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Fragmenting any parallelepiped into a signed tiling

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Abstract: It is broadly known that any parallelepiped tiles space by translating copies of itself along its edges. In earlier work relating to higher-dimensional sandpile groups, the second author discovered a novel construction which fragments the parallelpiped into a collection of smaller tiles. These tiles fill space with the same symmetry as the larger parallelepiped. Their volumes are equal to the components of the multi-row Laplace determinant expansion, so this construction only works when all these signs are non-negative (or non-positive).

In this work, we extend the construction to work for all parallelepipeds, without requiring the non-negative condition. This naturally gives tiles with negative volume, which we understand to mean canceling out tiles with positive volume. In fact, with this cancellation, we prove that every point in space is contained in exactly one more tile with positive volume than tile with negative volume. This is a natural definition for a signed tiling.

Our main technique is to show that the net number of signed tiles doesn't change as a point moves through space. This is a relatively indirect proof method, and the underlying structure of these tilings remains mysterious.

This extended abstract is made of three sections. In Section 1, we state our main theorem (Theorem 1.7) after providing necessary definitions. In Section 2, we give three examples of tilings obtained from our construction. Finally, in Section 3, we give a brief sketch of the outline of our proof of Theorem 1.7. For more details, see our full paper on ArXiv [DM23].

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1. Signed Tiling Construction

Fix positive integers r and k as well as an $(r+k) \times (r+k)$ matrix M with real entries. Additionally, fix a generic direction vector $\mathbf{w} \in \mathbb{R}^{r+k}$.

Definition 1.1. Let N be an $(r+k) \times (r+k)$ matrix with real entries. Define $\Pi(N)$ to be the set of $\mathbf{p} \in \mathbb{R}^{r+k}$ such that for all sufficiently small $\varepsilon > 0$, the point $\mathbf{p} + \varepsilon\mathbf{w}$ is in

$$\sum_{i \in [m]} \{x_i N_i : 0 \leq x_i \leq 1\}.$$

The set $\Pi(N)$ is called the *(half-open) parallelepiped of* N.

Although definition 1.1 depends on $\mathbf{w}$, we omit it for conciseness.

We present a simple observation about translating parallelepipeds, which will be the foundation of our construction.

Lemma 1.2. *For any choice of M, we have*

$$\mathbb{R}^{r+k} = \bigsqcup_{\mathbf{z} \in \mathbb{Z}^{r+k}} (\Pi(M) + M\mathbf{z}).$$

This lemma follows from the fact that the unit cube tiles space, and the displacement between cubes in this tiling is all $\mathbb{Z}$-valued vectors. The lemma describes this same tiling, after applying M as a linear transformation. Our main construction is of a more complicated tiling under the same translation lattice, which is formed by *fragmenting* M.

Definition 1.3. Let $\sigma \in \binom{[r+k]}{r}$, i.e., $\sigma \subset [r+k]$ with $|\sigma| = r$. The *σ-fragment matrix* of M, written $S_\sigma(M)$, is the matrix obtained from M by the following 3 step process:

(1) For each $i \notin \sigma$, replace the *first* r entries of column i with 0.
(2) For each $i \in \sigma$, replace the *last* k entries of column i with 0.
(3) Negate all of the entries in the last k rows.

Example 1.4. Let $r = k = 2$. Any $(r+k) \times (r+k)$ matrix M has 6 associated fragment matrices corresponding to the subsets of $\binom{[4]}{2}$. For example, if

$$M = \begin{bmatrix} 3 & 2 & -4 & 1 \\ 1 & 0 & 2 & 2 \\ 2 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 \end{bmatrix} \text{ and } \sigma = \{1,4\}, \text{ then } S_\sigma(M) = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}.$$

To form a signed tiling, we parameterize tiles formed by translating the fundamental parallelepiped of fragment matrices by integer combinations of the columns of M.

Definition 1.5. For any $\mathbf{z} \in \mathbb{Z}^{r+k}$ and $\sigma \in \binom{[r+k]}{r}$, the *tile* parameterized by the pair $(\mathbf{z}, \sigma)$ is defined as

$$\mathcal{T}(\mathbf{z}, \sigma) := \Pi(S_\sigma(M)) + M\mathbf{z}.$$

Using this parameterization, we can collect tiles into useful groups.

