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Interview with Gil Kalai

Toufik Mansour

Photo by Muli Safra

Gil Kalai received his Ph.D. at the Einstein Institute of the Hebrew University in 1983, under the supervision of Micha A. Perles. After a postdoctoral position at the Massachusetts Institute of Technology (MIT), he joined the Hebrew University in 1985, (Professor Emeritus since 2018) and he holds the Henry and Manya Noskwith Chair. Since 2018, Kalai is a Professor of Computer Science at the Efi Arazi School of Computer Science in IDC, Herzliya. Since 2004, he has also been an Adjunct Professor at the Departments of Mathematics and Computer Science, Yale University. He has held visiting positions at MIT, Cornell, the Institute of Advanced Studies in Princeton, the Royal Institute of Technology in Stockholm, and in the research centers of IBM and Microsoft. Professor Gil Kalai has made significant contributions to the fields of discrete mathematics and theoretical computer science. Two breakthrough contributions include his work in understanding randomized simplex algorithms and disproving Borsuk's famous conjecture of 1933. In recognition of his contributions, professor Kalai has received several awards including the Pólya Prize in 1992, the Erdős Prize of the Israel Mathematical Society in 1993, the Fulkerson Prize in 1994, and the Rothschild Prize in mathematics in 2012. He has given lectures and talks at many conferences, including a plenary talk at the International Congress of Mathematicians in 2018. From 1995 to 2001, he was the Editor-in-Chief of the Israel Journal of Mathematics. He is a member of the Israel Academy of Sciences and the Humanities, the European Academy, and an honorary member of the Hungarian Academy of Science.

Mansour: Professor Kalai, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Kalai: Hmm, you have started with a difficult question. Combinatorics is a very rich and inclusive subject that deals mainly, albeit not solely, with discrete objects. Can we say, perhaps, that combinatorics to mathematics is a bit like mathematics to science? Anyway, whatever it is, it was my choice for my professional life, and I am very happy about this choice.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Kalai: I personally think that many flagships' contributions in combinatorics are the ones developed within the field itself: Tutte's theory for enumeration of planar maps[1], Sémeredi's theorem,[2] the graph minors theorem,[3,4] etc. So I do not think that connections with other areas are needed to justify combinatorics. But still, there are beauti-

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[1] W. T. Tutte, *On the enumeration of planar maps*, Bull. Amer. Math. Soc. 74(1) (1968), 64–74.
[2] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. 27 (1975), 299–345.
[3] N. Robertson and P.D. Seymour, *Graph minors. XX. Wagner's conjecture*, J. Combin. Theory Ser. B 92 (2004),325–357.
[4] L. Lovász, *Graph minor theory*, Bull. Amer. Math. Soc. 43 (2005), 75–86.

ful connections with the rest of mathematics; sometimes a result of a method in combinatorics resonates with other areas, and at other times, methods from other areas are the only known ways to prove combinatorial results. These connections are beautiful as well. Richard Stanley[5], who was my postdoctoral advisor, found amazing applications in combinatorics of commutative algebra, algebraic geometry, and other areas.

As the field of combinatorics widens, it is also important to find relations between different areas of combinatorics itself. I remember that at MIT, I was quite happy to notice some unexpected connections between a problem of Stanley, motivated in algebraic combinatorics, and a method of Kleitman[6] from extremal combinatorics.

Mansour: What have been some of the main goals of your research?

Kalai: I started with combinatorial aspects of convex geometry, Helly-type theorems, and the theory of convex polytopes, and up till now, I keep thinking also about related problems. In these fields, the g-conjecture[7] for spheres was my personal holy grail, but I could not solve it. I also wanted (and indeed still want) to understand flag numbers of polytopes and related cellular objects. Later on, I was influenced by Nati Linial to work with him and Jeff Kahn[8] on a certain problem on Boolean functions and, since then, the analysis of Boolean functions has become a central part of my research together with applications to probability and computer science.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Kalai: My father showed me the formula for $(a+b)^2$ when I was very young and it impressed me. I think I knew that I was good in math pretty early on, and I was also interested at a young age in other areas of science (largely because of excellent popular books I read.) My mother gave me her calculus book from high school (she did not like mathematics that much, but realized that I liked it), and I remember that trying to read it, I could understand various things (like functions) but I got stuck on the expression $f(x+\Delta)-f(x)$. I knew that x is a variable that represents numbers but I did not understand how numbers could be added to triangles.

