Enumerative Combinatorics and Applications
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ECA **2:2** (2022) Interview #S3I8

Interview with Ira Gessel

Toufik Mansour

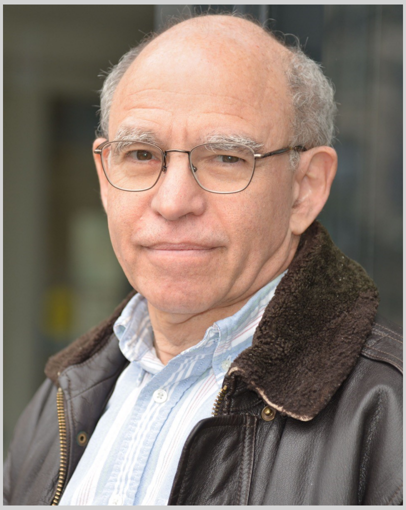

Photo by Michael Gessel

Ira Gessel completed his undergraduate studies at Harvard University in 1973. He obtained a Ph.D. from the Massachusetts Institute of Technology (MIT) in 1977, under the supervision of Richard Stanley. He was an Assistant Professor at the MI from 1980 until 1984. After that, he was an Assistant Professor at Brandeis University and has remained there as an Associate, Full Professor, and Emeritus Professor since 2015. Since 2012, he has been a Fellow of the AMS. Professor Gessel has given numerous invited talks in conferences and seminars. Professor Gessel has served as a member of the editorial board of *Discrete Mathematics*, the *Journal of Algebraic Combinatorics*, and *Algebraic Combinatorics*.

Mansour: Professor Gessel, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Gessel: As a first approximation, combinatorics is the study of finite or discrete structures. More precisely, we traditionally exclude certain kinds of finite structures, such as finite groups and fields, so combinatorics might be described in biological terminology as paraphyletic. My area, enumerative combinatorics, is easier to describe: it is the study of formulas for counting things. But even here, there are exceptions, as some parts of number theory also involve formulas for counting, but are not considered to be combinatorics.

Mansour: What have been some of the main goals of your research?

Gessel: My research has never had any overarching goals; I have just tried to prove interesting theorems and write interesting papers. I have mostly stuck to the parts of mathematics that I understand (primarily enumerative combinatorics, with a little bit of algebraic combinatorics, elementary number theory, identities). I have occasionally looked at problems outside these areas without success. I write very slowly so usually when I write a paper it is on something that I have been thinking about for years, and often the main results are quite old. At this point in my life, I am more concerned with finishing things that I have already started than with new projects.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Gessel: I have been interested in mathematics from an early age. My parents were supportive, but they did not know much about mathematics. I was pretty much self-taught until high school. After my freshman year in high school, I went to a summer program at Kenyon College, run by Daniel Finkbeiner, and the next two summers I went to a program at

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Ohio State run by Arnold E. Ross, now called the Ross Program[1], though it was not called that at the time. The Ross program had a big influence on me. I learned elementary number theory and met other high school students who had the same kind of interest in mathematics that I had. In addition to two years as a student in the Ross program, I was also a counselor there for four summers when I was in college. Dr. Ross's favorite piece of advice was "Think deeply of simple things," which I have always taken to heart, not so much as a guiding principle, but because that is the way I work. I work on simple things, like words and graphs, but find interesting things to say about them.

Mansour: Were there specific problems that made you first interested in combinatorics?

Gessel: No. In my first year as a graduate student, I was not sure what I wanted to study, and I did not know anything about combinatorics. I thought I might want to study logic, but as I recall, in my second year, when I had to pick a field, most of the logicians at MIT were away. I took Richard Stanley's combinatorics course and found it very interesting.

Mansour: What was the reason you chose MIT for your Ph.D. and your advisor Richard Stanley?

Gessel: I chose MIT only because it had a strong mathematics department. I was not sure what I wanted to specialize in—I thought maybe algebra, but nothing more specific than that. I took Richard Stanley's combinatorics class in my second year as a graduate student and I liked it. My officemate Emden Gansner was already working with Richard and thought he was a good advisor, so I decided to work with him.

Mansour: What was the problem you worked on in your thesis?

