

Counting on Matrices

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ABSTRACT: In this paper, we have found formulas for the number of rectangular and symmetric matrices with the line sums divisible by a given integer. As an application, we have derived an explicit formula enumerating the number of traceless $n \times n$, $(0,1)$ symmetric matrices having line sums divisible by a given integer, which leads to an enumeration of labeled regular graphs with n vertices. Also, we have found a formula for the weighted enumerator (in terms of rows and columns) of rectangular matrices, which subsequently yields some nice identities satisfying curious reciprocity phenomena.

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1. Introduction

Enumeration of matrices (arrays) are not new in the literature. For instance, in [2] Anand, Dumir and Gupta addressed the problem of enumerating non negative square matrices with fixed line sums. For instance, they proved that if $A(n) = \frac{H(n,2)}{(n!)^2}$, then

$$\sum_{n=0}^{\infty} A(n)x^n = (1-x)^{-\frac{1}{2}}e^{\frac{x}{2}},$$

where $H(n, r)$ denotes the number of $n \times n$ matrices with entries from \mathbb{N} and line sums (row sums and column sums) equal to r . For $n = 3$, they also found a generating function for $H(3, r)$. In [1] Abramson and Moser addressed the problem of counting matrices with fixed row sum and column sum. Carlitz in [3, 4] and Gupta in [7] considered analogous problems for symmetric matrices and found several remarkable results for particular values of n and r . In this paper, we have also addressed similar kind of problems for rectangular and symmetric matrices.

Unlike them, we have considered matrices with line sums divisible by a fixed positive integer k . For instance, one can think of the problem of finding the number of $m \times n$ matrices with entries from the set $\{0, 1, \dots, q\}$ such that all the row sums and column sums are even integers. It is not difficult to see that if q is odd, then one can find a pretty simple recursion which yields that the required number is $(q+1)^{(m-1)(n-1)}\left(\frac{q+1}{2}\right)^{(m+n-1)}$. But, the problem arises when q is even since that simple inductive argument becomes inapplicable if the number of odd terms and the number of even terms becomes different in the given set $\{0, 1, \dots, q\}$. In this paper, we have found explicit formulas for the number of $m \times n$ matrices and the number of $n \times n$ symmetric matrices having entries from $\{0, 1, \dots, q\}$ with line sums divisible by a given integer k . As a consequence of the results, we have also been able to find an explicit formula enumerating the number of traceless $n \times n$, $(0, 1)$ symmetric matrices having line sums divisible by a given integer, which leads to an enumeration of labeled regular graphs with n vertices. The latter is known to follow P-recursiveness from the works of Goulden, Jackson in [6] and Gessel in [5]. But our approach does not involve the use of recursion and leads to an explicit enumeration formula.

The other half of this paper is devoted to a weighted enumeration of rectangular matrices. Let us denote the given set of integers $\{0, 1, \dots, q\}$ by S . Let $M_{m \times n}(S)$ be the set of all $m \times n$ matrix with entries from the set $S := \{0, 1, \dots, q\}$. For a matrix $A \in M_{m \times n}(S)$, let the i th row sum and j th column sum be denoted by $r_i(A)$ and

$c_j(A)$ respectively. Now, if we concentrate on the weighted sum $\mathcal{S} := \sum_{A \in M_{m \times n}(S)} (\prod_{i=1}^m r_i(A)) (\prod_{j=1}^n c_j(A))$, a bare hand calculation yields that the sum is 34, if we take $m = n = 2$ and $q = 1$. But even for these small values of m, n, q , one has to check as many as 16 cases to get \mathcal{S} . The difficulty arises quickly if we increase m, n, q . For instance, if we choose $m = 4, n = 5, q = 1$, then one has to check $2^{20} = 1048576$ cases (in this case, Theorem 1.5 answers the sum to be 695664640). Choosing $q = 3$ under the same setup (i.e. $m = 4$ and $n = 5$), gives rise to a checking of 4^{20} cases and the difficulty keeps increasing. In this paper, we have found explicit formulae for \mathcal{S} . This enumeration also gives rise to a few interesting identities satisfying nice reciprocity phenomena.

Throughout this paper, we will be using a few notations defined below. We shall use the notation $p(m)$ to denote the set of all partitions of m . If σ is a partition of m , then the number of i 's occurring in σ is denoted by σ_i . For example, suppose $\sigma = 3 + 2 + 2 + 1$ is a partition of 8, then $\sigma_1 = 1, \sigma_2 = 2, \sigma_3 = 1$ and $\sigma_4, \sigma_5, \dots, \sigma_8 = 0$. Thus $\sum_{i=1}^m i\sigma_i = m$. Define $|\sigma| := \sum_{i=1}^m \sigma_i$. The minimum of two integers m and n will be denoted by $m \wedge n$. Let S be a set. Then the n -fold cartesian product $S \times S \times \dots \times S$ (n times) is denoted by S^n . For $z \in \mathbb{C}$, $Re(z)$ will denote the real part of z .

Now, we are in a position to state our results. Our first result will be on the enumeration of $m \times n$ matrices with entries from the set $S := \{0, 1, \dots, q\}$ such that the row sums and the column sums are divisible by a given integer k . Let $M(m, n, q, k)$ denote the number of matrices in $M_{m \times n}(S)$ with row sums and column sums divisible by k . Then we have the following.