Another source of information was a popular science journal called "The Young Technician"[9] that my uncle had. I remember that I read there about a young boy that had completed two patents three months after his eighth birthday. Since I was seven years old at the time, I was reassured that I had plenty of time, and need not worry about it. Then I forgot all about it until it was too late: at the age of nine I stumbled upon it again, and today, I have still not written any patent. I had a similar experience with the analysis of Boolean functions. I thought that it would be nice to extend the theory to the non-commutative representation theory, but I also thought that I should not start with such a non-commutative project before the age of 40. But then at the critical time, I totally forgot about it.

Mansour: Were there specific problems that made you first interested in combinatorics?

Kalai: I became interested in problems I had encountered in class. So in my first university year, I took a seminar on tree enumeration using J. W. Moon monograph[10] and I became intrigued with enumerating trees and related combinatorial identities. In my second year, I took a convexity class and became interested in Helly-type theorems[11]. (Talking about J. W. Moon, I was trying to find his picture online for my blog, but Google only gave me pictures of the moon. Any help would be wel-

[5]R. P. Stanley, *Combinatorics and Commutative Algebra*, 2nd ed., Progress in Mathematics, Volume 41, Birkhauser, Boston, 1996.

[6]M. Saks, *Kleitman and combinatorics*, Discrete Math. 257 (2002), 225–247.

[7]K. Adiprasito, *Combinatorial Lefschetz theorems beyond positivity*, arXiv:1812.10454.

[8]J. Kahn, G. Kalai, and N. Linial, *The influence of variables on Boolean functions*, 68–80, in Proc. 29th Annual Symposium on Foundations of Computer Science, 1988.

[9]https://he.wikipedia.org/wiki/%D7%94%D7%98%D7%9B%D7%A0%D7%90%D7%99_%D7%94%D7%A6%D7%A2%D7%99%D7%A8.

[10]J. W. Moon, *Counting labelled trees*, Canadian Mathematical Congress, Montreal, 1970. Available at https://www.math.ucla.edu/~pak/hidden/papers/Moon-counting_labelled_trees.pdf.

[11]L. Danzer, B. Grtinbaum, and V. Klee, *Helly's theorem and its relatives*, Proc. Sympos. Pure Math. 7, Amer. Math. Soc., Providence, R. I., 1963, 100–181.

come.)

Mansour: What was the reason you chose the Hebrew University for your Ph.D. and your advisor, Micha Perles?

Kalai: I was at the Hebrew University already as B.Sc. student and as an M.Sc. student and actually Micha Perles gave the two courses I mentioned in the previous question. Micha Perles was extremely generous and brilliant, and he was also very modest and I liked it; I even tried to adopt some of Micha's modesty in spite of my earlier reverse disposition.

Mansour: How was the mathematics in Jerusalem at that time?

Kalai: I liked the atmosphere in Jerusalem then and I still do now. There were many excellent students. As undergraduate students, our role models were mainly other students, and I remember that in my years as a student we all heard about a legendary student Abraham Neyman who was a great problem solver and studied a few years earlier. The faculty were very nice and also tolerated my tendency to jump over my head and were very generous in explaining things. Later in graduate school, Noga Alon did his Ph.D. with Micha at the same time I did, Nati Linial was his student a few years earlier and Yaacov Kupitz, Ido Shemer, and a few others were also Ph.D. students of Perles at that time.

