ICECA

International Conference Enumerative Combinatorics and Applications University of Haifa – Virtual – August 26-28, 2024

ASSOCIATIVE-COMMUTATIVE SPECTRA FOR SOME VARIETIES OF GROUPOIDS (EXTENDED ABSTRACT)

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Abstract The associative spectrum of a groupoid (i.e., a set with a binary operation) measures its nonassociativity while the associative-commutative spectrum measures both nonassociativity and noncommutativity of the groupoid. The two spectra are also the coefficients of the Hilbert series of certain operads. We establish upper bounds for the two spectra of various varieties of groupoids defined by different sets of identities and provide examples (often groupoids with three elements) for which the upper bounds are achieved. Our results have connections to many interesting combinatorial objects and integer sequences and naturally lead to some questions for future studies.

1. Introduction

A *groupoid* $(G, *)$ is a basic algebraic structure that consists of a set G together with a binary operation $*$ defined on G. Associativity and commutativity are common properties that could be satisfied by a groupoid. Csákány and Waldhauser [3] defined the *associative spectrum* (also called the *subassociativity type* by Braitt and Silberger [2]) to measure the failure of a groupoid to be associative, and we introduced the *associative-commutative spectrum*, or simply *ac-spectrum*, to measure both nonassociativity and noncommutativity of a groupoid in earlier work [6]; see the definition below.

Definition 1. Fix a countable list of distinct variables $x_1, x_2, \ldots$. Let $\mathcal{B}_n$ denote the set of all *bracketings* of $x_1, \ldots, x_n$, which are terms in the language of groupoids obtained by inserting pairs of parentheses into the word $x_1 x_2 \cdots x_n$ in all valid ways. Let $\mathcal{F}_n$ denote the set of *full linear terms* over $x_1, \ldots, x_n$, which are obtained by permuting the variables in the bracketings of $x_1, \ldots, x_n$. We can view $\mathcal{B}_n$ as a subset of $\mathcal{F}_n$. Every term $t \in \mathcal{F}_n$ induces an n-ary operation t^* on a groupoid $(G, *)$. It is often convenient to think about the terms

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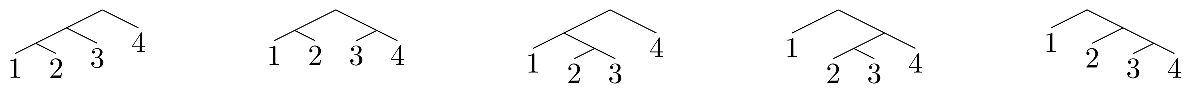
Key words and phrases. Associative-commutative spectrum; associative spectrum; binary operation; tree; 3-element groupoid.

in $\mathcal{F}_n$ or the n-ary operations induced by them in terms of the corresponding (*ordered, full*) *binary trees* with n labeled leaves; see the example below for $\mathcal{B}_4$, which can give $\mathcal{F}_4$ if the variables are permuted in all possible ways.



$((x_1{*}x_2){*}x_3){*}x_4$ $\quad (x_1{*}x_2){*}(x_3{*}x_4)$ $\quad (x_1{*}(x_2{*}x_3)){*}x_4$ $\quad x_1{*}((x_2{*}x_3){*}x_4)$ $\quad x_1{*}(x_2{*}(x_3{*}x_4))$

The *associative spectrum* (resp., *ac-spectrum*) of a groupoid $(G,*)$, or of its binary operation $*$, is a sequence whose nth term is $s_n^{\mathrm{a}}(*) := |P_n(*)|$ (resp., $s_n^{\mathrm{ac}}(*) := |\overline{P}_n(*)|$), where $P_n(*) := \{t^* : t \in \mathcal{B}_n\}$ (resp., $\overline{P}_n(*) := \{t^* : t \in \mathcal{F}_n\}$) for $n = 1,2,\ldots$. It turns out that $\{P_n(*)\}_{n\geq 1}$ (resp., $\{\overline{P}_n(*)\}_{n\geq 1}$) together with a composition function becomes a *nonsymmetric operad* (resp., *symmetric operad*) that satisfies certain coherence axioms [12], and the *Hilbert series* of this operad is the generating function (resp., exponential generating function) of the associative spectrum (resp., ac-spectrum) of $(G,*)$.

