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MATCHINGS IN MATROIDS OVER ABELIAN GROUPS

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Abstract We formulate and prove matroid analogues of results concerning matchings in groups. A matching in an abelian group $(G, +)$ is a bijection $f : A \to B$ between two finite subsets A, B of G satisfying $a + f(a) \notin A$ for all $a \in A$. A group G has the matching property if for every two finite subsets $A, B \subset G$ of the same size with $0 \notin B$, there exists a matching from A to B. In [14] it was proved that an abelian group has the matching property if and only if it is torsion-free or cyclic of prime order. Here we consider a similar question in a matroid setting. We introduce an analogous notion of matching between matroids whose ground sets are subsets of an abelian group G, and we obtain criteria for the existence of such matchings. Our tools are classical theorems in matroid theory, group theory and additive number theory. This is a joint work with Shira Zerbib.

1. Introduction

1.1. **Matchings in groups.** The notion of matching in groups was first introduced in [9] by Fan and Losonczy, who used matchings in $\mathbb{Z}^n$ to study an old problem of Wakeford [19] concerning canonical forms for symmetric tensors.

Let $(G, +)$ be an abelian group with neutral element 0. Let A, B be finite subsets of the same cardinality of G, so that $0 \notin B$. A *matching* from A to B is a bijection $f : A \to B$ such that $a + f(a) \notin A$ for every $a \in A$. Clearly, the conditions $|A| = |B|$ and $0 \notin B$ are necessary for the existence of such a bijection. A matching is called *symmetric* if $A = B$, and otherwise it is *asymmetric*. If there exists a matching from A to B, then A is said to be *matched* or *matchable* to B. The group G is said to satisfy the *matching property* if for every two finite subsets $A, B \subset G$ such that $|A| = |B|$ and $0 \notin B$, A is matched to B.

A characterization of abelian groups satisfying the matching property, as well as a necessary and sufficient condition for the existence of symmetric matchings, were obtained by Losonczy in [14]:

Theorem 1.1. [14] *Let G be an abelian group and let A be a nonempty finite subset of G. Then there is a matching from A to itself if and only if $0 \notin A$.*

Theorem 1.2. [14] *An abelian group G satisfies the matching property if and only if G is torsion-free or cyclic of prime order.*

These results were extended to arbitrary groups in [7] and to linear subspaces of a field extension in [8]. The linear version was further studied in [3]. See also [11] for enumerative aspect of matchings and [5] for the study of a certain subclass of matchings called *acyclic matchings*.

In this talk we introduce an analogous notion of matchings in matroids whose ground sets are subsets of an abelian group. We develop similar criteria for the existence of matching properties for such matroids. Our talk is based on a recent paper [4].

1.2. **Matching in matroids.** A *matroid* M is a pair $(E, \mathcal{I})$ where $E = E(M)$ is a finite *ground set* and $\mathcal{I}$ is a family of subsets of E, called *independent sets*, satisfying the following conditions:

- $\emptyset \in \mathcal{I}$.
- If $X \in \mathcal{I}$ and $Y \subset X$ then $Y \in \mathcal{I}$.
- The *augmentation property*: If $X, Y \in \mathcal{I}$ and $|X| > |Y|$ then there exists $x \in X$ so that $Y \cup \{x\} \in \mathcal{I}$.

Let $M = (E, \mathcal{I})$ be a matroid. The *rank* of a subset $X \subset E$ is given by

$$r_M(X) = r(X) = max\{|X \cap I| : I \in \mathcal{I}\}.$$

The rank of a matroid M, denoted by $r(M)$, is defined to be $r_M(E(M))$. A set $X \subset E$ is called *dependent* if it is not independent. A maximal independent set is called a *basis*. It follows from the augmentation property that every two bases have the same size. Given a matroid $M = (E, \mathcal{I})$, the *dual matroid* $M^* = (E, \mathcal{I}^*)$ is defined so that the bases in M^* are exactly the complements of the bases in M.

A matroid $M = (E, \mathcal{I})$ of rank n is said to be a *paving matroid* if every $(n-1)$-subset of E is independent. If M and M^* are both paving matroids then M is called a *sparse paving matroid*.