Here is a story about the atmosphere in Jerusalem some years later. Uli Wagner visited HUJI and one day we had coffee on campus and he told me that he had some problem related to the representation theory of S_n. I told Uli that in the next five minutes we would surely meet some local expert on the matter. Two minutes later we met Avital and asked him, and he said that he knows quite a bit about the representation of S_n but that his interest shifted to Hecke algebras. "No problem" I said, "we still have three more minutes." Sure enough in the next minute or so, we met another expert in representations of S_n that could help with the problem.

Mansour: What was the problem you worked on in your thesis?

Kalai: I worked on the following conjecture of Katchalski and Perles[12,13]: If you have n convex sets in R^d and no $d+k+1$ of them has a point in common then the number of $(d+1)$-tuples that has a point in common is at most $\binom{n}{d+1} - \binom{n-k}{d+1}$. After I solved the problem, I went on to solve a more general conjecture by Eckhoff[14] on face numbers of nerves of families of convex sets in R^d.

Mansour: What would guide you in your research, a general theoretical question or a specific problem?

Kalai: Largely, my research is problem-oriented (and I like to invent problems). In some cases, like the study of algebraic shifting[15] and the analysis of Boolean functions[16], pursuing the general theory got a life of its own.

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Kalai: Of course.

Mansour: What three results do you consider the most influential in combinatorics during the last thirty years?

Kalai: Three most amazing results during the last thirty years were Keevash's existence of designs, Chudnovsky, Robertson, Seymour, and Thomas' strong perfect graph conjecture[17], and Zeilberger's enumeration of alternating sign matrices[18].

For the first and last theorems, there are already additional proofs, which is a good place to mention how important it is to find more and more proofs also to already proven results.

Mansour: What are the top three open questions in your list?

Kalai: Let me try:

[12] G. Kalai, *Intersection Patterns of Convex Sets*, Israel J. Math. 48 (1984), 161–174.

[13] N. Alon and G. Kalai, *A Simple Proof of the Upper Bound Theorem*, European J. Comb. 6:3 (1985), 211–214.

[14] J. Eckhoff, *Über kombinatorisch-geometrische Eigenschaften von Komplexen and Familien knovexer Mengen*, J. Reine Angew. Math. 313 (1980), 171–188.

[15] A. Björner and G. Kalai, *An extended Euler-Poincaré formula*, Acta Math. 161 (1988), 279–303.

[16] R. O'Donnell, *Analysis of Boolean Functions*, Cambridge University Press, 2014.

[17] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The strong perfect graph theorem*, Ann. of Math. 164 (2006), 51–229.

[18] D. Zeilberger, *Proof of the alternating sign matrix conjecture*, Electronic J. Combin 3 (1996), R13.

[19] R. P. Stanley, *Generalized H-vectors, intersection cohomology of toric varieties, and related results*, In Commutative algebra and combinatorics (Kyoto, 1985), Adv. Stud. Pure Math. 11, 187–213, North-Holland, Amsterdam, 1987.

1) Is the toric g-vector an M-vector (for polytopes)?[19].
2) Turán (4,3)-problem[20]: What is the largest number of triples you can have from $[n] = \{1, 2, \ldots, n\}$ without having all four triples on some four vertices.
3) The polynomial Hirsch conjecture[21,22] (about the diameter of graphs of polytopes).

I also like the problems of better bounds for Roth's theorem, better asymptotic bounds for binary and spherical codes, and the dying percolation conjecture in 3-space.

Another nice problem is how to make the combinatorial, mathematical, and academic communities more diversified.

Mansour: What kind of mathematics would you like to see in the next ten-to-twenty years as the continuation of your work?

Kalai: I don't know; surprise me!

Mansour: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

Kalai: I suppose that there are core and mainstream areas in mathematics. And, for sure, some topics are more important than other topics. I must confess that I did not think in-depth about these matters, and I am not sure if importance is that important.