Theorem 1.1.

$$M(m, n, q, k) = \frac{m!}{k^{m+n}} \sum_{\substack{C=(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = m}} \frac{\left(\sum_{j=0}^{k-1} \prod_{l=1}^{k-1} \xi(j+l)^{c_l}\right)^n}{c_0!c_1! \dots c_{k-1}!}$$

where, ξ_k is a primitive k th root of unity and $\xi(x) := \sum_{t=0}^{q-1} \xi_k^{xt}$.

In the above case, if we concentrate only on the $n \times n$ symmetric matrices, then we obtain our next theorem. Let $H(n, q, k)$ denote the number of $n \times n$ symmetric matrices having entries from S (defined previously) with row sums and column sums divisible by k . Then we have the following.

Theorem 1.2.

$$H(n, q, k) = \frac{n!}{k^n} \sum_{\substack{C=(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{\prod_{j=0}^{k-1} \left(\xi(j)^{c_j} \xi(2j)^{\binom{c_j}{2}}\right) \prod_{r < s} \xi(r+s)^{c_r c_s}}{c_0!c_1! \dots c_{k-1}!}.$$

Now, in order to enumerate labelled regular graphs, we have to concentrate on traceless $(0, 1)$, $n \times n$ symmetric matrices. By a slight modification in the above setup and considering $q = 1$, we can in fact write down a ξ_k free expression which will help us enumerating labelled regular graphs. Our next theorem is on that. It says, if $\mathfrak{F}(n, k)$ denotes the set of traceless $(0, 1)$, $n \times n$ symmetric matrices with each line sum divisible by k and $F(n, k) := |\mathfrak{F}(n, k)|$, then we have the following.

Theorem 1.3.

$$F(n, k) = \frac{n!2^{\binom{n}{2}}}{k^n} \sum_{\substack{C=(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{\mathcal{C}(\mathfrak{C}(C)) \left(\prod_{j=0}^{k-1} \mathcal{C}(2j)^{\binom{c_j}{2}} \prod_{r < s} \mathcal{C}(r+s)^{c_r c_s}\right)}{c_0!c_1! \dots c_{k-1}!},$$

where $\mathcal{C}(x) := \cos\left(\frac{x\pi}{k}\right)$ and $\mathfrak{C}(C) := (n-1)(\sum_{j=0}^{k-1} j c_j)$ for any k -tuple $C = (c_0, c_1, \dots, c_{k-1})$ where $c_i \geq 0$ for all i and $\sum_i c_i = n$.

Now, as a consequence of Theorem 1.3, we can find an explicit formula for counting labeled k -regular graphs with n vertices. Suppose $R(n, k)$ denotes the number of labeled k -regular graphs with n vertices. Then we have the following.

Theorem 1.4.

$$R(n, k) = \begin{cases} \sum_{i=0}^n (-1)^i \binom{n}{i} F(n-i, k), & k > \frac{n-1}{2}, \\ F(n, k) + 2 \sum_{i=1}^n (-1)^i \binom{n}{i} F(n-i, k), & k = \frac{n-1}{2}. \end{cases}$$

When $k < \frac{n-1}{2}$, $R(n, k)$ can be evaluated from the above theorem using the trivial relation $R(n, k) = R(n, n-k-1)$.

Let us now come to the weighted enumeration of rectangular matrices, as mentioned earlier. Let us denote the given set of integers $\{0, 1, \dots, q\}$ by S . Let $M_{m \times n}(S)$ denote the set of all $m \times n$ matrix with entries from the set $S := \{0, 1, \dots, q\}$. For a matrix $A \in M_{m \times n}(S)$, let the i th row sum and j th column sum be denoted by $r_i(A)$ and $c_j(A)$ respectively. Then we have the following,

Theorem 1.5.

$$\begin{aligned} \sum_{A \in M_{m \times n}(S)} \left(\prod_{i=1}^m r_i(A) \right) \left(\prod_{j=1}^n c_j(A) \right) &= \frac{m!n!(q+1)^{mn}q^{(m+n)}}{2^{(m+n)}} \sum_{\substack{\sigma \in p(m), \\ |\sigma| \leq m \wedge n}} \frac{A(m, n, \sigma)}{q^{|\sigma|}} \prod_{r=1}^m \left(\frac{q(3m+r)+2r}{r!} \right)^{\sigma_r} \\ &\left(= \frac{m!n!(q+1)^{mn}q^{(m+n)}}{2^{(m+n)}} \sum_{\substack{\pi \in p(n), \\ |\pi| \leq m \wedge n}} \frac{A(n, m, \pi)}{q^{|\pi|}} \prod_{t=1}^n \left(\frac{q(3n+t)+2t}{t!} \right)^{\pi_t} \right) \\ &= \frac{q^{(m+n)}(q+1)^{mn}m^n n^m}{2^{(m+n)}} \sum_{i=0}^{m \wedge n} i! \binom{m}{i} \binom{n}{i} \left(\frac{q+2}{3mnq} \right)^i, \end{aligned}$$

where $A(x, y, \lambda) := \frac{x^{(y-|\lambda|)}}{\lambda_1! \lambda_2! \dots \lambda_x!(y-|\lambda|)! 3^{|\lambda|}}$ for non-negative integers x, y and $\lambda \in p(x)$.

Theorem 1.5, in its full generality exhibits a nice reciprocity phenomenon. In particular, letting $q = 1$, the above theorem specializes to the following nice identity.