Definition 1.6. Consider the sets of tiles

$$\mathbf{T}^+(M) := \bigsqcup_{\mathbf{z}\in\mathbb{Z}^{r+k}} \left(\bigsqcup_{\sigma\in\binom{[r+k]}{r},\ \det(S_\sigma(M))>0} \mathcal{T}(\mathbf{z},\sigma) \right),$$

$$\text{and } \mathbf{T}^-(M) := \bigsqcup_{\mathbf{z}\in\mathbb{Z}^{r+k}} \left(\bigsqcup_{\sigma\in\binom{[r+k]}{r},\ \det(S_\sigma(M))<0} \mathcal{T}(\mathbf{z},\sigma) \right).$$

The set $\mathbf{T}^+(M)$ is the set of *positive tiles*, and $\mathbf{T}^-(M)$ is the set of *negative tiles*. We also write $\mathbf{T}(M) := \mathbf{T}^+(M) \sqcup \mathbf{T}^-(M)$. Note that we don't include the tiles where $\det(S_\sigma(M)) = 0$, but in this case, $S_\sigma(M)$ is not invertible, and $\Pi(S_\sigma(M))$ is empty.

Definition 1.6 allows us to cleanly state our main result. Note that we write $\mathbb{1}_T$ for the *indicator function* of a tile T.

Theorem 1.7. *The function* $f(\mathbf{p}) : \mathbb{R}^{r+k} \to \mathbb{Z}$*, defined by*

$$f(\mathbf{p}) := \left(\sum_{T\in\mathbf{T}^+(M)} \mathbb{1}_T(\mathbf{p}) \right) - \left(\sum_{T\in\mathbf{T}^-(M)} \mathbb{1}_T(\mathbf{p}) \right),$$

is constant with value $(-1)^k \operatorname{sgn}(\det(M))$.

When one of $\mathbf{T}^+(M)$ or $\mathbf{T}^-(M)$ is empty, Theorem 1.7 specializes to a result about more traditional tilings. We state only the version where $\mathbf{T}^-(M)$ is empty, but the same statement holds if "non-negative" is replaced with "non-positive".

Corollary 1.8. [McD21b, Corollary 9.2.8] *If the sign of* $\det(S_\sigma(M))$ *is non-negative for each* $\sigma \in \binom{[r+k]}{r}$*, then*

$$\mathbb{R}^{r+k} = \bigsqcup_{\mathbf{z}\in\mathbb{Z}} \left(\bigsqcup_{\sigma\in\binom{[r+k]}{r}} \mathcal{T}(\mathbf{z},\sigma) \right).$$

Remark 1.9. The conditions required on M for Corollary 1.8 to apply are discussed in [McD21b, Section 6.7]. The original proof of the corollary relies on these properties, so we needed different methods to prove the more general Theorem 1.7. A special case of Corollary 1.8 was used in [McD21a] to define a family of *multijections* between the *sandpile group* and *cellular spanning forests* for a large class of cell complexes. This generalizes a construction of Backman Baker and Yuen which used *zonotopal tilings* to answer questions about *chip-firing* on *regular matroids*[BBY19].

2. Example Tilings

Example 2.1. Suppose that

$$M = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}. \qquad \text{Then,} \qquad S_{\{1\}}(M) = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \text{and} \qquad S_{\{2\}}(M) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

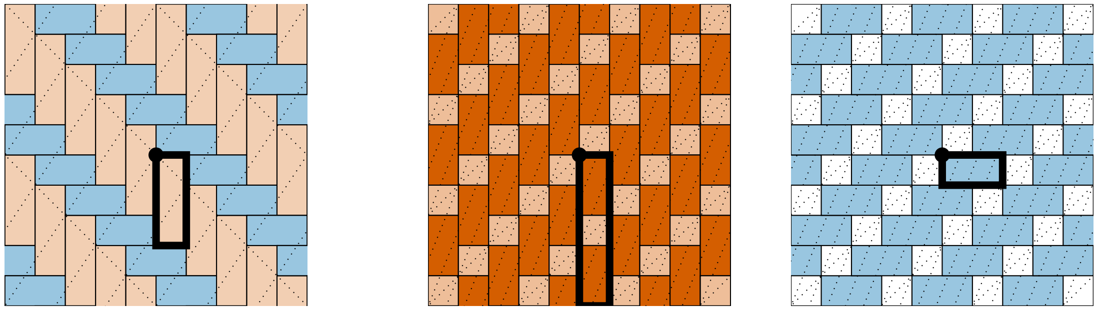


FIGURE 1. On the left is the tiling from Example 2.1. Translates of $S_{\{1\}}(M)$ are given in orange and translates of $S_{\{2\}}(M)$ are given in blue. When the partial tilings are combined, we get a full periodic tiling of $\mathbb{R}^2$. On the center/right is the signed tiling from Example 2.2. The darker regions indicate where two parallelepipeds overlap, while the lighter region is the portion covered by a single paralellepiped. By Theorem 1.7, the overlap region in the center image corresponds precisely to the shaded region on the rightmost image.