Mansour: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Kalai: For me personally, at present, the applied flavor of a problem is very appealing, but perhaps 40 years ago I was charmed by the idea of working on problems with no applications to "real life". Still, overall, I work on similar problems to those I worked on 40 years ago, which are largely combinatorial problems that I have some idea (or fantasy) about how to solve.

Generally speaking, I admire applied mathematics and statistics.

One more thing: in my (-1)th year at HUJI, I took a game theory class by Michael Maschler, and I have liked Game theory since then. In game theory and in other areas where theory meets practice, there is a lot of room for discussions and debates which I like. Another area where the relation between theory and practice is a great topic, is computer science, and my ICM 2018 proceeding paper[23] is largely devoted to this issue (as well as to the connection with combinatorics, of course).

Mansour: What advice would you give to young people thinking about pursuing a research career in mathematics?

Kalai: The two major activities of a career in mathematics are research and teaching, and it is hard for me to give universal advice on these core matters. I also cannot give general advice on the very important matter of interaction and collaboration. So let me mention two things:
a) Learn to use, to master, and to enjoy computer programming;
b) Learn to use, to master, and to enjoy the English language. Here one can even add
c) Learn touch typing.

Actually, these are pieces of advice that I would also give myself at present, and I hope they might be suitable for the elderly as well.

Mansour: You have been managing a wonderful blog, *Combinatorics and more*[24], since 2008 and have published very interesting notes there. What was your motivation for it?

Kalai: Thank you! I think the main motivation was that I like to write about the works of other people. I was also encouraged by several people that are mentioned on the "about" page and wanted to experience collective mathematical discussions.

In a sense, it is Aesop's litigious cat fable in reverse. In Aesop's story, a monkey helps two cats divide a piece of cheese by repeatedly eating from the larger piece. In my case, I think that I need to balance my blog and to do justice to areas, problems, and people that I have not mentioned yet, and the more I

[20] P. Turán, *Research problems*, Magyar Tud. Akad. Mat. Kutató Int. Közl. 6 (1961), 417–423.
[21] F. Eisenbrand, N. Hahnle, S. Razborov, and T. Rothvoss, *Diameter of Polyhedra: The Limits of Abstraction*, Mathematics of Operations Research 35 (2010), 786–794.
[22] G. Kalai and D. J. Kleitman, *A quasi-polynomial bound for the diameter of graphs of polyhedra*, Bull. Amer. Math. Soc. 26 (1992), 315–316.
[23] G. Kalai, *Three puzzles on mathematics, computation, and games*, Proceedings of the International Congress of Mathematicians (B. Sirakov, P. Ney de Souza, and M. Viana, eds.), 551–606, World Scientific, 2018.
[24] `https://gilkalai.wordpress.com/`.

write the more I feel that work is still required to make the blog's coverage (within subareas of combinatorics that I have some familiarity with) sufficiently complete, just and balanced. However, recently WordPress moved to an inconvenient editor, so this may be a sign from the universe to slow down matters on my blog.

Mansour: Would you tell us about your other interests besides mathematics?

Kalai: Besides the interest in family and friends, I don't have serious interests or notable hobbies. My wife and I like to dance (freestyle disco), but we almost only do it when we host the dancing party ourselves, which happened twice.

Mansour: Before we close this interview with one of the foremost experts in combinatorics, we would like to ask some more specific mathematical questions. Would you tell us more about the Borsuk problem in geometry, which is known as *Borsuk's conjecture*[25]? It was first proved affirmatively for several cases. But then, surprisingly, you and Kahn[26] constructed finite sets showing the contrary. How did you become interested in this problem? Did you also initially try to obtain an affirmative answer to it rather than looking for a counter-example?