Throughout the talk, $(G, +)$ is assumed to be an abelian additive group with neutral element 0, and let $p(G)$ denote the smallest cardinality of a non-zero subgroup of G. We may assume that all matroids are loopless. We say that $M = (E, \mathcal{I})$ is a matroid over G if E is a subset of G. We now introduce the definition of *matchings in matroids over an abelian group*. Let S_n denote the permutation group on n elements.

Definition 1.3. Let $(G, +)$ be an abelian group.

(1) Let M and N be two matroids over G with $r(M) = r(N) = n > 0$. Let $\mathcal{M} = \{a_1, \ldots, a_n\}$ and $\mathcal{N} = \{b_1, \ldots, b_n\}$ be ordered bases of M and N, respectively. We say $\mathcal{M}$ is *matched* to $\mathcal{N}$ if $a_i + b_i \notin E(M)$, for all $1 \leq i \leq n$.
(2) We say that M is *matched* to N if for every basis $\mathcal{M}$ of M there exists a basis $\mathcal{N}$ of N, so that $\mathcal{M}$ is matched to $\mathcal{N}$.
(3) The group G has the *matroid matching property* if for every two matroids M and N over G with $r(M) = r(N) = n > 0$ and $0 \notin E(N)$, M is matched to N.

Remark 1.4. Note that if $\mathcal{M}$ is matched $\mathcal{N}$, then $a_i + b_i \notin E(M)$, and in particular, $a_i + b_i \notin \mathcal{M}$, for all $1 \leq i \leq n$. It follows that the map $a_i \mapsto b_i$ is a matching in the group sense between the subsets $\mathcal{M}$ and $\mathcal{N}$ of G. In this sense, our definition of matchings in matroids is compatible with the definition of matchings in groups.

Furthermore, our definition is compatible with the definition of matchability in vector spaces over field extensions, as defined by Eliahou and Lecouvey in [8]. Let $K \subset F$ be fields,

$A, B \subset F$ n-dimensional K-subspaces of F, and $\mathcal{A} = \{a_1, \dots, a_n\}$, $\mathcal{B} = \{b_1, \dots, b_n\}$ ordered bases of A, B (as linear vector spaces), respectively. Then $\mathcal{A}$ is *matched* to $\mathcal{B}$ if

$$a_i^{-1}A \cap B \subset \langle b_1, \dots, \hat{b}_i, \dots, b_n\rangle, \tag{1}$$

for every $1 \leq i \leq n$, where $\langle b_1, \dots, \hat{b}_i, \dots, b_n\rangle$ is the vector space spanned by $\mathcal{B} \setminus \{b_i\}$. The vector space A is *matched* to the vector space B if every basis of A is matched to a basis of B.

Let G be the multiplicative group $F\setminus\{0\}$, and define the matroids over G $M = (A\setminus\{0\}, \mathcal{I})$, $N = (B \setminus \{0\}, \mathcal{I}')$, where $\mathcal{I}, \mathcal{I}'$ are the collections of linearly independent subsets of A, B, respectively. Then

$$r(M) = \dim_K(A) = n = \dim_K(B) = r(N).$$

Suppose $\mathcal{A}$ is matched to $\mathcal{B}$ as linear vector spaces. Then it follows from (1) that $a_ib_i \notin A$, and therefore $\mathcal{A}$ is matched to $\mathcal{B}$ also in the matroid sense.

The purpose of this talk is to investigate matroid analogs of Theorems 1.1 and 1.2. We study the following questions:

Question 1.5. *Let $M = (E, \mathcal{I})$ be a matroid over G. Is it true that M is matched to itself if and only if $0 \notin E$?*

Question 1.6. *Is it true that G has the matroid matching property if and only if it is torsion-free or cyclic of prime order?*

We will show that the "only if" part is true in both questions.

Proposition 1.7. *We have the following:*

1. *Let M be a matroid over G. If M is matched to itself then $0 \notin E(M)$.*
2. *If G satisfies the matroid matching property then G is either torsion-free or cyclic of prime order.*

We show in [4] that the "if" part is not true in general. However, if one restricts the discussion to some special (large) classes of matroids, then the answer becomes positive in both cases. Our main results give a partial characterization of the cases at which the answer is positive.