Gessel: I was studying enumeration of sequences (or words, as they are more likely to be called today) according to occurrences of consecutive pairs. My favorite result in this area is a formula that says that the sum of all words in which every consecutive pair lies in a set S of words of length 2 is the reciprocal of the sum of all words, with alternating signs, in which no consecutive pair is in S. For example, the sum of all words in x, y, and z with no occurrence of xy is $(1-x-y-z+xy)^{-1}$. I was a bit disappointed when I learned that this result had been found a little earlier by Carlitz, Roselle, and Scoville[2] (and many years later I learned that a version of it had been found even earlier by Ralf Fröberg). But Carlitz, Roselle, and Scoville applied this beautiful general result only to one very specialized problem, and in my thesis, I applied it to many other problems involving descents[3] of permutations, for example, counting permutations with a periodic pattern of ascents and descents. The ideas in my thesis were the basis for much of my later work on permutation enumeration.

Mansour: What would guide you in your research? A general theoretical question or a specific problem?

Gessel: Rather than pursuing general theoretical questions, I try to develop general methods that can be used to derive explicit interesting results. Many of my papers fit this description, such as work on symmetric functions, quasisymmetric functions, combinatorial species, graphical enumeration, umbral calculus, and combinatorial proofs of congruences. I can only think of one paper, on counting unlabeled k-trees[4] (with Andrew Dewar-Gainer), where I worked on a specific problem for a long time and eventually solved it.

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Gessel: Sometimes. I often do computations (usually with Maple) which sometimes lead to conjectures. But more often I don't quite know in advance what a line of reasoning will lead to.

Mansour: What are the top three open questions in your list?

Gessel: Most of my research time is spent not on solving specific open problems, but rather on organizing, clarifying, and writing about things that I more or less already understand. So the open problems on my "list" are not what

[1] `https://rossprogram.org/`.

[2] L. Carlitz, R. Scoville, and T. Vaughan, *Enumeration of pairs of sequences by rises, falls and levels*, Manuscripta Math. 19 (1976), 211–243.

[3] A descent of a permutation π is an index i such that $\pi(i) > \pi(i+1)$.

[4] A k-tree is either a complete graph on k vertices or a graph obtained from a smaller k-tree by adjoining a new vertex together with k edges connecting it to a k-clique. So a 1-tree is an ordinary tree.

I spend most of my time working on. But here are a few problems that I think about from time to time. They are pretty obscure and I don't think that too many people would be interested in them, but I find them intriguing. One of them is not really combinatorics at all and the other two are on the unfashionable topic of graphical enumeration.

(1) Let $\sum_{n=1}^{\infty} v_n x^n/n!$ be the compositional inverse of $\sqrt{2x - 2\log(1+x)}$ and let $B(x) = \sum_{k=2}^{\infty} \frac{B_k}{k(k-1)} x^{k-1}$, where B_k is the kth Bernoulli number. Then, by computing the asymptotic expansion for the gamma function in two different ways, deBruijn showed that $\sum_{k=0}^{\infty} 2^{-k} v_{2k+1} x^k/k! = e^{B(x)}$. Is there a formal power series proof?

(2) There are many formulas that cry out for combinatorial interpretations. Here is one that I have looked at many times without success. Let s_n be the number of strong (i.e., strongly connected) digraphs on $[n] = \{1, 2, \dots, n\}$ and let t_n be the number of strong tournaments on $[n]$. Then

$$\sum_{n=1}^{\infty} s_n \frac{x}{n!} = \log \frac{1}{1 - T(x)},$$

where $T(x) = \sum_{n=1}^{\infty} 2^{\binom{n}{2}} t_n\, x^n/n!$.

(3) It is almost always the case that if we can count a class of labeled graphs then with some more work we can count the corresponding unlabeled graphs. Here is a problem where I have not been able to do this. Ronald Read[5] counted both labeled and unlabeled graphs with the property that no two vertices have the same neighborhood. These graphs have been studied and named independently several times—they are called "point-determining," "mating-type," or "R-thin." The labeled exponential generating function $P(x)$ for point-determining graphs is $G\big(\log(1+x)\big)$ where $G(x) = \sum_{n=0}^{\infty} 2^{\binom{n}{2}} x^n/n!$ is the exponential generating function for all graphs. The equivalent formula $G(x) = P(e^x - 1)$ has a very simple combinatorial interpretation: every graph can be reduced to a point-determining graph by identifying vertices with the same neighborhood, and this decomposition can be used to count unlabeled point-determining graphs. I looked at an analogous problem, counting graphs in which no two vertices have complementary neighborhoods, and showed, using inclusion-exclusion, that the exponential generating function for these graphs is $G\big(\frac{1}{2}\log(2e^x - 1)\big)$. But this approach does not seem to work for counting the corresponding unlabeled graphs, and I have no idea how to count them.