Kalai: The problem asks if every set of diameter 1 in R^d could be divided to $d+1$ sets of smaller diameter. Let us write $f(d)$ to be the smallest integer such that every set of diameter 1 in R^d can be divided to $f(d)$ sets of smaller diameter. Let $g(d)$ be the same number when you consider only finite sets. Let me make a few brief comments:

a) Jeff Kahn and I believed all along that the answer is negative.

b) It is not known if $f(d) = g(d)$ and my guess is that, except in low dimensions, they are different.

c) We proved that

$$f(d) \geq (1+\delta)^{\sqrt{d}},$$

and it is not known if $f(d) \geq (1+\delta)^d$, for some $\delta > 0$. The best upper bound on $f(d)$ for large d is from 1988 by Oded Schramm[27]:

$$f(d) \leq \sqrt{4/3}^{\,d(1+o(1))}.$$

d) I found a way to (perhaps) save the conjecture! Here it is for the finite case. (It can also be stated for the general situation.) In rough terms, it says that counter-examples to the Borsuk conjecture are just coincidental.

You look at a finite set of points of diameter 1 in R^d and let G be the graph of distance-one edges. The original Borsuk conjecture asserts that G is $(d+1)$-colorable. My "saved-Borsuk conjecture"[28] is that if G is stress-free then it is $(d+1)$-colorable. Here, stress is an assignment of weights to the edges so that every vertex is in "equilibrium"; and a framework in R^d is stress-free of it has no non-zero stress.

Mansour: You have a long-standing conjecture in geometry, known as *Kalai's* 3^d *conjecture*[29]. Would you tell us about it? What was the motivation behind this conjecture? What best results do we have so far? Do you want to see a resolution for it soon, or would you rather it remains a mystery for a long time?

Kalai: The conjecture asserts that every centrally symmetric d-polytope P has at least 3^d non-empty faces (P itself is considered a face). The d-cube, and, in fact, all Hanner polytopes have precisely 3^d non-empty faces.

I think the best results as well as the motivation were prior to the conjecture. Stanley[30] proved a much stronger result for centrally symmetric simplicial polytopes and Fiegiel, Lindenstrauss, and Milman[31] proved that $\log f_0(P) \log f_o(P^*) \geq \gamma d$, for some absolute constant γ. (P^* is the dual of P.) Sanyal, Werner, and Ziegler[32] proved the conjecture for $d \leq 4$, disproved some stronger conjectures that I made in my original paper, and also discussed connections with Mahler's conjecture.

[25] K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fundamenta Mathematicae (in German) 20 (1933), 177–190

[26] J. Kahn and G. Kalai, *A counterexample to Borsuk's conjecture*, Bull. Amer. Math. Soc. 29(1) (1993), 60–62

[27] O. Schramm, *Illuminating sets of constant width*, Mathematika 35(2) (1988), 180–189.

[28] https://gilkalai.wordpress.com/2013/09/04/around-borsuks-conjecture-3-how-to-save-borsuks-conjecture/.

[29] G. Kalai, *The number of faces of centrally-symmetric polytopes*, Graphs and Combin. 5(1) (1989), 389-391

[30] R. P. Stanley, *On the number of faces of centrally-symmetric simplicial polytopes*, Graphs and Combin. 3(1) (1987), 55–66.

[31] T. Figiel, J. Lindenstrauss, and V. D. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. 139(1-2) (1977), 53–94.

[32] R. Sanyal, A. Werner, and G. Ziegler, *On Kalai's conjectures concerning centrally symmetric polytopes*, Discrete & Computational Geometry 41:2 (2009), 183–198.

To your second question: I certainly want it to be resolved. I would be happy to see it solved soon in an embarrassingly simple way.

Mansour: You claim that "quantum computers can't possibly work, even in principle". Would you briefly explain what a quantum computer is and why you think that it won't work?

Kalai: Yes, that's correct and my claim applies not only to the ultimate goal of operating quantum computers but also to very short-term tasks that people try to achieve:
Task A) To demonstrate "quantum computational advantage" in noisy intermediate-scale quantum (NISQ) systems. And
Task B) To build good-quality qubits based on quantum error correction.

The short version of my argument is
Claim 1) Task A is beyond reach because of the computational complexity power of NISQ systems. And
Claim 2) Task B is harder than Task A.

