ICECA

International Conference
Enumerative Combinatorics and Applications
University of Haifa – Virtual – August 26-28, 2024

SHUFFLE THEOREMS AND SANDPILES

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Abstract We provide an explicit description of the recurrent configurations of the sandpile model on a family of graphs $\widehat{G}_{\mu,\nu}$, which we call *clique-independent* graphs, indexed by two compositions μ and ν. Moreover, we define a *delay* statistic on these configurations, and we show that, together with the usual *level* statistic, it can be used to provide a new combinatorial interpretation of the celebrated *shuffle theorem* of Carlsson and Mellit. More precisely, we will see how to interpret the polynomials $\langle \nabla e_n, e_\mu h_\nu \rangle$ in terms of these configurations.

1. Introduction

1.1. **Shuffle theorem.** The *shuffle theorem* of Carlsson and Mellit [4] is a recent breakthrough that provided a positive solution to a long-standing conjecture about a combinatorial formula for the Frobenius characteristic of the so-called diagonal harmonics. More precisely, this theorem provides the monomial expansion of the symmetric function ∇e_n, where e_n is the elementary symmetric function of degree n in the variables $x_1, x_2, \dots$, and ∇ is the famous *nabla* operator introduced by Bergeron and Garsia in the 90's. In this formula, to each *labelled Dyck path* of size n corresponds a monomial, where the variables $x_1, x_2, \dots$ keep track of the labels, while the variables q and t keep track of the bistatistic $(\mathsf{dinv}, \mathsf{area})$.

In [6] Loehr and Remmel provided an alternative combinatorial interpretation of the same symmetric function in terms of the same objects, but using the bistatistic $(\mathsf{area}, \mathsf{pmaj})$. In particular, they showed bijectively that the two combinatorial formulas coincide. In our work we provide a new combinatorial interpretation of this symmetric function by proving a bijection to the Loehr–Remmel model.

1.2. **Sandpile model.** The *(abelian) sandpile model* is a combinatorial dynamical system on graphs first introduced by Bak, Tang and Wiesenfeld [2] in the context of "self-organized criticality" in statistical mechanics. The sandpile model (and variants of it) have found applications in a wide variety of mathematical contexts including enumerative combinatorics, tropical geometry, and Brill–Noether theory, among others: see [5] for a nice introductory monograph. *For now we only consider the sandpile model with a sink.*

A well-known link between the combinatorics of this dynamical system and that of the underlying graph is given by the so-called *recurrent configurations* (see Definition 3.3). For example, the recurrent configurations of the sandpile model are in bijection with the spanning trees of the graph (see e.g. [3]). If the underlying graph presents some symmetries, then it is natural to look at the recurrent configurations "modulo" those symmetries. For example, for the complete graph we can identify recurrent configurations that are the same up to a permutation of the vertices (not moving the sink); perhaps unsurprisingly, we still get interesting combinatorics, as in this case we find Catalan many such "sorted" configurations.

More formally, consider the sandpile model on a graph G, and let $\mathsf{Aut}(G)$ be the automorphism group of G. Consider a subgroup Γ of the stabilizer of the sink. Now Γ acts naturally on the set $\mathsf{Rec}(G)$ of recurrent configurations: we are interested in the orbits of this action, that we will call *sorted recurrent configurations*.

1.3. **Main result.** We will consider an explicit family of graphs $\widehat{G}_{\mu,\nu}$ indexed by pairs of compositions μ and ν. For such a graph $\widehat{G}_{\mu,\nu}$ we will look at a subgroup Γ of its automorphism group that will be isomorphic to the Young subgroup $\mathfrak{S}_\mu \times \mathfrak{S}_\nu$ of the symmetric group $\mathfrak{S}_n$, where $n = |\mu| + |\nu|$. We denote by $\mathsf{SortRec}(\mu, \nu)$ the set of the corresponding sorted recurrent configurations of $\widehat{G}_{\mu,\nu}$.