Mansour: Would you tell us about your interests besides mathematics?

Gessel: My main interest other than mathematics is folk dancing, mostly Balkan and Hungarian. I also sing in Bulgarian and Croatian music groups, and play the Macedonian tambura, though not very well. Interestingly, Bruce Sagan and Tom Roby, two other students of Richard Stanley's, have also sung in the same Bulgarian chorus. I also like to take photos, though I am not a serious photographer.

Mansour: For about ten years, you have been active in the mathematical forum, MathOverflow. How can you describe your activity there? How important do you see the interaction of mathematicians in such platforms?

Gessel: MathOverflow is a convenient place for people to ask questions without knowing in advance who to ask. Some of the questions are not very interesting, but some are quite interesting, and I have learned some things there about areas of mathematics that I do not think about very much. One MathOverflow question that I worked on [Liviu Nicolaescu. Combinatorial Morse functions and random permutations, `http://mathoverflow.net/questions/86193`, 2012] led to a publication of mine[6].

Mansour: You frequently propose problems to some math magazines, especially to the Monthly. How do you find these problems? Do you specifically dedicate some time to create such interesting questions?

Gessel: No, the problems that I propose all come from my work. Every once in a while (not all that often) I come across something interesting that I think might make a good problem and then I submit it. To take a recent example, for some reason (I don't remember

[5] R. C. Read, *The enumeration of mating-type graphs*, Research Report CORR 38, Department of Combinatorics and Optimization, University of Waterloo, 1989, `https://oeis.org/A006023/a006023.pdf`.

[6] I. M. Gessel and Y. Zhuang, *Counting permutations by alternating descents*, Electronic J. Combin. 21(4) (2014), Paper #P4.23.

why) I was looking at $(1 + x + x^2 + x^3)^4 \equiv 1 - x + x^4 - x^6 + x^8 - x^{11} + x^{12} \pmod 5$, and the fact that the nonzero coefficients were alternately 1 and -1 seemed interesting. I investigated further, and found a nice generalization that seemed to be at the right level for a Monthly problem. (It remains to be seen whether the Monthly agrees with me!)

Mansour: You have mentored around thirty graduate students. Would you tell us about your experience as a supervisor? In general, what do you think about mentorship?

Gessel: I have no particular insight into supervising graduate students and I am not sure why I have had more students than average. I think one reason might be that the problems I work on are more accessible than those of many of my colleagues. Mentorship is generally rewarding, though sometimes challenging. Most of my students went on to become contributing members of society, many of them used part of their mathematical training in some way, and a few are still doing research in enumerative combinatorics!

Mansour: It is becoming common that undergraduate students undertake some research works. How important is it for an undergraduate student to have research experience under an experienced researcher's supervision for their future research career? What would you suggest for an undergraduate student who wants to work in your field?

Gessel: Undergraduate research is not necessary for a successful research career. Mathematicians have been doing research for centuries without organized undergraduate research programs. On the other hand, undergraduate research experience can help in gaining admission to a graduate program. My only advice for an undergraduate student who wants to do research as an undergraduate in my field (or in any particular field) is obvious: Look for an advisor in your institution, look into summer REU[7] programs (if in the US), and look into the Budapest Semesters in Mathematics.

Mansour: You are known for the *invention* of quasisymmetric functions. What motivated their introduction to your research?

Gessel: Quasisymmetric functions are the natural generating functions for Stanley's P-partitions, and they actually appear briefly in Stanley's work. The basic idea of P-partitions is that for a finite poset P, the set of P-partitions (weakly order-reversing maps from P to the positive integers) can be partitioned into sets corresponding to the linear extensions of P, in a way that is closely related to the descents of the linear extensions. Quasisymmetric functions encode descent sets of these permutations; more precisely the quasisymmetric P-partition generating function contains exactly the same information as the multiset of descent sets of the linear extensions of P. So quasisymmetric function arise very naturally in counting permutations according to their descent sets. The paper in which I introduced quasisymmetric functions[8] grew out of some work that I did with Adriano Garsia[9] on counting k-tuples of permutations whose product is the identity, according to their descent number and major index by decomposing multipartite partitions. I realized that the same decomposition could be used to count these permutations by their descent sets if we weighted the multipartite partitions differently—this weighting gives quasisymmetric functions. So this paper was about a somewhat specialized aspect of quasisymmetric functions; the decomposition of multipartite partitions corresponds to the "inner coproduct" of quasisymmetric functions. In this paper, I did not even discuss how multiplication of the fundamental quasisymmetric functions corresponds to shuffles of permutations, though it is an immediate consequence of the theory of P-partitions.

