ICECA

International Conference Enumerative Combinatorics and Applications University of Haifa – Virtual – August 26-28, 2024

PARITY STATISTICS ON RESTRICTED PERMUTATIONS AND THE CATALAN–SCHETT POLYNOMIALS

ZHICONG LIN, JING LIU, AND SHERRY H.F. YAN

ABSTRACT. Motivated by Kitaev and Zhang's recent work on non-overlapping ascents in stack-sortable permutations and Dumont's permutation interpretation of the Jacobi elliptic functions, we investigate some parity statistics on restricted permutations. Some new related bijections are constructed and two refinements of the generating function for descents over 321-avoiding permutations due to Barnabei, Bonetti and Silimbanian are obtained. In particular, an open problem of Kitaev and Zhang about non-overlapping ascents on 321-avoiding permutations is solved and several combinatorial interpretations for the Catalan–Schett polynomials are found. The stack-sortable permutations are at the heart of our approaches.

1. Introduction

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] := \{1, 2, \dots, n\}$. A permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$ is said to *avoid* pattern $\sigma \in \mathfrak{S}_k$ if there does not exist $i_1 < i_2 < \cdots < i_k$ such that the subsequence $\pi_{i_1}\pi_{i_2}\dots\pi_{i_k}$ of π is order isomorphic to σ. Let

$$\mathfrak{S}_n(\sigma) := \{\pi \in \mathfrak{S}_n : \pi \text{ avoids } \sigma\}.$$

An element in $\mathfrak{S}_n(\sigma)$ is called a *σ-avoiding permutation*. One of the classical enumerative results in pattern avoiding permutations, attributed to MacMahon and Knuth (see [7]), is that $|\mathfrak{S}_n(\sigma)| = C_n$ for each pattern $\sigma \in \mathfrak{S}_3$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th *Catalan number*. Because of Knuth's work [5], the class of 231-avoiding permutations was also known as *stack-sortable permutations*.

The MacMahon–Knuth result has been refined several times in the literature. Robertson, Saracino and Zeilberger [13] refined this result by proving that the distribution of "number of fixed points" is the same in 321-avoiding as in 132-avoiding permutations. Elizalde and Pak [4] further refined Robertson et al.'s result by taking into account the "number of excedances" in a permutation. Barnabei, Bonetti and Silimbanian [1] showed that

$$(t^2x - t^2 + t)A^2 + (2t^2x^2 - 2t^x + 2xt - x)A + t^2x^3 - t^2x^2 + tx^2 = 0, \tag{1.1}$$

Date: June 6, 2024.

Key words and phrases. Pattern avoidance, Non-overlapping ascents, Odd descent compositions, Left peaks, Bijections.

where A is the enumerator of 321-avoiding permutations by the length (variable t) and the number of descents (variable x). The main objective of this paper is to present new refinements, from the aspects of both bijective combinatorics and the generating functions, of the MacMahon–Knuth result using two classes of parity statistics on permutations.

We begin by introducing the first class of parity statistics. Given $\pi \in \mathfrak{S}_n$, an index $i \in [n-1]$ is called an *ascent* of π if $\pi_i < \pi_{i+1}$ and a *descent* of π if $\pi_i > \pi_{i+1}$. Let $\mathrm{ASC}(\pi)$ (resp., $\mathrm{DES}(\pi)$) be the set of ascents (resp., descents) of π. To a subset $S \subseteq [n-1]$ with elements $s_1 < s_2 < \cdots < s_k$, we associate the multiset

$$\mathsf{M}(S) = \begin{cases} \{s_1, s_2 - s_1, \ldots, s_k - s_{k-1}, n - s_k\}, & \text{if } S \neq \varnothing; \\ \{n\}, & \text{if } S = \varnothing. \end{cases}$$