Now, my second claim is largely agreed upon by experts. The first claim is largely disagreed upon by experts and some research groups, including one in Google, claim that they already experimentally achieved quantum computational advantage.

So let me add three quick remarks:
a) In my view what was achieved by Google and other groups could be regarded as spectacular mock-up demos, but not yet as reliable scientific experimental evidence for "quantum advantage".
b) Impossibility in principle to achieve stable qubits may ease the old-time tension between determinism and free will. The free will problem is an old philosophical problem that has some aspects in the interface of philosophy and physics.
c) It is an interesting experience to have a view/theory that goes against what most experts think especially since I personally know, like, and appreciate many of these experts.

There are various places to read about my theory in more detail. It is related to a paper I wrote in 2014 with Guy Kindler[33] and to a theory of noise sensitivity and noise stability that goes back to my work with Benjamini and Schramm.

Mansour: In your research, you have extensively used combinatorial reasoning to address important problems in probability theory, theoretical computer science, etc. Do enumerative techniques play an important role in your research?

Kalai: I admire enumerative combinatorics and I have a nice early result about enumerating high dimensional trees[34] (in fact, my master's thesis was in enumerative combinatorics). But I am an amateur in this area, and enumerative combinatorics did not play an important role in my research since then. Of course, the line between probabilistic methods and enumerative methods is not so firm.

This brings me to a general question about enumeration that I find (from an amateur standpoint) very interesting. It is intractable to count certain combinatorial objects precisely. Can you get some mileage by weighted enumeration? Some years ago, I interviewed enumerative combinatorics friends, and got a few cool examples where this happens. The very last question I pose in this interview about Laman's graphs is of this nature.

Mansour: One of your most influential works is a joint paper with Jeff Kahn and Nathan Linial, on Boolean functions. It seems that this was an effort to apply Fourier analysis for a problem in theoretical computer science. Would you tell us about it?

Kalai: In this joint work, we solve a problem by Michael Ben-Or and Nati Linial[35], and among the three of us, I personally represented an effort not to use Fourier methods. The success of Fourier methods and hypercontractive inequalities in our work had a great influence on me later on. What we proved can be stated as follows: Consider a family $\mathcal{F}$ of subsets of $[n] = \{1, 2, \ldots, n\}$, and suppose that $\mathcal{F}$ is monotone, namely, if $R \subset S$, and $R \in \mathcal{F}$ then also $S \in \mathcal{F}$. Suppose further that $|\mathcal{F}| = 2^{n-1}$. Then there exists an index $k \in [n]$ such that

$$|\{S \in \mathcal{F} : k \in S\}| \geq 2^{n-2}(1 + C \log n/n).$$

Another way to say it is as follows: monotonicity implies that every element $k \in [n]$

[33]G. Kalai and G. Kindler, *Gaussian noise sensitivity and BosonSampling*, arxiv:1409.3093.
[34]G. Kalai, *Enumeration of Q-acyclic simplicial complexes*, Israel J. Math. 45:4 (1983), 337–351
[35]M. Ben-Or and N. Linial, *Collective coin flipping, robust voting schemes and minima of Banzhaf values*, In: 26th Annual Symposium on Founda tions of Computer Science, 408–416.

belongs to at least half the sets in the family and our theorem asserts that some k belongs to substantially more than half.

An example where the bound is tight, is Ben-Or–Linial "tribe example"[35]. It is very interesting to ask for which families this bound is tight (for some value of the constant C). Ehud Friedgut[36] has a very nice conjecture about it.

Mansour: The union-closed sets conjecture, posed by Péter Frankl in 1979, is an elementary problem in combinatorics, but is still open. Would you tell us about this conjecture? What are the recent important related works in this direction?

Kalai: The problem is very simple to state: If a family $\mathcal{F}$ of subsets of $\{1, 2, \ldots, n\}$ is closed under union, then there is an element k that belongs to more than half the sets in the family. You may recall from the last question that if $\mathcal{F}$ is monotone, then every k belongs to at least half the sets. So Frankl's conjecture asks for a weaker conclusion that applies to the more general and mysterious "union-closed" property.