By the above definition, we have (i) $s_n^{\mathrm{a}}(*) = 1$ for $n = 1,2$, (ii) $\mathrm{s}_1^{\mathrm{ac}}(*) = 1$, and (iii) $\mathrm{s}_2^{\mathrm{ac}}(*)$ is either 1 or 2, depending on whether $*$ is commutative. Thus we may assume $n \geq 3$ when necessary. It is easy to see that isomorphic or anti-isomorphic groupoids have the same associative spectrum and the same ac-spectrum, where two groupoids $(G,*)$ and $(H,\otimes)$ are said to be *anti-isomorphic*, denoted by $G \simeq H^{\mathrm{op}}$, if there is a bijection $f : G \to H$ such that $f(a*b) = f(b) \otimes f(a)$ for all $a,b \in G$.

It is clear that $s_n^{\mathrm{a}}(*) = 1$ for all $n \in \mathbb{N}$ if and only if $*$ associative and that $s_n^{\mathrm{ac}}(*) = 1$ for all $n \in \mathbb{N}$ if and only if $*$ is associative and commutative, where $\mathbb{N} := \{1,2,\ldots\}$. On the other hand, we have $s_n^{\mathrm{a}}(*) \leq C_{n-1}$, where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the ubiquitous *Catalan number*, and thus $s_n^{\mathrm{ac}}(*) \leq n!C_{n-1}$. We showed in previous work [6] that a commutative groupoid $(G,*)$ must have $s_n^{\mathrm{ac}}(*) \leq D_{n-1}$, where $D_n := (2n!)/(2^n n!)$ is the solution to Schröder's third problem [14, A001147], and that an associative groupoid $(G,*)$ must have $s_n^{\mathrm{ac}}(*) \leq n!$, which holds as an equality if the groupoid is noncommutative and has an identity element.

In addition, the precise values of the associative spectrum and ac-spectrum have been determined for various groupoids [3, 4, 5, 6, 9, 10], including 2-element groupoids, generalizations of addition and subtraction, exponentiation, arithmetic/geometric/harmonic mean, cross product, Lie algebras with an $\mathfrak{sl}_2$-triple, graph algebras, and so on. The results show connections with interesting combinatorial objects, avoided patterns, and integer sequences. However, the ac-spectra of 3-element groupoids are largely undetermined.

According to the Siena Catalog [1], there are 3330 non-isomorphic 3-element groupoids, which are indexed from 1 to 3330. Each of these groupoids is determined by a binary operation $*$ defined on the set $\{0,1,2\}$. We write them as $\mathsf{SC1}, \mathsf{SC2}, \ldots, \mathsf{SC3330}$. There are 729 *idempotent* 3-element groupoids, which can be labeled in a different way: $\mathsf{ID0}, \mathsf{ID1}, \ldots, \mathsf{ID728}$. Csákány and Waldhauser [3] showed the following (see Table 1).

- Both $\mathsf{ID35} = \mathsf{SC271}(\simeq \mathsf{SC1610}^{\mathrm{op}})$ and $\mathsf{ID68} = \mathsf{SC356}(\simeq \mathsf{SC2032}^{\mathrm{op}})$ have associative spectrum $s_n^{\mathrm{a}}(*) = 2^{n-2}$ for $n \geq 2$.
- Both $\mathsf{SC1066}$ and $\mathsf{SC10}(\simeq \mathsf{SC367}^{\mathrm{op}})$ have associative spectrum $s_n^{\mathrm{a}}(*) = n-1$ for $n \geq 1$.
- Both $\mathsf{SC405}$ and $\mathsf{SC3242}(\simeq \mathsf{SC3302}^{\mathrm{op}})$ have associative spectrum $s_n^{\mathrm{a}}(*) = 3$ for $n > 3$ (it is easy to check that $s_n^{\mathrm{a}}(*) = 1$ for $n = 1,2$ and $s_n^{\mathrm{a}}(*) = 2$ for $n = 3$).

- The groupoid SC79 has associative spectrum $s_n^{\mathrm{a}}(*) = F_{n+1} - 1$ for $n \geq 2$, where F_{n+1} is the *Fibonacci number* defined by $F_{n+1} := F_n + F_{n-1}$ for $n \geq 1$ and $F_i = i$ for $i = 0, 1$,

Our original motivation for this work was to determine the ac-spectra of the above 3-element groupoids, whose Cayley tables are given in Table 1.

