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COUNTING CONJUGACY CLASSES OF ELEMENTS OF FINITE ORDER IN LIE GROUPS

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Abstract In this talk, I will discuss two numbers that are associated with Lie groups. The first number is $N(G, m)$, the number of conjugacy classes of elements in G whose order divides m. The second number is $N(G, m, s)$, the number of conjugacy classes of elements in G whose order divides m and which have s distinct eigenvalues, where we view G as a matrix group in its smallest-degree faithful representation.

Much of my work on this topic was originally motivated by an enumerative problem that arose in string theory. I will describe some of this motivation, as well as the history of studying these numbers. I will then discuss recent results, concentrating mainly on exceptional Lie groups.

This talk is based mostly on joint work with Qidong He which has recently been accepted at *Combinatorial Theory* [7].[1]

1. Background and motivation

Let G be a complex, simply-connected Lie group, viewed as a matrix group via its standard representation (i.e., its smallest-degree, faithful representation), and let m and s be positive integers. We study the number of conjugacy classes of elements in G whose order divides m, as well as the number of such classes whose elements have s distinct eigenvalues. That is, we define

$$E(G, m) = \{x \in G \mid x^m = 1\},$$
$$E(G, m, s) = \{x \in E(G, m) \mid x \text{ has } s \text{ distinct eigenvalues}\},$$

and study

$$N(G, m) = \text{number of conjugacy classes of } G \text{ in } E(G, m),$$
$$N(G, m, s) = \text{number of conjugacy classes of } G \text{ in } E(G, m, s).$$

The study of the number of conjugacy classes of elements of finite order in Lie groups has an interesting history that combines mathematical and physical approaches and applications.

The story begins in the 1980's with a pair of papers by Djokovic [3, 4], where nice formulas for $N(G, m)$ were obtained for any connected semisimple complex Lie group G that is simply

[1]This short summary paper heavily relies on [7], [8], [9].

connected or adjoint, using a generating function approach. In [12, 2], the case of certain prime power orders was computed; and in [10], $N(\mathrm{SU}(n), m)$ was obtained. This topic was revived in [8, 9], where $N(G, m)$ was obtained for unitary, orthogonal, and symplectic Lie groups using simple combinatorial methods that apply to groups that are not necessarily connected, simply connected, or adjoint.

In [8, 9], $E(G, m, s)$ and $N(G, m, s)$ were introduced for the first time,[2] motivated by an explicitly enumerative problem in string theory: counting the number of certain vacua in the quantum moduli space of M-theory compactifications on manifolds of G_2 holonomy. In that context, the numbers $N(SU(p), q)$ and $N(SU(p), q, s)$, where q and p are relatively prime, were computed in [6]. These numbers are related to symmetry breaking patterns in grand unified theories, with the number $N(SU(p), q, s)$ being particularly significant as s is related to the number of massless fields in the gauge theory that remains after the symmetry breaking. In [8, 9], formulas for $N(G, m, s)$ were obtained for the unitary, orthogonal, and symplectic Lie groups.

Most recently in [7], methods are developed to obtain the numbers $N(G, m)$ and $N(G, m, s)$ for G an exceptional Lie group. Results in [7] for $N(G, m)$ coincide with those in [4], but are obtained in a different way. Results in [7] for $N(G, m, s)$ are new. They are obtained in closed form for the smallest exceptional groups G_2 and F_4 (for the latter, the table required to present the results is very large). For E_6, E_7, E_8, the results can in principle be obtained algorithmically but require significant computational power.

In this talk, we will briefly review the earlier results mentioned above, and then concentrate on the recent results announced in [7].

2. Counting conjugacy classes

We begin by reducing the computation of $N(G, m)$ and $N(G, m, s)$ to the computation of $N(T, m)$ and $N(T, m, s)$, where T is a maximal torus of G: we have

$$N(G, m) = N(T, m),$$
$$N(G, m, s) = N(T, m, s). \tag{2.1}$$

These equalities can be proven using the torus theorem when G is compact and connected, or using the Jordan decomposition $x = su$ for $x \in G$, s semisimple, and u unipotent when G is complex and semi-simple [8, 9, 7].