Mansour: Some of your research is related to *permutation statistics*. Why is the study of permutation statistics interesting?

Gessel: Counting permutations is one of the very basic problems of enumerative combinatorics. When ordinary people (non-mathematicians) ask me what I do, I tell them that I count things like permutations and combinations–only more complicated–and they usually understand. In counting any combinatorial objects, it is natural to count them according to various parameters. For example,

[7] Research Experience for Undergraduates

[8] I. M. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Combinatorics and algebra (Boulder, Colo., 1983), 289–317, Contemp. Math. 34, Amer. Math. Soc., Providence, RI, 1984.

[9] A. M. Garsia and I. Gessel, *Permutation statistics and partitions*, Adv. in Math. 31 (1979), no. 3, 288–305.

if we are counting graphs, it is very natural to keep track of the number of edges or the number of connected components. The number of cycles and the number of inversions of a permutation arise in many places. Some permutation statistics are more subtle. It is not obvious that the descents of permutations are worth studying. But it turns out that they are extremely interesting. One of the most beautiful formulas in enumerative combinatorics is

$$\sum_{k=0}^{\infty} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}},$$

where the coefficient of t^i in $A_n(t)$ (the nth Eulerian polynomial) is the number of permutations of $[n]$ with $i-1$ descents. Understanding this formula is the key to much of my work on permutations.

Mansour: In joint work with Christian Krattenthaler, *Cylindric partitions*[10], you introduced a new object that generalizes the plane partitions, namely *cylindric partitions*. Would you tell us about this work and the critical ideas behind it?

Gessel: This grew out of my work with Viennot[11] on non-intersecting paths. Plane partitions can be represented by configurations of non-intersecting lattice paths, and they can be counted by determinants which are alternating sums over the symmetric group of counts of configurations of paths without the non-intersecting condition. This is analogous to counting single lattice paths in n dimensions restricted to the region $x_1 > x_2 > \cdots > x_n$ where we can apply the reflection principle to obtain an alternating sum over the symmetric group, generated by reflections in the hyperplanes $x_i = x_j$. We can count paths restricted to the region $x_1 > x_2 > \cdots > x_n > x_1 - m$ for some m by enlarging the group of reflections to include the reflection in the hyperplane $x_n = x_1 - m$, and we obtain a sum over the affine symmetric group. We can then do something very similar with nonintersecting paths; in addition to requiring that the paths be nonintersecting, we require that the last path not intersect a translation of the first path. We then obtain a sum over the affine symmetric group, and the configurations of nonintersecting paths can be transformed into cylindric partitions.

Mansour: Another great work of yours, co-authored with Yan Zhuang, is *Shuffle-compatible permutation statistics*. Therein, you introduced the shuffle algebra of a shuffle-compatible permutation statistic. Would you elaborate on this result and point out some related research directions?

Gessel: First, let me define shuffles of words. If u and v are words with disjoint letters then the shuffles of u and v are the interleavings of u and v. For example, the shuffles of 13 and 42 are 1342, 1423, 1432, 4123, 4213, and 4132. A permutation statistic st is called *shuffle compatible* if for any disjoint permutations π and σ (viewed in one-line notation as words), the multiset $\{\, \mathrm{st}(\tau) : \tau \text{ is a shuffle of } \pi \text{ and } \sigma \,\}$ depends only on $\mathrm{st}(\pi)$ and $\mathrm{st}(\sigma)$ (and the lengths of π and σ[12]). Some examples of shuffle-compatible permutation statistics are the descent set, the descent number des, the major index (sum of the descents) maj, and the ordered pair (des, maj), all of which were proved by Stanley[13] using P-partitions, and the peak[14] set and a number of peaks, proved by John Stembridge[15].