Corollary 1.1.

$$\begin{aligned} \sum_{\substack{\sigma \in p(m), \\ |\sigma| \leq m \wedge n}} \frac{m^{(n-|\sigma|)}}{\sigma_1! \sigma_2! \dots \sigma_m!(n-|\sigma|)!} \prod_{r=1}^m \left(\frac{m+r}{r!} \right)^{\sigma_r} &= \sum_{\substack{\pi \in p(n), \\ |\pi| \leq m \wedge n}} \frac{n^{(m-|\pi|)}}{\pi_1! \pi_2! \dots \pi_n!(m-|\pi|)!} \prod_{t=1}^n \left(\frac{n+t}{t!} \right)^{\pi_t} \\ &= \frac{m^n n^m}{m!n!} \sum_{i=0}^{m \wedge n} \binom{m}{i} \binom{n}{i} \frac{i!}{(mn)^i}. \end{aligned}$$

2. Proofs of the results

Proof of Theorem 1.1. Before getting into the proof, let us make a quick observation first.

Observation 1: Let $\mathcal{P}(x_1, x_2, \dots, x_s)$ be a polynomial in $\mathbb{C}[x_1, x_2, \dots, x_s]$ for some $s \in \mathbb{N}$ and $a \in \mathbb{N}$ be any given positive integer. Suppose we are interested in finding the sum of the coefficients of the monomials in which every variable x_i has exponent divisible by a . Let us denote this required sum by $S(\mathcal{P}, a)$. To face the above scenario, let us consider the set $\mathcal{R}_a := \{\xi_a^0, \xi_a^1, \dots, \xi_a^{(a-1)}\}$, where ξ_a denotes a primitive a th root of unity in \mathbb{C} .

Let us define a map $\mathcal{D} : \mathbb{C}[x_1, x_2, \dots, x_n] \rightarrow \mathbb{C}[x_1, x_2, \dots, x_{n-1}]$ for all $n \in \mathbb{N}$, as follows,

$$\mathcal{D}(f(x_1, x_2, \dots, x_n)) := \frac{1}{a} \sum_{z \in \mathcal{R}_a} f(x_1, x_2, \dots, x_{n-1}, z).$$

It is easy to see that if

$$f = \sum_{(m_1, m_2, \dots, m_n) \in (\mathbb{N} \cup \{0\})^n} c_{m_1, m_2, \dots, m_n} x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$$

then,

$$\mathcal{D}(f) = \sum_{(m_1, m_2, \dots, m_n) \in (\mathbb{N} \cup \{0\})^n} \bar{c}_{m_1, m_2, \dots, m_n} x_1^{m_1} x_2^{m_2} \dots x_{n-1}^{m_{n-1}}$$

where $\bar{c}_{m_1, m_2, \dots, m_n} = c_{m_1, m_2, \dots, m_n}$ when m_n is divisible by a and $\bar{c}_{m_1, m_2, \dots, m_n} = 0$ otherwise. So, it is also easy to see that $S(\mathcal{P}, a) = \mathcal{D}^s(\mathcal{P})$, where $\mathcal{D}^s(\mathcal{P})$ means \mathcal{D} acts iteratively s -times over \mathcal{P} . On the other hand, from the definition of \mathcal{D} it is clear that $\mathcal{D}^t(\mathcal{P}) = \frac{1}{a^t} \sum_{z \in \mathcal{R}_a^t} \mathcal{P}(x_1, x_2, \dots, x_{s-t}, z)$ for all $0 \leq t \leq s$. So, it follows that,

$$S(\mathcal{P}, a) = \mathcal{D}^s(\mathcal{P}) = \frac{1}{k^s} \sum_{z \in \mathcal{R}_a^s} \mathcal{P}(z).$$

Now consider the polynomial

$$P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) := \prod_{i=1}^m \prod_{j=1}^n \left(\sum_{t=0}^q (x_i y_j)^t \right).$$

In order to find $M(m, n, q, k)$, we have to concentrate on the set \mathcal{R}_k (defined in *observation 1*). With a slight abuse of notation, we will proceed considering \mathcal{R} to be same as \mathcal{R}_k , unless otherwise mentioned. Now, in light of *observation 1*, one can see that

$$M(m, n, q, k) = \frac{1}{k^{m+n}} \sum_{\mathbf{Z} \in \mathcal{R}^{m+n}} P(\mathbf{Z}).$$

Again, it is quite clear that

$$\frac{1}{k^{m+n}} \sum_{\mathbf{Z} \in \mathcal{R}^{m+n}} P(\mathbf{Z}) = \frac{1}{k^{m+n}} \sum_{\substack{\mathbf{Z}_m \in \mathcal{R}^m \\ \mathbf{Z}_n \in \mathcal{R}^n}} P(\mathbf{Z}_m, \mathbf{Z}_n).$$

So, first let us try to compute $\sum_{\mathbf{Z}_n \in \mathcal{R}^n} P(X, \mathbf{Z}_n)$, where $X = (x_1, x_2, \dots, x_m)$. Now, for a particular $\mathbf{Z}_n = (z_1, z_2, \dots, z_n) \in \mathcal{R}^n$,

$$P(X, \mathbf{Z}_n) = \prod_{i=1}^m \prod_{j=1}^n (\sum_{t=0}^q (x_i z_j)^t).$$

So,

$$\sum_{\mathbf{Z}_n \in \mathcal{R}^n} P(X, \mathbf{Z}_n) = (\sum_{z \in \mathcal{R}} \prod_{i=1}^m (\sum_{t=0}^q (x_i z)^t))^n.$$