2.1. Classical Lie groups.

With the simplification to the torus, the computation of $N(G, m)$ and $N(G, m, s)$ for unitary, orthogonal, and symplectic groups can be obtained in a relatively straightforward manner by careful study of their respective tori. We illustrate this with an example taken from [8]. Let $G = SO(2n + 1)$. Its torus is

$$T_{SO(2n+1)} = \{\mathrm{diag}(A(\theta_1), A(\theta_2), \ldots, A(\theta_n), 1)\}, \tag{2.2}$$

where

$$A(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

[2] In [8, 9], the number s denotes the number of distinct *pairs* of eigenvalues.

and note that

$$BA(\theta)B^{-1} = A(-\theta).$$

Since $\mathrm{diag}(B, I_{2n-2}, -1) \in SO(2n+1)$, and since we can always reorder the θ_l via conjugation by a permutation within $SO(2n+1)$, it follows that two elements x and x' of $T_{SO(2n+1)}$ that differ by $\theta_l' = -\theta_l$ for any $l = 1, \dots, n$ belong to the same conjugacy class. We therefore consider only elements of $T_{SO(2n+1)}$ with $\theta_l \in [0, 1/2]$. For elements of order dividing m, we have $\theta_l \in \frac{1}{m}(0, 1, \dots, [\frac{m}{2}])$, and we order the θ_l's in nondecreasing order. It follows that $N(SO(2n+1), m)$ is the number of weak $([\frac{m}{2}]+1)$-compositions of n:

$$N(SO(2n+1), m) = \binom{n + \left[\frac{m}{2}\right]}{\left[\frac{m}{2}\right]}.$$

We now turn to $N(SO(2n+1), m, s)$, where s denotes the number of distinct conjugate pairs of eigenvalues of the elements. There are n θ_l's and $\binom{n-1}{s-1} = \frac{s}{n}\binom{n}{s}$ ways to partition them into s nonzero parts. There are $\left[\frac{m}{2}\right] + 1$ possible values for the θ_l, yielding

$$N(SO(2n+1), m, s) = \frac{s}{n}\binom{n}{s}\binom{\left[\frac{m}{2}\right]+1}{s}.$$

Similar, though in many cases more subtle, computations lead to results for other orthogonal groups, as well as for symplectic groups and unitary groups. Interesting subtleties arise from tori that are not connected, and from the interplay between those and the parity of m. All the results are rather pleasing linear combinations of one or two binomial coefficients, with the exception of $N(SU(n), m)$ and $N(SU(n), m, s)$, which are given by a sum over divisors of $\gcd(n, m)$. For more detail, see [8].

2.2. **Exceptional Lie groups.** For the exceptional Lie groups, the tori and the conjugation of torus elements are not as simple to work with as for the classical Lie groups, so we need additional insights in order to compute $N(G, m)$ and $N(G, m, s)$ for these.

Let W be the Weyl group of G. Let $T(G, m)$ be the subgroup of the torus of G consisting of elements of order dividing m, and let $T_s(G, m)$ be the subset of $T(G, m)$ of elements that have s distinct eigenvalues when G is considered to be in its smallest faithful representation.

Recall that two elements $t, s \in T$ are conjugate in G iff there exists $w \in W$ satisfying $s = w \cdot t$, where W acts on T by conjugation. It follows that $N(T, m)$ (resp. $N(T, m, s)$) is the number of orbits of elements in $T(G, m)$ (resp. $T_s(G, m)$) under the action of W. By Burnside's Lemma,

$$N(T, m) = \frac{1}{|W|}\sum_{w \in W} |\mathrm{Fix}(w)|,$$

and

$$N(T, m, s) = \frac{1}{|W|}\sum_{w \in W} |\mathrm{Fix}_s(w)|,$$

where $\mathrm{Fix}(w) = \{t \in T(G, m) : w \cdot t = t\}$ and $\mathrm{Fix}_s(w) = \{t \in T_s(G, m) : w \cdot t = t\}$.

