilon, \epsilon, 1)^{(21,1,2)}}(Q_0(x)_{123,123})\text{Imm}^{i_1}_{1}(Q_1(x)_{6,6})\text{Imm}^{i_2}_{2}(Q_2(x)_{45,45}) + \text{Imm}^g_{(\epsilon, \epsilon, 1)^{(21,1,2)}}(Q_0(x)_{124,124})\text{Imm}^{i_1}_{1}(Q_1(x)_{3,3})\text{Imm}^{i_2}_{2}(Q_2(x)_{56,56}) \\
\vdots
\]

(23)

each corresponding to an ordered set partition of \([6]\) of type \((3, 1, 2)\). Consider the term corresponding to the ordered set partition \((136, 4, 25)\). It is a product of the three factors

\[
\text{Imm}^{i_1}_{1}(Q_1(x)_{136,136}) = \det \begin{bmatrix}
x_{1,1} + x_{1,1} + x_{1,1} & x_{1,3} + x_{1,3} + x_{1,3}
x_{3,1} + x_{3,1} + x_{3,1} & x_{3,3} + x_{3,3} + x_{3,3}
\end{bmatrix} (x_{6,6} + x_{6,6} + x_{6,6})
\]

\[
+ \det \begin{bmatrix}
x_{1,1} + x_{1,1} + x_{1,1} & x_{1,6} + x_{1,6} + x_{1,6}
x_{6,1} + x_{6,1} + x_{6,1} & x_{6,6} + x_{6,6} + x_{6,6}
\end{bmatrix} (x_{3,3} + x_{3,3} + x_{3,3})
\]

\[
+ \det \begin{bmatrix}
x_{3,3} + x_{3,3} + x_{3,3} & x_{3,6} + x_{3,6} + x_{3,6}
x_{6,3} + x_{6,3} + x_{6,3} & x_{6,6} + x_{6,6} + x_{6,6}
\end{bmatrix} (x_{1,1} + x_{1,1} + x_{1,1}),
\]

(24)

It is easy to see that this term, like all others in (23), contributes 3 to the coefficient of \(x_{1,1}x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,6}\). Thus we have

\[
(\epsilon, \epsilon, 1)^{(21,1,2)}(123456) = 180.
\]

Now consider the computation of \((\epsilon, \epsilon, 1)^{(21,1,2)}(623451)\). Terms in (23) with nonzero contributions to the coefficient of

\[
x_{1,6}x_{2,2}x_{3,3}x_{4,4}x_{5,5}x_{6,1}
\]

are those corresponding to ordered set partitions in which 1 and 6 belong to the same block. (Otherwise the variables \(x_{1,6}, x_{6,1}\) will not appear in the term.) Each such ordered set partition has one of the forms

\[
(1a6, 4, bc), \quad (1a6, b, 4c), \quad (ab4, c, 16), \quad (235, 4, 16).
\]

There are three terms corresponding to ordered set partitions of the first form, including (24). Multiplying the three factors in (24), we find the desired monomial (25) as

\[
(-x_{1,6}x_{6,1})(x_{3,3})(\zeta^2 x_{4,4})(x_{2,2}x_{5,5}),
\]

i.e., the term contributes \(-\zeta^2\) to the coefficient. The remaining two terms having ordered set partitions of the form \((1a6, 4, bc)\) contribute \(-\zeta^2\) as well. Terms corresponding to the six ordered set partitions \((1a6, b, 4c)\) contribute \(-\zeta\) each,

\[
(-x_{1,6}x_{6,1})(x_{a,a})(x_{b,b})(\zeta x_{4,4}x_{c,c}),
\]

terms corresponding to the three ordered set partitions \((ab4, c, 16)\) contribute 3\(\zeta\) each,

\[
(3x_{a,a}x_{b,b}x_{4,4})(x_{c,c})(x_{1,6}x_{6,1}),
\]

and the term corresponding to the ordered set partition \((235, 4, 16)\) contributes \(3\zeta^2\) as

\[
(3x_{2,2}x_{3,3}x_{5,5})(\zeta^2 x_{4,4})(x_{1,6}x_{6,1}).
\]

Thus we have

\[
(\epsilon, \epsilon, 1)^{(21,1,2)}(623451) = -3\zeta^2 - 6\zeta + 9\zeta + 3 = 6 + 6\zeta.
\]

It would be interesting to extend Theorem 3.1 to obtain a generating function for the monomial characters of Hecke algebras of wreath products [1], as was done for monomial characters of the Hecke algebra of \(S_n\) in [6, Thm. 2.1].
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References