In this example, both fragment matrices have negative determinant. Thus, by Corollary 1.8, we get the traditional tiling presented on the left in Figure 1.

Example 2.2. Suppose that

$$M = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}. \qquad \text{Then,} \quad S_{\{1\}}(M) = \begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix} \quad \text{and} \quad S_{\{2\}}(M) = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}.$$

In this example, $\det(S_{\{1\}}(M)) < 0$ while $\det(S_{\{2\}}(M)) > 0$. This means that Corollary 1.8 no longer applies and the tiles in $\mathbf{T}(M)$ contain some overlap. Nevertheless, the overlap of the negative tiles is precisely the region that is "cancelled out" by the positive tiles. This tiling is shown in the center/right of Figure 1).

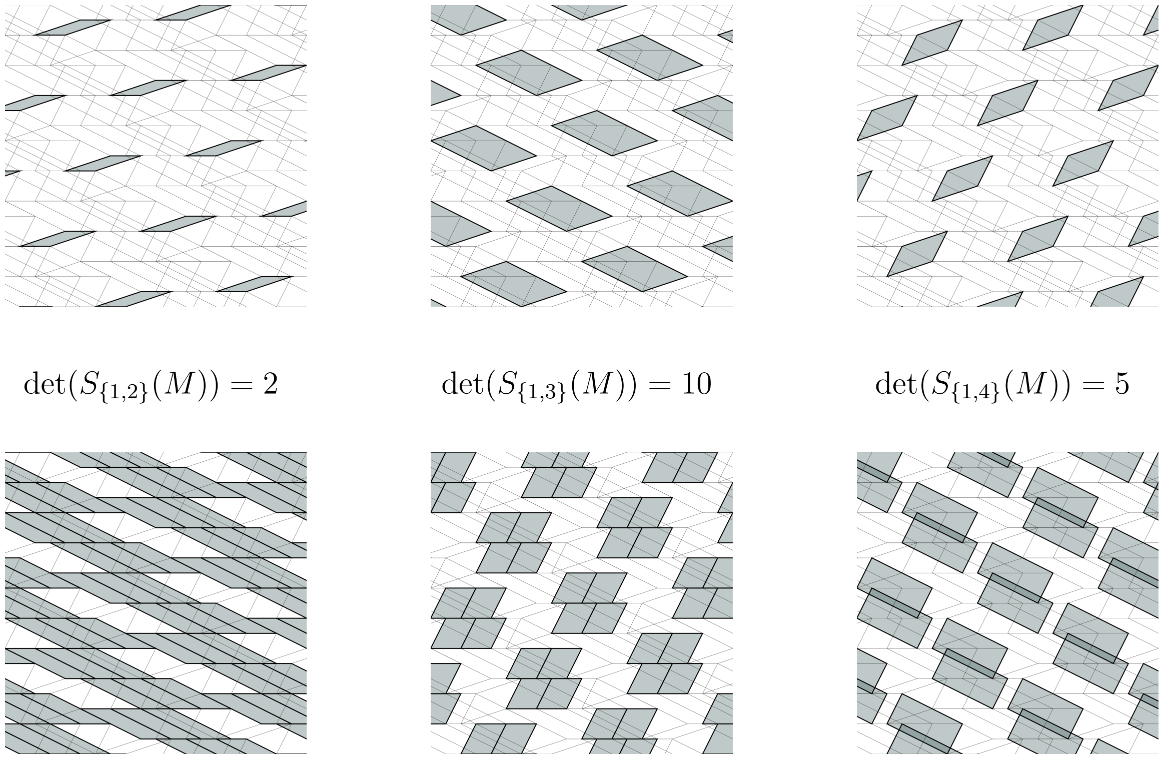
Example 2.3. For the matrix M from Example 1.4, the set $\mathbf{T}(M)$ consists of 6 families of 4-dimensional parallelepipeds, where each family contains infinitely many translations of a single fragment.