$*$	0	1	2
0	0	0	0
1	1	1	0
2	2	2	2

SC271 $\simeq$ SC1610$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	0
1	2	1	1
2	1	2	2

SC356 $\simeq$ SC2032$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	1

SC1066 $=$ SC1066$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	0
1	0	0	0
2	1	0	0

SC10 $\simeq$ SC367$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	1
1	0	0	1
2	1	1	0

SC405 $\simeq$ SC405$^{\mathrm{op}}$

$*$	0	1	2
0	1	1	1
1	2	2	2
2	0	0	0

SC3242 $\simeq$ SC3302$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	1

SC79 $\simeq$ SC79$^{\mathrm{op}}$

$*$	0	1	2
0	0	1	1
1	0	1	2
2	0	1	2

SC1610

$*$	0	1	2
0	0	1	2
1	0	1	2
2	1	0	2

SC2032

$*$	0	1	2
0	0	0	1
1	0	0	0
2	0	0	0

SC367

$*$	0	1	2
0	1	2	0
1	1	2	0
2	1	2	0

SC3302

TABLE 1. Some 3-element groupoids

However, we are able to establish more general results on various varieties of groupoids, where a *variety* of groupoids axiomatized by a set Σ of identities is the family of all groupoids satisfying the identities in Σ. For each variety of groupoids considered in this paper, we establish an upper bound for the associative spectra and an upper bound for the ac-spectra of the groupoids belonging to this variety; if the latter upper bound is reached by a member of the variety, so is the former. Moreover, we show that both upper bounds are attained by at least one 3-element groupoid.

For example, we showed in earlier work [6] that a commutative groupoid must have $s_n^{\mathrm{ac}}(*) \leq D_{n-1}$ and if the equality in this upper bound holds, so does the equality in the upper bound $s_n^{\mathrm{a}}(*) \leq C_{n-1}$. In the same paper, we showed that $s_n^{\mathrm{ac}}(*) = D_{n-1}$ for a 3-element groupoid called the *rock-paper-scissors groupoid*, which turns out to be isomorphic to SC1108, and the proof is also valid for SC2407 and SC3093. Therefore, we have the following result.

$*$	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

SC1108

$*$	0	1	2
0	1	0	0
1	0	2	0
2	0	0	0

SC2407

$*$	0	1	2
0	1	1	0
1	1	2	0
2	0	0	1

SC3093

Theorem 2 ([6]). *A groupoid $(G, *)$ satisfying the identity $xy \approx yx$ must have $s_n^{\mathrm{ac}}(*) \leq C_{n-1}$ and $s_n^{\mathrm{ac}}(*) \leq D_{n-1}$ for $n = 1, 2, \ldots$, where the first inequality holds as an equality whenever the second does and both equalities hold for the 3-element groupoids* SC1108, SC2407, *and* SC3093.

In this paper, we provide a series of results that are similar to the above one. A summary of our results is given by Table 2, where we use the well-known *Bell number* B_n counting partitions of the set $\{1, 2, \ldots, n\}$ into unordered nonempty blocks, the *restricted Bell number* $B_{n,m}$ counting partitions of $\{1, 2, \ldots, n\}$ into unordered nonempty blocks of size at most m [13], and the *ordered Bell number* or *Fubini number* B'_n counting partitions of $\{1, 2, \ldots, n\}$