Hence,

$$\sum_{\substack{\mathbf{Z}_m \in \mathcal{R}^m \\ \mathbf{Z}_n \in \mathcal{R}^n}} P(\mathbf{Z}_m, \mathbf{Z}_n) = \sum_{\mathbf{Z}_m \in \mathcal{R}^m} (\sum_{z \in \mathcal{R}} \prod_{i=1}^m (\sum_{t=0}^q (\mathfrak{z}_i z)^t))^n$$

where, $\mathbf{Z}_m = (\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_m) \in \mathcal{R}^m$. So, for a particular $\mathbf{Z}_m = (\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_m) \in \mathcal{R}^m$, if ξ_k^i occurs c_i times in the multiset $\{\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_m\}$, then we have the following,

$$\begin{aligned} \sum_{\substack{\mathbf{Z}_m \in \mathcal{R}^m \\ \mathbf{Z}_n \in \mathcal{R}^n}} P(\mathbf{Z}_m, \mathbf{Z}_n) &= \sum_{\mathbf{Z}_m = (\mathfrak{z}_1, \mathfrak{z}_2, \dots, \mathfrak{z}_m) \in \mathcal{R}^m} \left(\sum_{z \in \mathcal{R}} \prod_{i=1}^m (\sum_{t=0}^q (\mathfrak{z}_i z)^t) \right)^n \\ &= \sum_{\substack{(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = m}} \frac{m!}{c_0! c_1! \dots c_{k-1}!} \left(\sum_{j=0}^{k-1} \prod_{l=1}^{k-1} (\sum_{t=0}^q (\xi_k^{(j+l)})^t)^{c_l} \right)^n. \end{aligned}$$

□

Remark: Theorem 1.1 can be directly used to answer the following question raised in the introduction. What is the number of $m \times n$ matrices with entries from the set $\{0, 1, \dots, q\}$ such that all the row sums and column sums are even integers, if q is even? The answer turns out to be

$$\frac{1}{2^{m+n}} \sum_{t=0}^n \binom{n}{t} ((q+1)^t + (q+1)^{n-t})^m = \frac{1}{2^{m+n}} \sum_{t=0}^m \binom{m}{t} ((q+1)^t + (q+1)^{m-t})^n.$$

Proof of Theorem 1.3. Consider the polynomial

$$G(x_1, x_2, \dots, x_n) := \prod_{1 \leq i < j \leq n} (1 + x_i x_j).$$

Let $\mathcal{R} := \{\xi_k^0, \xi_k^1, \dots, \xi_k^{(k-1)}\}$, where ξ_k denotes a primitive k -th root of unity. Then again *observation 1* enlightens us to see that

$$F(n, k) = \frac{1}{k^n} \sum_{\mathbf{Z} \in \mathcal{R}^n} G(\mathbf{Z}).$$

For a particular $\mathbf{Z} = (z_1, z_2, \dots, z_n) \in \mathcal{R}^n$, suppose the number of occurrences of ξ_k^i in the multiset $\{z_1, z_2, \dots, z_n\}$ be c_i . Then

$$G(\mathbf{Z}) = \prod_{j=0}^{k-1} (1 + \xi_k^{2j})^{\binom{c_j}{2}} \prod_{r < s} (1 + \xi_k^{(r+s)})^{c_r c_s}.$$

So we have

$$F(n, k) = \frac{1}{k^n} \sum_{\substack{(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{n!}{c_0! c_1! \dots c_{k-1}!} \prod_{j=0}^{k-1} (1 + \xi_k^{2j})^{\binom{c_j}{2}} \prod_{r < s} (1 + \xi_k^{(r+s)})^{c_r c_s}$$

$$= \operatorname{Re} \left(\frac{1}{k^n} \sum_{\substack{(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{n!}{c_0! c_1! \dots c_{k-1}!} \prod_{j=0}^{k-1} (1 + \xi_k^{2j})^{\binom{c_j}{2}} \prod_{r < s} (1 + \xi_k^{(r+s)})^{c_r c_s} \right)$$

(since $F(n, k)$ is real)

$$\begin{aligned} &= \frac{1}{k^n} \sum_{\substack{(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{n!}{c_0! c_1! \dots c_{k-1}!} \operatorname{Re} \left(\prod_{j=0}^{k-1} (1 + \xi_k^{2j})^{\binom{c_j}{2}} \prod_{r < s} (1 + \xi_k^{(r+s)})^{c_r c_s} \right) \\ &= \frac{1}{k^n} \sum_{\substack{(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{n!}{c_0! c_1! \dots c_{k-1}!} \operatorname{Re} \left(\prod_{j=0}^{k-1} 2^{\binom{c_j}{2}} \left(\cos \frac{2\pi j}{k} \right)^{\binom{c_j}{2}} \left(\cos \frac{2\pi j}{k} + \iota \sin \frac{2\pi j}{k} \right)^{\binom{c_j}{2}} \right. \\ &\quad \left. \times \prod_{r < s} 2^{c_r c_s} \left(\cos \frac{\pi(r+s)}{k} \right)^{c_r c_s} \left(\cos \frac{\pi(r+s)}{k} + \iota \sin \frac{\pi(r+s)}{k} \right)^{c_r c_s} \right) \\ &= \frac{n! 2^{\binom{n}{2}}}{k^n} \sum_{\substack{(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{\cos \left(\left(\sum_{j=0}^{k-1} 2j \binom{c_j}{2} + \sum_{r < s} (r+s) c_r c_s \right) \frac{\pi}{k} \right)}{c_0! c_1! \dots c_{k-1}!} \\ &\quad \times \left(\prod_{j=0}^{k-1} \left(\cos \frac{2\pi j}{k} \right)^{\binom{c_j}{2}} \prod_{r < s} \left(\cos \frac{\pi(r+s)}{k} \right)^{c_r c_s} \right). \end{aligned}$$