By taking the determinant of each fragment, we find that

$$\mathbf{T}^+(M) = \bigsqcup_{\mathbf{z}\in\mathbb{Z}^{r+k}} \left(\bigsqcup_{\sigma\in\{(1,2),(1,3),(1,4),(2,3),(2,4)\}} \mathcal{T}(\mathbf{z},\sigma) \right), \text{ and}$$
$$\mathbf{T}^-(M) = \bigsqcup_{\mathbf{z}\in\mathbb{Z}^{r+k}} \mathcal{T}(\mathbf{z},\{3,4\}).$$

Confirming that Theorem 1.7 holds for this example is not a completely straightforward task, even with the help of a computer. Nevertheless, regardless of the choice of $\mathbf{w}$, one can show that each $\mathbf{p} \in \mathbb{R}^4$ is contained in

- one tile in $\mathbf{T}^+(M)$ and no tiles in $\mathbf{T}^-(M)$,
- two tiles in $\mathbf{T}^+(M)$ and one tile in $\mathbf{T}^-(M)$, or
- three tiles in $\mathbf{T}^+(M)$ and two tiles in $\mathbf{T}^-(M)$.



$\det(S_{\{1,2\}}(M)) = 2$ $\det(S_{\{1,3\}}(M)) = 10$ $\det(S_{\{1,4\}}(M)) = 5$

$\det(S_{\{2,3\}}(M)) = 24$ $\det(S_{\{2,4\}}(M)) = 16$ $\det(S_{\{3,4\}}(M)) = -20$

FIGURE 2. Here we show the contributions of each of the six classes of tiles in Example 2.3 to a 2-dimensional slice of the tiling.

In each case, the value of $f(\mathbf{p})$ is 1, which is also the sign of $\det(M)$.

It is possible to visualize this tiling by taking a 2-dimensional slice which fixes the last 2 coordinates in $\mathbb{R}^4$. Each of the six families of tiles are given in Figure 2.

Recall that the first 5 families are made up of positive tiles, while the last is made up of negative tiles. Figure 3 gives an enlarged view of the collection of all positive tiles. By Theorem 1.7, this is the same picture obtained by adding the negative tiles to the set of all points in $\mathbb{R}^2$.

3. An Outline of the Proof

Our proof of Theorem 1.7 is structured in the following way.

(1) First, we show that the average value of f is $(-1)^k \operatorname{sgn}(\det(M))$.
(2) Next, we group the facets of the tiles into collections that lie in the same hyperplane.
(3) After this, we imagine a particle crossing a point contained in one of these collections of facets. We show that when doing so, it crosses exactly two facets. Furthermore, in one crossing it *enters* a *positive* tile or *exits* a *negative* tile, while in the other crossing, it *exits* a *positive* tile or *enters* a *negative* tile.

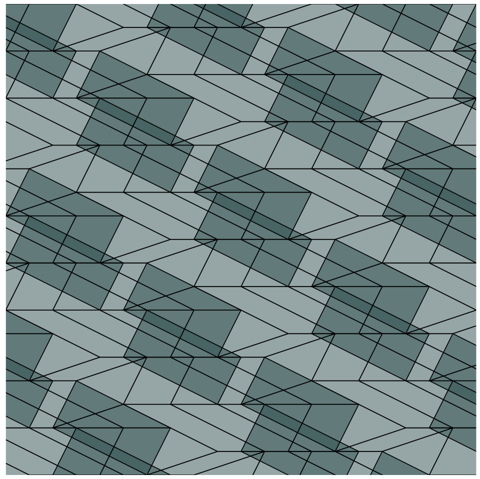


FIGURE 3. This figure is formed by overlapping the first 5 images in Figure 2. We showed in Example 2.3 that these are precisely the tiles in $\mathbf{T}^+(M)$. If we "subtract" the final image in Figure 2 from this picture, each point in $\mathbb{R}^2$ would be covered once.

(4) From these observations, we conclude that f is constant. Theorem 1.7 then follows from our first observation.

To find the average value of f, we use the multiple row version of Laplace's determinant expansion formula as well as some basic calculus techniques. One important observation is the following chain of equalities, which holds for any $\sigma \in \binom{[r+k]}{r}$.

$$\sum_{\mathbf{z}\in\mathbb{Z}^{r+k}} \int_{\Pi(M)} \mathbb{1}_{\mathcal{T}(\mathbf{z},\sigma)}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{r+k}} \mathbb{1}_{\mathcal{T}(\mathbf{0},\sigma)}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^{r+k}} \mathbb{1}_{S_\sigma(M)}(\mathbf{x})\mathrm{d}\mathbf{x} = |\det(S_\sigma(M))|.$$

The longest and most technical part of our proof is the facet grouping result. This argument required careful bookkeeping and several applications of Cramer's rule. After this hurdle, the final steps of the proof were relatively straightforward.

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