For every recurrent configuration κ of $\widehat{G}_{\mu,\nu}$, we will define a new statistic, called the *delay* of κ (denoted $\mathsf{delay}(\kappa)$), which we will couple with the usual *level* statistic (denoted $\mathsf{level}(\kappa)$). To state our main result, we need a few more definitions.

Given a composition $\mu = (\mu_1, \mu_2, \dots)$, we denote by e_μ the product $e_{\mu_1} e_{\mu_2} \cdots$, and similarly $h_\mu = h_{\mu_1} h_{\mu_2} \cdots$, where h_n is the complete homogeneous symmetric function of degree n. Finally, we denote by $\langle -, - \rangle$ the Hall scalar product on symmetric functions.

Theorem 1.1. *For every pair of compositions μ, ν such that $n = |\mu| + |\nu|$ we have*

$$\langle \nabla e_n, e_\mu h_\nu \rangle = \sum_{\kappa \in \mathsf{SortRec}(\mu,\nu)} q^{\mathsf{level}(\kappa)} t^{\mathsf{delay}(\kappa)}.$$

Notice that for $\mu = \varnothing$, the coefficient $\langle \nabla e_n, h_\nu \rangle$ is simply the coefficient of $x^\nu = x_1^{\nu_1} x_2^{\nu_2} \cdots$ in ∇e_n, hence this formula gives in particular a new combinatorial interpretation of the monomial expansion of the symmetric function ∇e_n in terms of the sandpile model.

In the rest of this extended abstract we introduce all of the definitions necessary for Theorem 1.1.

2. The clique-independent graphs $\widehat{G}_{\mu,\nu}$

Definition 2.1. Let μ, ν be two compositions (i.e. tuples of positive integers). Set $n = |\mu| + |\nu|$. We define a graph $G_{\mu,\nu}$ with set of vertices $[n] := \{1, 2, \dots, n\}$ consisting of the following *components*:

- $\ell(\mu)$ *clique components*, i.e. complete graphs, $K_{\mu_1}, K_{\mu_2}, \dots$, on $\mu_1, \mu_k, \dots$ vertices respectively. The vertices of K_{μ_1} are $n, n-1, \dots, n-\mu_1+1$; the vertices of K_{μ_2} are $n-\mu_1, n-\mu_1-1, \dots, n-\mu_1-\mu_2+1$; and so on.

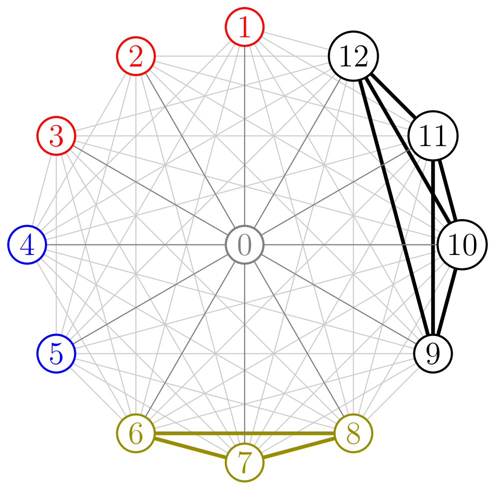

FIGURE 1. The graph $\widehat{G}_{(4,3),(3,2)}$.

- $\ell(\nu)$ *independent components*, i.e. graphs without edges, $I_{\nu_1}, I_{\nu_2}, \ldots$, on $\nu_1, \nu_2, \ldots$ vertices respectively; the vertices of I_{ν_1} are $1, 2, \ldots, \nu_1$; the vertices of I_{ν_2} are $\nu_1 + 1, \nu_1 + 2, \ldots, \nu_1 + \nu_2$; and so on.

Finally, two vertices in distinct components are always connected by an edge.