We can simplify the above relation using the following simple identity

$$\sum_{j=0}^{k-1} 2j \binom{c_j}{2} + \sum_{r < s} (r+s) c_r c_s = (n-1) \sum_{j=0}^{k-1} j c_j,$$

for all k -tuples $(c_0, c_1, \dots, c_{k-1})$ satisfying $c_i \geq 0$ for all i and $\sum_i c_i = n$. To prove this identity, take a particular $(c_0, c_1, \dots, c_{k-1}) \in (\mathbb{N} \cup \{0\})^k$ satisfying $\sum_i c_i = n$. Now consider the polynomial $p(x)$ in $\mathbb{Z}[x]$, defined as follows

$$p(x) := \left(\sum_j c_j x^j \right)^2 - \left(\sum_j c_j x^{2j} \right)$$

One can easily see that

$$\left(\sum_j c_j x^j \right)^2 - \left(\sum_j c_j x^{2j} \right) = \sum_j (c_j^2 - c_j) x^{2j} + 2 \sum_{r < s} c_r c_s x^{r+s}$$

Taking the derivative with respect to x on both sides at $x = 1$ and dividing both sides by 2 and using the condition $\sum_i c_i = n$, one gets the desired relation.

In the light of the above identity and all the previous calculations, one can get

$$F(n, k) = \frac{n! 2^{\binom{n}{2}}}{k^n} \sum_{\substack{(c_0, c_1, \dots, c_{k-1}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{\cos \left((n-1) \left(\sum_{j=0}^{k-1} j c_j \right) \frac{\pi}{k} \right)}{c_0! c_1! \dots c_{k-1}!} \times \left(\prod_{j=0}^{k-1} \left(\cos \frac{2\pi j}{k} \right)^{\binom{c_j}{2}} \prod_{r < s} \left(\cos \frac{\pi(r+s)}{k} \right)^{c_r c_s} \right).$$

□

A class of identities can be deduced from the Theorem 1.3. For instance, the following nice identity is an immediate corollary of the above.

Corollary 2.1.

$$2 = \frac{n! 2^{\binom{n}{2}}}{(n-1)^n} \sum_{\substack{(c_0, c_1, \dots, c_{n-2}) \\ c_i \geq 0; \forall i \\ \sum_i c_i = n}} \frac{(-1)^{\left(\sum_{j=0}^{n-2} j c_j \right)}}{c_0! c_1! \dots c_{n-2}!} \left(\prod_{j=0}^{n-2} \left(\cos \frac{2\pi j}{n-1} \right)^{\binom{c_j}{2}} \prod_{r < s} \left(\cos \frac{\pi(r+s)}{n-1} \right)^{c_r c_s} \right)$$

Proof. Putting $k = n - 1$ in Theorem 1.3, yields the result, since the following two are the only matrices in $\mathfrak{F}(n, n - 1)$,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix}.$$

□

Using similar techniques, used in the proof of Theorems 1.1 and 1.3, we obtain Theorem 1.2. So the details are omitted.

Proof of Theorem 1.4. With Theorem 1.3 in hand, we are in a state to count labelled k -regular graphs in n vertices. Theorem 1.3 gives us the number of traceless $n \times n$, $(0, 1)$ symmetric matrices with line sums divisible by k . Now, one can see that, for a traceless $n \times n$, $(0, 1)$ matrix, the maximum possible line sum is $n - 1$. So if we choose k to be greater than $\frac{n-1}{2}$, the only possible line sums, divisible by k are 0 and k itself. So, in order to find $R(n, k)$, we have to eliminate the matrices having at least one row (equivalently column) sum zero from $\mathfrak{F}(n, k)$. Using the *principle of inclusion and exclusion*, we get the following

$$R(n, k) = \sum_{i=0}^n (-1)^i \binom{n}{i} F(n - i, k).$$

When $k = \frac{n-1}{2}$, it is not difficult to see that $\mathfrak{F}(n, k)$ consists of precisely three types of matrices, *Type A*, *Type B* and *Type C*, say, where *Type A* consists of the matrices with at least one line sum 0, *Type B* consists of matrices with at least one line sum $2k$ and *Type C* consists of matrices with line sums precisely equal to k . Now the number of *Type A* matrices is $\sum_{i=1}^n (-1)^{i+1} \binom{n}{i} F(n - i, k)$. Also it is easy to see that *Type A* and *Type B* matrices are equinumerous. So, by the *principle of inclusion and exclusion*, we have

$$R(n, k) = F(n, k) + 2 \left(\sum_{i=1}^n (-1)^i \binom{n}{i} F(n - i, k) \right).$$

□

Proof of Theorem 1.5. The proof of this theorem is divided into two parts. The first part is devoted to proving the first two equalities of Theorem 1.5 and the second part is devoted to proving the last equality of the same theorem. The main task in the whole proof is to evaluate an iterated partial derivative. In the first part of the proof, we have used Faà di Bruno’s formula [8] proposition 1] and in the second part, we have used the partial differential operator technique to evaluate the iterated partial derivative.