Now, if $u, v \in W$ are conjugate, there is a canonical bijection between the fixed points of $T(G, m)$ under u and under v, so $|\mathrm{Fix}(u)| = |\mathrm{Fix}(v)|$; similarly, $|\mathrm{Fix}_s(u)| = |\mathrm{Fix}_s(v)|$. Therefore, we can simplify the above sum over the Weyl group to get

$$N(G, m) = N(T, m) = \frac{1}{|W|}\sum_{c \in \mathrm{Cl}(W)} |c|\, |\mathrm{Fix}(w_c)|, \tag{2.3}$$

and

$$N(G,m,s) = N(T,m,s) = \frac{1}{|W|} \sum_{c \in \mathrm{Cl}(W)} |c| \, |\mathrm{Fix}_s(w_c)| \,, \tag{2.4}$$

where $\mathrm{Cl}(W)$ is the collection of conjugacy classes in W.

The utility of our simplified formulas stems from the fact that while the Weyl group of an exceptional Lie group may be forbiddingly large, the number of conjugacy classes inside the Weyl group is generally small. Hence, provided that we know the size and a representative of each conjugacy class in the Weyl group, the sums above will be much easier to evaluate than the ones given by Burnside's Lemma. Fortunately, this information has been completely determined in [1] and translated into an accessible form in GAP 3 [5, 11].

Due to space constraints, we give only a brief description of the methods we developed to compute $|\mathrm{Fix}(w_c)|$ and $|\mathrm{Fix}_s(w_c)|$ for the exceptional groups. We will elaborate further in the talk. For more detail, see [7].

2.2.1. $N(G,m)$. For each exceptional Lie group, we use the set of matrix generators provided by [13] both for a concrete representation of the torus elements themselves and for a concrete representation of the generators of W that act on the torus. This allows us to compute the conditions for $t \in T$ to be fixed under some $w \in W$. These conditions take the form of homogeneous linear equations with integer coefficients represented by a matrix $A \in M_{n\times\ell}(\mathbb{Z})$, where $n = 3, 12, 27, 28, 120$ for $G = G_2, F_4, E_6, E_7, E_8$, respectively, and ℓ is the rank of the Lie group. At most ℓ of the n equations are linearly independent.

The number of torus elements that satisfy the conditions, i.e. $|\mathrm{Fix}(w)|$, is the number of elements in the kernel of A over $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$. This number can be computed using the Smith Normal Form (SNF) of A. We carry out the computation of $|\mathrm{Fix}(w_c)|$ for a representative w_c of each conjugacy class of W. We obtain complete results for $N(G,m)$ that are presented in the form of five tables for $G = G_2, F_4, E_6, E_7, E_8$, each organized by congruence classes of m mod $6, 12, 6, 12$, or 60, respectively. The numbers 6, 12, 6, 12, and 60 are the least common multiples of the elementary divisors that come up in the relevant SNF's.

2.2.2. $N(G,m,s)$. The main questions that need to be addressed here are: when does a torus element $t \in T$ have s distinct eigenvalues? How many of those are fixed under the action of some $s \in W$?

We develop methods to detect when an eigenvalue is repeated by introducing a matrix for which the kernels of its submatrices are associated with repeated eigenvalues. We define a partial order on the set of submatrices. The structure of the resulting partially ordered set, along with the principle of inclusion and exclusion, ultimately leads to an algorithm for computing $|\mathrm{Fix}_s(w_c)|$. We use this algorithm to obtain complete results for G_2 and F_4. The results for G_2 are presented in a table organized by congruence classes of $m \mod 12$. For F_4, the results have to be given in terms of $m \mod 12252240$. For the remaining three exceptional groups, we lack the computational power required. Finding an efficient way to compute those is an open problem.

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