Example 2.2. If $\mu = \emptyset$, then $G_{\emptyset,\nu}$ is the complete multipartite graph $K_{\nu_1,\nu_2,\ldots}$. If $\nu = \emptyset$, then $G_{\mu,\emptyset}$ is isomorphic to the complete graph $K_{|\mu|}$; however, for our purposes we will distinguish between $G_{(|\mu|),\emptyset}$ and $G_{(\mu_1,\mu_2,\ldots),\emptyset}$, as we will consider the action of different groups of automorphisms, which will lead to different sorted configurations.

Given one of our labelled graphs $G_{\mu,\nu}$, we define the graph $\widehat{G}_{\mu,\nu}$ simply as $G_{\mu,\nu}$ to which we add a vertex 0, and we connect it with every other vertex. We will consider the sandpile on $\widehat{G}_{\mu,\nu}$, where 0 is the sink. Figure 1 is an illustration of the graph $\widehat{G}_{(4,3),(3,2)}$.

3. Basics of the sandpile model

Definition 3.1. Let G be a finite, undirected, simple graph on the vertex set $\{0, 1, \ldots, n\}$.

A *configuration* of the *sandpile (model) on* G is a map $\kappa : [n] \cup \{0\} \to \mathbb{Z}$ that assigns a (integer) number of "grains of sand" to each nonzero vertex of G.

If $0 \leq \kappa(v) \leq \deg(v)$, we say that v is *stable*, and otherwise it is *unstable*. Any vertex can *topple* (or *fire*), and "donate a single grain" to each of its neighbors: the result is a new configuration κ' in which $\kappa'(v) = \kappa(v) - \deg(v)$ and for any $w \neq v$

$$\kappa'(w) = \begin{cases} \kappa(w) + 1, & \text{if } (v,w) \text{ is an edge} \\ \kappa(w), & \text{otherwise.} \end{cases}$$

For any $v \in \{0, \ldots, n\}$ we write ϕ_v for the *toppling operator* at vertex v. That is $\phi_v(\kappa)$ is a new configuration obtained from κ by toppling the vertex v.

The vertex 0 is special in this model, and we call it the *sink*, while we call all the others *nonsink* vertices. We say that a configuration κ is *non-negative* if all of its nonsink vertices are non-negative, *stable* if all of its nonsink vertices are stable, and *unstable* if at least one of its nonsink vertices is unstable.

Remark 3.2. Notice that the notion of stable configuration has no dependency on the value on the sink. Therefore, as it is customary, we will ignore the value of a configuration on the sink, and consider the configurations as restricted on the nonsink vertices. Moreover, we will identify every configuration κ with the word $\kappa(n)\kappa(n-1)\cdots\kappa(2)\kappa(1)$.

Definition 3.3. Let κ be a stable configuration, and consider the configuration $\phi_0(\kappa)$. We say that κ is *recurrent*[1] if there is an order of all the nonsink vertices such that toppling the vertices in that order we always stay non-negative. Of course at the end of this sequence of topplings we will be back to κ. More precisely, a configuration κ is recurrent if there is a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ such that

$$\phi_0(\kappa), (\phi_{\sigma(1)} \circ \phi_0)(\kappa), (\phi_{\sigma(2)} \circ \phi_{\sigma(1)} \circ \phi_0)(\kappa), \dots, (\phi_{\sigma(n)} \circ \cdots \circ \phi_{\sigma(1)} \circ \phi_0)(\kappa) = \kappa$$

are all non-negative configurations. In this case, σ is the *toppling word* of this sequence of topplings, and we say that this sequence *verifies the recurrence* of κ.

Remark 3.4. It is well known (see e.g. [1, Theorem 2.4]) that the condition for κ to be recurrent is equivalent to say that starting from $\phi_0(\kappa)$ there is no proper (possibly empty) subset A of $[n]$ such that toppling all the vertices of A brings $\phi_0(\kappa)$ to a stable configuration.