Proof of the first part: Recall that we have defined

$$\mathcal{S} = \sum_{A \in M_{m \times n}(S)} \left(\prod_{i=1}^m r_i(A) \right) \left(\prod_{j=1}^n c_j(A) \right)$$

in the introduction. Now consider the polynomial

$$P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) := \prod_{i=1}^m \prod_{j=1}^n \left(\sum_{t=0}^q (x_i y_j)^t \right).$$

Then one can see that

$$\mathcal{S} = \frac{\partial^{m+n}}{\partial x_1 \partial x_2 \cdots \partial x_m \partial y_1 \partial y_2 \cdots \partial y_n} P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$$

evaluated at $x_i = 1 = y_j$ for all $1 \leq i \leq m, 1 \leq j \leq n$.

Let $Q(x_1, x_2, \dots, x_m) := \frac{\partial^m}{\partial y_1 \partial y_2 \cdots \partial y_n} P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$ evaluated at $y_1 = y_2 = \dots = y_n = 1$.

Now, $P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$ can be written as

$$P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = \prod_{j=1}^n \left(\sum_{k=0}^{mq} \widetilde{h}_k(\bar{X}) y_j^k \right)$$

where, $\widetilde{h}_k(\bar{X}) := \sum_{\psi \in \beta_m(k)} X^\psi$, $\beta_m(k)$ is the set of all ordered tuples of integers $\psi = (b_1, b_2, \dots, b_m)$ such that $b_1 + b_2 + \dots + b_m = k$ with $0 \leq b_i \leq q$ for all $i = 1, 2, \dots, m$ and $X^\psi = x_1^{b_1} x_2^{b_2} \dots x_m^{b_m}$. We call $\widetilde{h}_k(\bar{X})$ as *restricted homogeneous symmetric function* of degree k in m variables.

So, $Q(x_1, x_2, \dots, x_m)$ becomes $\left(\sum_{k=0}^{mq} k \widetilde{h}_k(\bar{X})\right)^n$.

Thus we have $\mathcal{S} = \frac{\partial^m}{\partial x_1 \partial x_2 \dots \partial x_m} Q(x_1, x_2, \dots, x_m)$, evaluated at $x_1 = x_2 = \dots = x_m = 1$.

Let $z = \sum_{k=0}^{mq} k \widetilde{h}_k(\bar{X})$ and $f(z) = z^n (= Q(x_1, x_2, \dots, x_m))$.

We have to evaluate $\frac{\partial^m f(z)}{\partial x_1 \partial x_2 \dots \partial x_m}$ at $x_1 = x_2 = \dots = x_m = 1$. We shall use the following generalized version of Faà di Bruno's formula [8] proposition 1] to accomplish this task:

$$\frac{\partial^m f(z)}{\partial x_1 \partial x_2 \dots \partial x_m} = \sum_{\eta} f^{|\eta|}(z) \prod_{B \in \eta} \frac{\partial^{|B|} z}{\prod_{i \in B} \partial x_i} \tag{1}$$

where the summation runs over all partitions η of the set $\{1, 2, \dots, m\}$ (not to be confused with the integer partitions, previously dealt with) and the product is over all of the parts (or blocks) B of the partition η and we denote the number of members of any set T by $|T|$ (the same notation has already occurred previously, but in a different context. So when the context is clear, no confusion should arise). Note that $f^{|\eta|}(z) = \frac{n!}{(n-|\eta|)!} z^{(n-|\eta|)}$, when $|\eta| \leq n$; $f^{|\eta|}(z) = 0$, otherwise. So, the only set partitions η with $|\eta| \leq n$ will be our matter of interest.

Now, for a particular set partition η of the set $\{1, 2, \dots, m\}$, let us denote the product, $\prod_{B \in \eta} \frac{\partial^{|B|} z}{\prod_{i \in B} \partial x_i}$ evaluated at $x_1 = x_2 = \dots = x_m = 1$, by $\partial(\eta)$.

Since, $\widetilde{h}_k(\bar{X})$ is symmetric, z is also symmetric. So the value of $\frac{\partial^{|B|} z}{\prod_{i \in B} \partial x_i}$ when evaluated at $x_1 = x_2 = \dots = x_m = 1$ does not depend on the choice of the elements in B but only on $|B|$. So if $|B| = l$ it suffices to evaluate $\frac{\partial^l z}{\partial x_1 \partial x_2 \dots \partial x_l}$ at $x_1 = x_2 = \dots = x_m = 1$. Now

$$\begin{aligned} \frac{\partial^l z}{\partial x_1 \partial x_2 \dots \partial x_l} &= \frac{\partial^l}{\partial x_1 \partial x_2 \dots \partial x_l} \sum_{k=0}^{mq} k \widetilde{h}_k(\bar{X}) \\ &= \sum_{k=0}^{mq} \frac{\partial^l}{\partial x_1 \partial x_2 \dots \partial x_l} \left(\sum_{\psi \in \beta_m(k)} X^\psi \right) = \sum_{k=0}^{mq} k \left(\sum_{\psi=(b_1, \dots, b_m) \in \beta_m(k)} b_1 b_2 \dots b_l \right) \end{aligned}$$

(after evaluating the partial derivative at $x_1 = x_2 = \dots = x_m = 1$).

Now, let us simplify the summation above in a different manner. Let us define the polynomial

$$R(x_1, x_2, \dots, x_m, t) := \sum_{i=1}^m (1 + x_i t + x_i^2 t^2 + \dots + x_i^q t^q).$$

So, it evidently follows that

$$\sum_{k=0}^{mq} k \left(\sum_{\psi=(b_1, \dots, b_m) \in \beta_m(k)} b_1 b_2 \dots b_l \right) = \frac{\partial^{l+1}}{\partial t \partial x_1 \partial x_2 \dots \partial x_l} R(x_1, x_2, \dots, x_m, t)$$

evaluated at $x_1 = x_2 = \dots = x_m = t = 1$.