Ilan Karpas[37] used Fourier methods to prove that this is true if the family has more than 2^{n-1} sets, and this is a lovely result.

Mansour: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have an "eureka moment"?

Kalai: Let me talk about the diameter of d-polytopes with n facets. Every vertex v can be regarded as a d-set S_v indexed by the faces. Two vertices u and v are adjacent if $|S_v \cap S_u| = d - 1$. We know that

(*) For every set R of facets the graph induced on all vertices v such that $R \subset S_v$ is a connected graph.

Step 1: Try to relate it to f-vector theory, and also to the expansion of graphs. This led to some nice results, but not to major progress on the diameter problem itself.

Next I tried to use purely combinatorial reasoning only based on condition (*). We need now another definition: given a family $\mathcal{F}$ of d-sets a (d, k)-path between S and T is a sequence of sets $S = S_1, S_2, \ldots, S_m = T$ such that $|S_i \cap S_{i+1}| \geq k$.

Step 2: Let us move from a S to T by short (d, s) paths, and then move to the "links". This improved the bound from $2^d \cdot n$ to roughly

$$\exp(\sqrt{n}).$$

Step 3: I considered some more complicated paths from S to T. Unfortunately, I do not remember it so well but only that it provided substantial improvement, an upper bound

$$\exp\exp((\log\log n)^{2/3}).$$

Step 4: Observation: Let us now move from S and T by various such paths that we considered before. Once we reach more than half the vertices from both sides, we can move to the link of some vertex and get some new recurrence relation which gave roughly

$$d^{2\log n}.$$

Step 5: (with Danny Kleitman): the "book proof" for Step 4 that also improved the constant from 2 to 1.

I often thought whether the thought process behind the crucial steps, 2, 3, 4, and 5 could be automated, and then perhaps improved. This would be very nice!

Mansour: Is there a specific problem you have been working on for many years? What progress have you made?

Kalai: I already mentioned some well-known problems that occupied me over the years, so let me mention three of my own problems that I keep returning to.

The first problem[38] is from 1974. For a set X in R^d, let $T_r(X)$ be the set of all points in R^d that belong to the convex hull of r pairwise disjoint subsets of X. Let $tv_r(X) = 1 + \dim T_r(X)$. Then

$$\sum_{r\geq 0} tv_r(X) \geq |X|.$$

The second problem[39] is a conjecture, by Friedgut and myself, known under the name

[36] E. Friedgut, *Influences in product spaces, KKL and BKKKL revisited*, Combinatorics Probability and Computing 13:1 (2004), 17–29.

[37] I. Karpas, *Two Results on Union-Closed Families*, arXiv:1708.01434.

[38] Section 5 of: I. Barany and G. Kalai, *Helly-type problems*, Bull. Amer. Math. Soc., to appear.

[39] E. Friedgut and G. Kalai, *Every monotone graph property has a sharp threshold*, Proc. Amer. Math. Soc. 124 (1996), 2993–3002.

of the “influence–entropy conjecture”. It asserts that for some constant C, if f is a Boolean function then

$$\sum_{S\subset[n]} \hat{f}^2(S)|S| \geq C \sum_{S\subset[n]} \hat{f}^2(S)\log(|\hat{f}^2(S)|).$$

The third problem is the following: A Laman graph is a graph on n vertices with $2n-3$ edges such that every subgraph with m vertices, $m \geq 2$ has at most $2m-3$ edges. Find a (weighted) enumeration of Laman graphs with n labelled vertices that gives

$$\binom{n}{2}^{n-3}.$$

Mansour: Professor Gil Kalai, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

Kalai: I would like to use this opportunity, Professor Toufik Mansour, to thank you for the beautiful journal and interviews. This is a most valuable contribution to our community.