Definition 3.5. Given a recurrent configuration κ of G, we define its *level* as

$$\mathsf{level}(\kappa) := -|E_s(G)| + \sum_{i=1}^{n} \kappa(i)$$

where $E_s(G)$ is the set of edges of G that are not incident to the sink.

It is well-known that $\mathsf{level}(\kappa) \geq 0$, and there exists a recurrent configuration of level 0 if G is connected [7].

Remark 3.6. For $\widehat{G}_{\mu,\nu}$ with $|\mu| + |\nu| = n$ we have

$$|E_s(\widehat{G}_{\mu,\nu})| = \binom{n}{2} - \sum_{i \geq 0} \binom{\nu_i}{2}.$$

Example 3.7. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ for $\widehat{G}_{(4,3),(3,2)}$ has level

$$\mathsf{level}(\kappa) = -\binom{12}{2} + \binom{3}{2} + \binom{2}{2} + 97 = 35.$$

Definition 3.8. A *sorted configuration*[2] of the sandpile on $\widehat{G}_{\mu,\nu}$ is a configuration κ that is weakly decreasing inside each clique component of $\widehat{G}_{\mu,\nu}$ and weakly increasing inside each independent component of $\widehat{G}_{\mu,\nu}$: if $i, j \in K_{\mu_r}$ and $i < j$, then $\kappa(i) \leq \kappa(j)$; if $i, j \in I_{\nu_s}$ and $i < j$, then $\kappa(i) \geq \kappa(j)$.

Example 3.9. The configuration $\kappa = 3\overline{10}\,\overline{11}\,\overline{11}8\overline{10}\,\overline{11}\,\overline{10}4973$ is a sorted recurrent configuration for $\widehat{G}_{(4,3),(3,2)}$ (recall that in our notation $\kappa = \kappa(n)\kappa(n-1)\cdots\kappa(1)$).

[1] In the literature "recurrent" is sometimes used in a broader sense than in this paper. Configurations that are recurrent in our sense are called *critical* in these settings.

[2] The relation with the general definition of *sorted configuration* given in Section 1.2 is simply that we are picking a specific convenient element in each orbit.

4. Toppling algorithm and delay

Consider the sandpile on a graph G with vertices $\{0\} \cup [n]$, where 0 is the sink. Let κ be a recurrent configuration of G. Consider Algorithm 1.

Algorithm 1 Toppling algorithm

Input: A graph G and a recurrent configuration κ
Output: The word of nonsink vertices in the order they have been toppled

```
Topple the sink, i.e. compute φ_0(κ)
Initialize the output word as empty
while there are nonsink vertices that are untoppled do
    for i going from n to 1 (in decreasing order) do
        if vertex i is unstable then
            Topple vertex i
            Append i to the output word
        end if
    end for
end while
```

Observe that by construction the algorithm terminates: since κ is recurrent, $\phi_0(\kappa)$ is non-negative and at least one of the vertices adjacent to the sink is unstable; then every time we topple we stay non-negative, and since κ is recurrent the process must go through all the nonsink vertices (otherwise we found a subset A of nonsink vertices such that after we topple its vertices we are in a stable configuration, cf. Remark 3.4).

By construction the algorithm outputs a toppling sequence that verifies the recurrence of κ. We can now define our new statistic on recurrent configurations.

Definition 4.1. Let κ be a recurrent configuration of G. For every $i \in [n]$, let $r_i(\kappa)$ be the number of **for** loop iterations in Algorithm 1 that occurred before the one in which the vertex i is toppled (so if i is toppled in the first iteration, then $r_i(\kappa) = 0$). Then we define the *delay* of κ as

$$\mathsf{delay}(\kappa) := \sum_{i=1}^{n} r_i(\kappa).$$

Remark 4.2. If σ is the output of Algorithm 1 applied to κ, then clearly

$$\mathsf{delay}(\kappa) = \mathsf{maj}(\sigma_n \sigma_{n-1} \cdots \sigma_1).$$

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