Now

$$\frac{\partial^l R}{\partial x_1 \partial x_2 \dots \partial x_l} = \prod_{i=1}^l \frac{\partial}{\partial x_i} \left(\sum_{p=0}^q (x_i t)^p \right) \prod_{i=l+1}^m \left(\sum_{p=0}^q (x_i t)^p \right).$$

Evaluating the above derivative at $x_1 = x_2 = \dots = x_m = 1$ we get, $(\sum_{p=1}^q p t^p)^l (\sum_{p=0}^q t^p)^{(m-l)}$. Hence, we have $\frac{\partial^{l+1}}{\partial t \partial x_1 \partial x_2 \dots \partial x_l} R(x_1, x_2, \dots, x_m, t)$ evaluated at $x_1 = x_2 = \dots = x_m = t = 1$ is same as $\frac{d}{dt} (\sum_{p=1}^q p t^p)^l (\sum_{p=0}^q t^p)^{(m-l)}$ evaluated at $t = 1$.

Now

$$\frac{d}{dt} \left(\sum_{p=1}^q p t^p \right)^l \left(\sum_{p=0}^q t^p \right)^{(m-l)} = l \left(\sum_{p=1}^q p t^p \right)^{(l-1)} \left(\sum_{p=1}^q p^2 t^{(p-1)} \right) \left(\sum_{p=0}^q t^p \right)^{(m-l)} + (m-l) \left(\sum_{p=1}^q p t^p \right)^l \left(\sum_{p=0}^q t^p \right)^{(m-l-1)} \left(\sum_{p=1}^q p t^{(p-1)} \right).$$

Evaluating the above derivative at $t = 1$, we get

$$l \left(\frac{q(q+1)}{2} \right)^{(l-1)} \left(\frac{q(q+1)(2q+1)}{6} \right) (q+1)^{(m-l)} + (m-l) \left(\frac{q(q+1)}{2} \right)^{(l+1)} (q+1)^{(m-l-1)}$$

$$= \frac{q^l(q+1)^m}{2^{l+1}} \binom{q(l+3m)+2l}{3}.$$

So, finally we have the following

$$\frac{\partial^l z}{\partial x_1 \partial x_2 \cdots \partial x_l} = \frac{q^l(q+1)^m}{2^{l+1}} \binom{q(l+3m)+2l}{3}. \tag{2}$$

when evaluated at $x_1 = x_2 = \cdots = x_m = 1$. Note that, if we put $l = 0$ in the above expression, we get

$$z = \frac{mq}{2}(q+1)^m \tag{3}$$

when evaluated at $x_1 = x_2 = \cdots = x_m = 1$.

As we have already mentioned, if $\zeta(\in p(m))$ has l_1, l_2, \dots, l_d as parts and $\eta = \{B_1, B_2, \dots, B_d\}$ and $\dot{\eta} = \{\dot{B}_1, \dot{B}_2, \dots, \dot{B}_d\}$ are any two set partitions of $\{1, 2, \dots, m\}$ with $|B_i| = |\dot{B}_i| = l_i$ for all $i = 1, 2, \dots, d$, then $\partial(\eta) = \partial(\dot{\eta})$. So, for any particular partition σ of m , there will be $\frac{m!}{\sigma_1! \sigma_2! \cdots \sigma_m! (1!)^{\sigma_1} (2!)^{\sigma_2} \cdots (m!)^{\sigma_m}}$ no of partitions η of the set $\{1, 2, \dots, m\}$, contributing same $\partial(\eta)$. Choose such an η . Then its contribution to (1), that is, $f^{|\eta|}(z) \prod_{B \in \eta} \frac{\partial^{|B|} z}{\prod_{i \in B} \partial x_i}$ when evaluated at $x_1 = x_2 = \cdots = x_m = 1$ and remembering that $|\eta| = |\sigma|$, gives

$$\begin{aligned} & \frac{n!}{(n-|\sigma|)!} \left(\frac{mq(q+1)^m}{2} \right)^{(n-|\sigma|)} \prod_{r=1}^m \frac{q^{r\sigma_r} (q+1)^{m\sigma_r}}{2^{(r\sigma_r + \sigma_r)}} \binom{q(3m+r)+2r}{3}^{\sigma_r} \quad (by(2), (3)) \\ &= \frac{n!}{(n-|\sigma|)!} \frac{m^{(n-|\sigma|)} q^{(m+n-|\sigma|)} (q+1)^{mn}}{2^{(m+n)}} \prod_{r=1}^m \binom{q(3m+r)+2r}{r!}^{\sigma_r} \end{aligned}$$

when $|\sigma| (= |\eta|) \leq n$ and 0 otherwise.

So, finally we have the following

$$S = \frac{m!n!(q+1)^{mn}}{2^{(m+n)}} \sum_{\substack{\sigma \in p(m), \\ |\sigma| \leq m \wedge n}} \frac{m^{(n-|\sigma|)} q^{(m+n-|\sigma|)}}{\sigma_1! \sigma_2! \cdots \sigma_m! (n-|\sigma|)! 3^{|\sigma|}} \prod_{r=1}^m \binom{q(3m+r)+2r}{r!}^{\sigma_r}.$$

This completes the proof of the first part.