2. Main results

A matroid $M = (E, \mathcal{I})$ of rank n is said to be a *paving matroid* if all its circuits are of size at least n, that is, if every $(n-1)$-subset of E is independent. If M and M^* are both paving matroids then M is called a *sparse paving matroid*. We start by showing that the answer to Question 1.5 is positive whenever M is a sparse paving matroid.

Theorem 2.1. *Let M be a sparse paving matroid over G. If $0 \notin E(M)$ then M is matched to itself.*

The rest of our results concern asymmetric matchings. We begin this discussion by investigating the following question:

Question 2.2. *Let G be an abelian torsion-free group or a cyclic group of prime order. Let M be a matroid over G and let N be a sparse paving matroid over G, both of the same rank n, so that $|E(M)| \leq |E(N)|$ and $0 \notin E(N)$. Is M matched to N?*

Remark 2.3. Notice that in Question 2.2, the condition $|E(M)| \leq |E(N)|$ is imposed as otherwise if $|E(M)|$ is arbitrarily larger than $|E(N)|$, then the matchability condition $a_i+b_i \notin E(M)$ may not hold in most situations.

In the next theorem we provide four different conditions for M to be matched to N. We need the following definitions. Let G be an abelian group, $x, a \in G$ and let k be a positive integer. A *progression* of length k with difference x and initial term a is a subset of G of the form $\{a, a+x, a+2x, ..., a+(k-1)x\}$.. We say A is a *semi-progression* if $A \setminus \{a\}$ is a progression, for some $a \in A$.

Theorem 2.4. *Let M be a matroid over G and let N be a sparse paving matroid over G, both of the same rank n, so that $0 \notin E(N)$. Assume further that one of the following conditions holds:*

(1) $|E(M)| < \min\{|E(N)| - 1, p(G)\}$, or
(2) $E(M)$ is not a progression, G is finite, $|E(M)| = |E(N)| - 1$ and $|E(N)| < p(G)$, or
(3) $E(M)$ is neither a progression nor a semi-progression, G is finite and $|E(M)| = |E(N)| < p(G)$, or
(4) $|E(M)| < |E(N)| - n - 1$.

Then M is matched to N

Conditions (1) implies:

Corollary 2.5. *Suppose G is an abelian torsion-free group or a cyclic group of prime order. Let M be a matroid over G and N be a sparse paving matroid over G, both of the same rank n, so that $|E(M)| < |E(N)| - 1$. If $0 \notin E(N)$, then M is matched to N.*

Conditions (2) and (3) imply:

Corollary 2.6. *Let p be a prime number. Let M be a matroid over $\mathbb{Z}/p\mathbb{Z}$ and N be a sparse paving matroid over $\mathbb{Z}/p\mathbb{Z}$, both of the same rank n, so that*

(1) $E(M)$ is not a progression, and $|E(M)| = ||E(N)| - 1$, or
(2) $E(M)$ is neither a progression nor a semi-progression, and $|E(M)| = |E(N)|$.

If $0 \notin E(N)$, then M is matched to N.

3. Our tools

Theorems 1.1 and 1.2 are proved using Hall's marriage theorem [10] together with a result due to Kneser (see [16, Theorem 4.3]) known as Kneser's additive theorem. A matroidal generalization of Hall's theorem was proved by Rado in [17], where a necessary and sufficient condition for the existence of independent transversals in matroids is given. Rado's theorem plays a role in our proofs. Various matroidal analogues of Kneser's additive theorem appear in the literature [18, 6], but none of them are applicable in our setting. We fill this gap to some extent by using the group version of Kneser's theorem as well as results in group theory that enable us to equip a group G with a total order that preserves the group's additive properties. In the cases where such order does not exist we use a rectification principle due to Lev [13]. A corollary due to Eliahou and Lecouvey [7] of Kemperman's additive theorem [12], is also invoked in our proofs.

4. A concluding remark

We hope that the techniques presented here have more general applicability, especially in the direction of generalizing these statements to abstract simplicial complexes. In this case, various substitutes for Rado's theorem on the existence of transversals in simplicial complexes have the potential to be used. Such alternatives may be found in [1, 2, 15].

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