If st is a shuffle-compatible permutation statistic, we can define an associated algebra in the following way: for a permutation π we define the equivalence class $[\pi]_{\mathrm{st}}$ to be the set of all π' with $\mathrm{st}(\pi') = \mathrm{st}(\pi)$. Then we take these equivalence classes as basis elements and multiply them by $[\pi]_{\mathrm{st}}[\sigma]_{\mathrm{st}} = \sum_\tau [\tau]_{\mathrm{st}}$ where the sum is over all shuffles τ of π and σ. The shuffle compatibility of st ensures that this product is well defined. If st is a descent statistic, i.e., it depends only on the descent set, then the shuffle algebra is a quotient algebra of the algebra of quasisymmetric functions, and in most cases that we know of, we can describe the shuffle algebra explicitly.

One big open question is whether there is anything interesting that we can say about the structure of the set of all shuffle-compatible

[10] I. M. Gessel and C. Krattenthaler, *Cylindric partitions*, Transactions Amer. Math. Soc. 349:2 (1997), 429–479.
[11] I. M. Gessel and G. Viennot, *Binomial determinants, paths and hook length formulae*, Adv. Math. 58 (1985), 300–321.
[12] It is convenient to incorporate the length of a permutation into the statistic, so that if $\mathrm{st}(\pi) = \mathrm{st}(\sigma)$ then π and σ necessarily have the same length.
[13] R. P. Stanley, *Ordered structures and partitions*, Mem. Amer. Math. Soc. 119 (1972).
[14] A peak of π is an i for which $\pi(i-1) < \pi(i) > \pi(i+1)$.
[15] J. Stembridge, *Enriched P-partitions*, Transactions Amer. Math. Soc. 349 (1997), 763–788.

permutation statistics. For example, we can partially order shuffle-compatible statistics by refinement, but we know nothing about this partial order. Yan and I conjectured that every shuffle-compatible statistic is a descent statistic, but this was disproved by Ezgi Kantarcı Oğuz.[16]. Another direction for research is the study of shuffle-compatible permutation statistics for signed or colored permutations. For example, the "flag descent number" is a shuffle-compatible statistic on signed permutations.

Mansour: In your very recent work, *Plethystic formulas for permutation enumeration*, you and Yan Zhuang, among others, introduced some new ideas and obtained exciting results regarding permutations statistics. Would you tell us about the main ideas behind this paper? What would be some possible interesting follow-up research directions?

Gessel: The basic idea behind this paper is that if we can count a set of permutations by their descent sets, then we should be able to count them by any statistics that are determined by the descent set, such as the number of descents or the number of peaks. We also consider two other statistics, the number of left peaks and the number of up-down runs[17]. The quasisymmetric generating function for a set of permutations is the generating function that keeps track of the descent sets, so, for example, to count a set of permutations by peaks we apply a linear transformation on quasisymmetric functions that gives a generating function by peaks. If the quasisymmetric generating function is symmetric then these linear transformations are given by fairly simple operations on symmetric functions. The quasisymmetric generating functions for permutations by cycle type are symmetric and can be described explictly and in the special cases of cyclic permutations, involutions, and derangements, they are given by simple formulas. So we can count permutations of these types by descents, peaks, left peaks, and up-down runs.

Additional classes of permutations with symmetric quasisymmetric generating functions that we did not consider in this paper, but to which our method could be applied, are inverse descent classes, Knuth classes, sets of permutations with fixed inversion number, and certain classes of pattern-avoiding permutations. So, for example, we could apply our method to count permutations by peaks and inverse peaks (i.e., peaks of the inverse permutation).

Mansour: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have a "Eureka moment"?

Gessel: One of my favorite results is the enumeration of unlabeled k-trees. A big challenge in graphical enumeration is finding interesting problems that are tractable but have not already been solved. When I first started thinking about the enumeration of unlabeled graphs many years ago, it eventually dawned on me that counting unlabeled k-trees was an interesting open problem that should be solvable. (Counting labeled k-trees is fairly straightforward.) The standard approach to counting trees or tree-like graphs is to first reduce the problem to counting rooted trees, using either a "dissimilarity characteristic theorem" (due to Richard Otter) or a "dissymmetry theorem" (due to Pierre Leroux). We can then decompose the rooted trees by removing the root to get a functional equation for their generating function. For k-trees, reducing to the enumeration of (several types of) rooted k-trees is fairly straightforward. But the decomposition step is not so easy. To see why, consider the case of 2-trees, which can be thought of as triangles stuck together on edges, rooted at an edge. If we remove the root edge, what remains is almost a set of ordered pairs of edge-rooted 2-trees, but not quite—if we switch the order of the entries in all the ordered pairs we get the same object. So what we really have is an orbit of a set of ordered pairs of edge-rooted 2-trees under the action of the two-element group. My coauthors and I[18] were able to count 2-trees (which had been counted earlier by Harary and

[16]E. K. Oğuz, *A counterexample to the shuffle compatibility conjecture*, arXiv:1807.01398 [math.CO].