Proof of the second part: Let $g(y) = \prod_{i=1}^m (\sum_{k=0}^q (x_i y)^k)$. So

$$P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = \prod_{j=1}^n g(y_j).$$

Hence

$$\frac{\partial^n}{\partial y_1 \partial y_2 \cdots \partial y_n} P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n) = \prod_{j=1}^n \frac{\partial}{\partial y_j} g(y_j).$$

Again

$$\frac{g'(y)}{g(y)} = \frac{\sum_{k=0}^q k x_i^k y^{k-1}}{\sum_{k=0}^q x_i^k y^k}.$$

Let $X_i := x_i + x_i^2 + \cdots + x_i^q$ and $\bar{X}_i := x_i + 2x_i^2 + \cdots + qx_i^q$. Therefore we have,

$$g'(y) = \left(\prod_{i=1}^m (1 + X_i) \right) \left(\sum_{i=1}^m \frac{\bar{X}_i}{1 + \bar{X}_i} \right),$$

when evaluated at $y = 1$. Hence, $\frac{\partial^n}{\partial y_1 \partial y_2 \cdots \partial y_n} P(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_n)$, when evaluated at $y_1 = y_2 = \cdots = y_n = 1$, gives

$$(1 + X_1)^n (1 + X_2)^n \cdots (1 + X_m)^n \left(\frac{\bar{X}_1}{1 + \bar{X}_1} + \frac{\bar{X}_2}{1 + \bar{X}_2} + \cdots + \frac{\bar{X}_m}{1 + \bar{X}_m} \right)^n.$$

Define, $F_i := (1 + X_i)^n \cdots (1 + X_m)^n$ and $\mathcal{F} := \frac{\bar{X}_1}{1 + \bar{X}_1} + \frac{\bar{X}_2}{1 + \bar{X}_2} + \cdots + \frac{\bar{X}_m}{1 + \bar{X}_m}$. Our task is to find $\frac{\partial^m}{\partial x_1 \partial x_2 \cdots \partial x_m} F_1 \mathcal{F}^n$ at $x_1 = x_2 = \cdots = x_m = 1$. Note that,

$$\frac{\partial}{\partial x_1} F_1 \mathcal{F}^n = n(1 + X_1)^{n-1} F_2 \mathcal{F}^n \frac{\partial X_1}{\partial x_1} + n(1 + X_1)^n F_2 \mathcal{F}^{n-1} \frac{\partial}{\partial x_1} \left(\frac{\bar{X}_1}{1 + \bar{X}_1} \right).$$

Define $\mathcal{F}_1 := \mathcal{F}$ and $\mathcal{F}_i := \mathcal{F}$ evaluated at $x_1 = x_2 = \cdots = x_{i-1} = 1$ for all $i > 1$. Define, $S_1 := (1 + X_i)^{n-1} \frac{\partial X_i}{\partial x_i}$ evaluated at $x_i = 1$ and $S_2 := (1 + X_i)^n \frac{\partial}{\partial x_i} \left(\frac{\bar{X}_i}{1 + \bar{X}_i} \right)$ evaluated at $x_i = 1$ (It is easy to see that S_1 and S_2 are

well defined, i.e. they do not depend on the choice of i). In fact, a little computation yields $S_1 = \frac{q(q+1)^n}{2}$ and $S_2 = \frac{q(q+2)(q+1)^n}{12}$.

Now, for any $k \in \mathbb{N}$, we have

$$\frac{\partial}{\partial x_i} F_i \mathcal{F}_i^k = k S_1 F_{i+1} \mathcal{F}_{i+1}^k + S_2 F_{i+1} k \mathcal{F}_{i+1}^{k-1} \tag{4}$$

when evaluated at $x_i = 1$. Suppose, D is an operator, satisfying $D\Omega^n = n\Omega^{n-1}$, when $n \geq 0$ and $D\Omega^n = 0$, when $n < 0$, where Ω is a rational function in m variables. I is defined to be the identity operator on the set of all rational functions in m variables.

So, in the light of (4), the iterated derivative $\frac{\partial^m}{\partial x_1 \partial x_2 \dots \partial x_m} F_1 \mathcal{F}^n$ at $x_1 = x_2 = \dots = x_m = 1$, becomes $(n S_1 I + S_2 D)^m \mathcal{F}^n$ at $x_1 = x_2 = \dots = x_m = 1$, which is equal to (via a bit computation),

$$\frac{q^{(m+n)}(q+1)^{mn} m^n n^m}{2^{(m+n)}} \sum_{i=0}^{m \wedge n} i! \binom{m}{i} \binom{n}{i} \left(\frac{q+2}{3mnq}\right)^i.$$

□

3. Further explore

The following problems naturally arise from our work.

- (1) Can one write down ξ_k free formulas for $M(m, n, q, k)$ and $H(n, q, k)$ appeared in Theorem 1.1 and 1.2 respectively? It is worth mentioning that one can actually find ξ_k free formulas for these quantities, if $q \equiv 0, 1, -1 \pmod k$ but we have not been able to find a ξ_k free formula in general.
- (2) We would like to see a combinatorial (or direct algebraic) proof of the identity that occurred in Corollary 1.6. Also, a direct algebraic proof of the identity that occurred in Corollary 2.1 would be of interest.
- (3) Also it would be worth exploring whether the techniques of the present paper could be applied to find coefficients or any kind of relations between them for general symmetric polynomials.

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