Identities satisfied by $(G,*)$	$n \geq$	$s_n^{\mathrm{a}}(*) \leq$	$s_n^{\mathrm{ac}}(*) \leq$	Examples for $=$
(1)	$1,1$	1	n	SC275($\simeq$ SC2029$^{\mathrm{op}}$)
(3), (4), (5), (7)	$3,3$	2	$n+1$	SC7($\simeq$SC4$^{\mathrm{op}}$) SC28($\simeq$SC5$^{\mathrm{op}}$)
(2), (7), (15)	$4,4$	3	$n+1$	SC405
(3), (5), (7), (8), (9)	$3,3$	2	$2n$	SC189($\simeq$ SC170$^{\mathrm{op}}$)
(5), (7), (10), (11), (12), (16)	$4,4$	3	$3n$	SC3242($\simeq$ SC3302$^{\mathrm{op}}$)
(5), (7), (11), (13), (17), (18)	$4,4$	4	$2n^2$	SC3162($\simeq$ SC2467$^{\mathrm{op}}$)
(2), (7)	$2,2$	$n-1$	$2^{n-1}-1$	SC1066
(4), (5), (7)	$2,1$	$n-1$	$n! + \sum_{k=0}^{n-3} k! \binom{n}{k}$	SC367($\simeq$ SC10$^{\mathrm{op}}$)
(3), (6), (14)	$2,2$	2^{n-2}	2^n-2	SC2302($\simeq$ SC2155$^{\mathrm{op}}$)
(3), (7), (12)	$2,2$	2^{n-2}	$n(2^{n-1}-1)$	SC271($\simeq$SC1610$^{\mathrm{op}}$) SC356($\simeq$SC2032$^{\mathrm{op}}$)
(2), (11)	$2,2$	$F_{n+1}-1$	$B_{n,2}-1$	SC79, SC1701
(3), (5)	$2,1$	2^{n-2}	nB_{n-1}	SC41($\simeq$SC398$^{\mathrm{op}}$) SC96($\simeq$SC1069$^{\mathrm{op}}$)
(5), (7)	$2,1$	2^{n-2}	nB'_{n-1}	SC262($\simeq$SC1441$^{\mathrm{op}}$) SC1812($\simeq$SC1793$^{\mathrm{op}}$) SC2446($\simeq$SC2430$^{\mathrm{op}}$)

(1) $xy \approx x$ (2) $xy \approx yx$ (3) $(xy)z \approx (xz)y$ (4) $x(yz) \approx y(xz)$ (5) $x(yz) \approx x(zy)$ (6) $x(yz) \approx z(yx)$

(7) $w(x(yz)) \approx w((xy)z)$ (8) $(wx)(yz) \approx (w(xy))z$ (9) $w(x(yz)) \approx ((wx)y)z$

(10) $((wx)y)z \approx ((wy)x)z$ (11) $((wx)y)z \approx ((wx)z)y$ (12) $(wx)(yz) \approx (wy)(xz)$

(13) $(w(xy))z \approx (w(xz))y$ (14) $w(x(yz)) \approx (w(xy))z$ (15) $(v(wx))(yz) \approx (vw)(x(yz))$

(16) $((vw)x)y)z \approx v(w(x(yz)))$ (17) $v(w(x(yz))) \approx ((v(wx))y)z$ (18) $((vw)(x(yz)) \approx (((vw)x)y)z$

Table 2. Summary of results

into ordered nonempty blocks [14, A000670]. The "$n \geq$" column in Table 2 gives the smallest values of n for which the upper bounds of $s_n^{\mathrm{a}}(*)$ and $s_n^{\mathrm{ac}}(*)$ are valid and sharp. Note that different varieties of groupoids in the table may have the same associative spectrum upper bound but different ac-spectrum upper bounds. Therefore, the ac-spectrum often offers a finer distinction than the associative spectrum between groupoids satisfying different sets of identities.

It is sometimes convenient to use not only identities but other conditions to describe a family of groupoids satisfying certain upper bounds for their spectra. Recall that every term $t \in \mathcal{F}_n$ corresponds to a binary tree with n leaves labeled by $1, \ldots, n$. Each leaf i has its *depth* $d_i(t)$ (resp. *left depth* $\delta_i(t)$ or *right depth* $\rho_i(t)$) defined as the number of edges (resp., left/right edges) in the unique path to the root of t. By abuse of notation, we also speak of these three kinds of depths for the variables in t. Previous work [4, 6] used the congruence modulo m relation on depths to study the associative spectra and ac-spectra of certain groupoids, and some of the results there can be rephrased to include our result on the variety of groupoids satisfying the identities (3), (6), (14) as a special case. We can also similarly generalize our results on the two varieties of groupoids satisfying identities (3), (5), (7), (8), (9) and (5), (7), (10), (11), (12), (16), respectively to the following theorem.

Theorem 3. *Let $(G,*)$ be a groupoid such that for all $s,t \in F_n$, we have $s^* = t^*$ whenever s and t have the same leftmost variable x_i, whose left depths in s and t are congruent modulo k. Then $s_n^{\mathrm{a}}(*) \le k$ and $s_n^{\mathrm{ac}}(*) \le kn$ for $n = k+1, \ldots$, where the first inequality holds as an equality if the second does. Moreover, both upper bounds are reached if "whenever" can be replaced with "if and only if" in the above condition.*

The first author, Mickey, and Xu [8] used the depth to find the associative spectrum of the *double minus* operation $a*b := -a-b$, and we determined the ac-spectrum of this operation in previous work [6]. Both proofs are valid for any field with at least three elements, giving the following result.