[17]A left peak of π is a peak of 0π (π with a prepended 0) and the up-down runs of π are the maximal monotonic consecutive subsequences of 0π.

[18] T. Fowler, I. M. Gessel, G. Labelle, and P. Leroux, *The specification of 2-trees*, Adv. in Appl. Math. 28 (2002), no. 2, 145–168.

[19]F. Harary and E. Palmer, *On acyclic simplicial complexes*, Mathematika 15 (1968), 115–122.

Palmer[19]) in a reasonably simple way, but our approach did not seem to generalize to k-trees. I thought about the problem for a long time and had a vague idea about coloring the vertices to destroy the symmetry of a k-tree in a way that would allow it to be decomposed, but would also allow the symmetry to be put back by group action. My student Andrew Gainer worked on the problem and succeeded, as part of his Ph.D. thesis, in finding a complicated way of counting k-trees using combinatorial species[20]. This inspired me to go back to my idea of coloring the vertices, and I thought about it carefully, and it worked![21]

Mansour: Is there a specific problem you have been working on for many years? What progress have you made?

Gessel: One class of problems that I have thought about a lot with very little success is that of finding combinatorial interpretations of nonnegative integers. Some problems of this type come from representations of finite groups, especially symmetric groups. We know that every complex representation of a finite group can be decomposed uniquely as a sum of irreducible representations and in some cases, especially when the representation is in some sense combinatorial, we can give a combinatorial interpretation to the multiplicities of the irreducible representations. But in most cases, we do not know how to do this. The most famous case of this problem, which may be intractable, is that of finding a combinatorial interpretation for the Kronecker coefficients, in which the representation is the tensor product of two irreducible representations of a symmetric group. This is equivalent to finding the expansion of the Kronecker product of two Schur functions into Schur functions, so it can be stated with no reference to representation theory.

A closely related problem, which can be described more combinatorially is that of finding the coefficients of the expansion of a cycle index of a group action into fundamental quasisymmetric functions.

A more down-to-earth problem that I have been intrigued by, but gotten nowhere with, is finding a combinatorial interpretation for the generalized Catalan numbers with generating function

$$C_m(x) = \frac{1-(1-m^2x)^{1/m}}{mx}.$$

For $m = 2$ these are the Catalan numbers. There's a very simple formula for the coefficient of x^n in $C_m(x)$,

$$(-1)^n m^{2n+1}\binom{1/m}{n+1} = m^n\frac{(m-1)(2m-1)\cdots(nm-1)}{(n+1)!}.$$

This formula shows that these coefficients are positive, but a separate argument is needed to show that they are integers.

Another example of positive integers is the power series coefficients of

$$\frac{1}{1-2(x+y+z)+3(xy+xz+yz)},$$

which were shown to be positive by Szegő in 1933. Reciprocals of power series with alternating coefficients often have combinatorial interpretations, but methods that I know do not seem to work here.

I am also intrigued by integer quotients of factorials, especially when they have a reasonably simple generating function. For example, I would like to find a combinatorial interpretation for

$$\frac{1}{2}\left(\frac{1}{\sqrt{1-4x}}+\frac{1}{\sqrt{1-4y}}\right)\left(\frac{2}{1+\sqrt{(1-4x)(1-4y)}}\right)^{l+1} = \sum_{m,n\geq 0}\frac{(l+2m)!\,(l+2n)!}{l!\,m!\,n!\,(l+m+n)!}x^m y^n.$$

Mansour: Professor Ira Gessel, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

[20] A. Gainer-Dewar, *Γ-species and the enumeration of k-trees*, Electron. J. Combin. 19 (2012), no. 4, Paper 45.

[21] A. Gainer-Dewar and I. M. Gessel, *Counting unlabeled k-trees*, J. Combin. Theory Ser. A 126 (2014), 177–193.