Then, $\mathsf{DR}(\pi) := \mathsf{M}(\mathrm{ASC}(\pi))$ is the *descending run multiset* of π and $\mathsf{AR}(\pi) := \mathsf{M}(\mathrm{DES}(\pi))$ is the *ascending run multiset* of π. Note that $\mathsf{DR}(\pi)$ records the lengths of the decreasing runs of π, while $\mathsf{AR}(\pi)$ records the lengths of the increasing runs of π. For instance, if $\pi = 318972456 \in \mathfrak{S}_9$, then we have $\mathsf{DR}(\pi) = \{3, 2, 1^4\}$ as $\mathrm{ASC}(\pi) = \{2, 3, 6, 7, 8\}$, while $\mathsf{AR}(\pi) = \{4, 3, 1^2\}$ as $\mathrm{DES}(\pi) = \{1, 4, 5\}$. Introduce four parity statistics of π as

- $\mathsf{odr}(\pi)$, the number of odd elements in $\mathsf{DR}(\pi)$;
- $\mathsf{edr}(\pi)$, the number of even elements in $\mathsf{DR}(\pi)$;
- $\mathsf{oar}(\pi)$, the number of odd elements in $\mathsf{AR}(\pi)$;
- $\mathsf{ear}(\pi)$, the number of even elements in $\mathsf{AR}(\pi)$.

Continuing with the running example, we have $\mathsf{odr}(\pi) = 5$, $\mathsf{edr}(\pi) = 1$, $\mathsf{oar}(\pi) = 3$ and $\mathsf{ear}(\pi) = 1$.

Another class of parity statistics that we consider are the even/odd left peaks of permutations. Recall that a *left peak* of a permutation $\pi \in \mathfrak{S}_n$ is a value π_i, $i \in [n-1]$, such that $\pi_{i-1} < \pi_i > \pi_{i+1}$ with the convention $\pi_0 = 0$. Let $\mathrm{LPK}(\pi)$ be the set of left peaks of π. Introduce the two parity statistics of π as

- $\mathsf{lpk}_o(\pi)$, the number of odd elements in $\mathrm{LPK}(\pi)$;
- $\mathsf{lpk}_e(\pi)$, the number of even elements in $\mathrm{LPK}(\pi)$.

For example, if $\pi = 3271654$, then $\mathrm{LPK}(\pi) = \{3, 6, 7\}$, so $\mathsf{lpk}_o(\pi) = 2$ and $\mathsf{lpk}_e(\pi) = 1$.

Our motivation to consider these parity statistics on restricted permutations comes from the recent work by Kitaev and Zhang [8]. The notion of the *maximum number of non-overlapping occurrences of a consecutive pattern* in a permutation was first considered by Kitaev [6]. Kitaev and Zhang [8] focused on the *maximum number of non-overlapping descents* (resp., *ascents*), denoted $\mathsf{mnd}(\pi)$ (resp., $\mathsf{mna}(\pi)$), of a permutation π. In other words, we have

$$\mathsf{mnd}(\pi) = \sum_{i \in \mathsf{DR}(\pi)} \left\lfloor \frac{i}{2} \right\rfloor \quad \text{and} \quad \mathsf{mna}(\pi) = \sum_{i \in \mathsf{AR}(\pi)} \left\lfloor \frac{i}{2} \right\rfloor.$$

Notice that

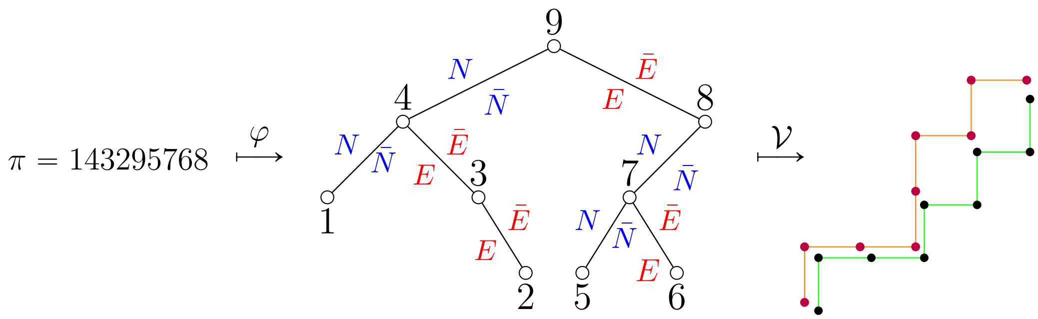
$$\mathsf{mnd}(\pi) = \frac{n - \mathsf{odr}(\pi)}{2} \quad \text{and} \quad \mathsf{mna}(\pi) = \frac{n - \mathsf{oar}(\pi)}{2}. \tag{1.2}$$

The main results in [8] are outlined as follows.