Theorem 4 ([8])**.** *Suppose that two terms $s,t \in \mathcal{F}_n$ induce the same n-ary operation on a groupoid $(G,*)$ whenever $d_i(s) \equiv d_i(t) \pmod 2$ for $i = 1,\ldots,n$. Then $s_n^{\mathrm{a}}(*) \le \lfloor 2^n/3 \rfloor$ and $s_n^{\mathrm{ac}}(*) \le (2^n - (-1)^n)/3$ for $n = 1,2,\ldots$, where the first equality holds as an equality if the second one does. Moreover, both upper bounds are reached if "whenever" can be replaced with "if and only if" in the above condition. In particular, both upper bounds are achieved by the double minus operation on any field with at least three elements.*

The two upper bounds in the above theorem are both well studied [14, A000975, A001045] from many other perspectives; the latter is known as the *Jacobsthal sequence*. The double minus operation on a field of three elements is actually the 3-element groupoid $\mathsf{SC}2346$.

$*$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

$\mathsf{SC}2346$

To generalize the above theorem, one could use a primitive root of unity $\omega := e^{2\pi i/k}$ to define an operation $a*b := \omega a + \omega b$ on the field of complex numbers, which reduces to the double minus operation when $k = 2$; for $k \ge 3$, the n-th term of the associative spectrum was shown in [11] to coincide with the number of equivalence classes of the equivalence relation on n-leaf binary trees that relates two trees if the depths of corresponding leaves are congruent modulo k. Closed formulas for the associative spectrum and the ac-spectrum of this operation are yet to be determined.

We use Sage (`https://www.sagemath.org/`) to help discover and verify the results in this paper. Computations in Sage also give the initial terms of the ac-spectra of several other varieties of groupoids, which coincide with some interesting sequences in OEIS [14]. The reader is referred to the full-length paper of this work [7] for more details.

References

[1] J. Berman and S. Burris, A computer study of 3-element groupoids, in: Logic and Algebra (Pontignano, 1994), Lecture Notes in Pure and Appl. Math., 180, Dekker, 1996. (pp. 379–429)

[2] M. S. Braitt and D. Silberger, Subassociative groupoids, Quasigroups Related Systems **14** (2006), no. 1, 11–26.

[3] B. Csákány and T. Waldhauser, Associative spectra of binary operations, Mult.-Valued Log. **5** (2000), no. 3, 175–200.

[4] N. Hein and J. Huang, Modular Catalan numbers, European J. Comb. 61 (2017), 197–218.

[5] N. Hein and J. Huang, Variations of the Catalan numbers from some nonassociative binary operations, Discrete Math. **345** (2022), no. 3, Paper No. 112711, 18 pp.

[6] J. Huang and E. Lehtonen, The associative-commutative spectrum of a binary operation, Discrete Math. **346** (2023), no. 10, Paper No. 113535, 22 pp.

[7] J. Huang and E. Lehtonen, Associative-commutative spectra for some varieties of groupoids, arXiv:2401.15786.

[8] J. Huang, M. Mickey and J. Xu, The nonassociativity of the double minus operation, J. Integer Seq. **20** (2017), no. 10, Art. 17.10.3, 11 pp.

[9] E. Lehtonen and T. Waldhauser, Associative spectra of graph algebras I. Foundations, undirected graphs, antiassociative graphs, J. Algebraic Combin. **53** (2021), no. 3, 613–638.

[10] E. Lehtonen and T. Waldhauser, Associative spectra of graph algebras II. Satisfaction of bracketing identities, spectrum dichotomy, J. Algebraic Combin. **55** (2022), no. 2, 533–557.

[11] E. Lehtonen and T. Waldhauser, Associativity conditions for linear quasigroups and equivalence relations on binary trees, arXiv: 2310.09184.

[12] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften, 346, Springer, Heidelberg, 2012.

[13] I. Mező, Periodicity of the last digits of some combinatorial sequences, J. Integer Seq. **17** (2014), no. 1, Article 14.1.1, 18 pp.

[14] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.