Theorem 1.1 (Kitaev and Zhang [8]). *The following two results hold:*

(i) *the pair* $(\mathsf{mna}, \mathsf{mnd})$ *is symmetric over* $\mathfrak{S}_n(231)$*;*

(ii) $|\{\pi \in \mathfrak{S}_n(231) : \mathsf{mnd}(\pi) = k\}| = \frac{1}{n+1}\binom{n+1}{2k+1}\binom{n+k}{k}$.



$$\mu = \bar{N}\bar{E}\bar{E}\bar{N}\bar{N}\bar{E}\bar{N}\bar{E}$$
$$\nu = \bar{N}EE\bar{N}E\bar{N}E\bar{N}$$

FIGURE 1. An example of the bijection $\Theta = \mathcal{V} \circ \varphi$.

Let $\mathsf{des}(\pi) = |\mathrm{DES}(\pi)|$ be the number of descents of π. As $\mathsf{mnd}(\pi) = \mathsf{des}(\pi)$ for any $\pi \in \mathfrak{S}_n(321)$, the distribution of "$\mathsf{mnd}$" over 321-avoiding permutations has been computed by Barnabei et al. [1]. Kitaev and Zhang [8] posed the following open problem, resolving which would complete the enumeration of "mnd" and "mna" over permutations avoiding a pattern of length 3.

Problem 1.2 (Kitaev and Zhang [8]). *Find the distribution of "mna" over 321-avoiding permutations.*

The refinement of Catalan numbers

$$\frac{1}{n+1}\binom{n+1}{2k+1}\binom{n+k}{k} \tag{1.3}$$

appears in Theorem 1.1 also has other known combinatorial interpretations. A *binary tree* is a special type of rooted tree in which every internal node has either one left child or one right child or both. Let $\mathcal{B}_n$ be the set of all binary trees with n nodes. Given a binary tree, its *left chain* (resp., *right chain*) is any maximal path composed of only left (resp., right) edges. The *order* of a left/right chain is the number of its nodes. Let $\mathsf{LC}(T)$ (resp., $\mathsf{RC}(T)$) be the multiset of all orders of the left (resp., right) chains of a tree $T \in \mathcal{B}_n$. For instance, the binary tree in Fig. 1 has 5 right chains, which are $4-3-2$, $9-8$, $7-6$, 1 and 5, and so $\mathsf{RC}(T) = \{3, 2^2, 1^2\}$. Define two statistics of T as

$$\mathcal{X}(T) = \sum_{i \in \mathsf{LC}(\pi)} \lfloor \frac{i}{2} \rfloor \quad \text{and} \quad \mathcal{Y}(T) = \sum_{i \in \mathsf{RC}(\pi)} \lfloor \frac{i}{2} \rfloor.$$

For the running example tree T, we have $\mathcal{X}(T) = 2$ and $\mathcal{Y}(T) = 3$. Sun [16] proved combinatorially that the number (1.3) enumerates binary trees $T \in \mathcal{B}_n$ with $\mathcal{X}(T) = k$. Kitaev and Zhang [8] constructed a bijection between $\{T \in \mathcal{B}_n : \mathcal{X}(T) = k\}$ and $\{\pi \in \mathfrak{S}_n(231) : \mathsf{mnd}(\pi) = k\}$, providing the first proof of Theorem 1.1 (ii).

The original impetus of this work lies in an unexpected connection between the joint distribution of $(\mathcal{X}, \mathcal{Y})$ over binary trees and the multiset Schett polynomials introduced by Ma and the first author [11]. The *Jacobi elliptic function* (see [11]) $\mathrm{sn}(u, \alpha)$ may be defined by the inverse of an elliptic integral:

$$\mathrm{sn}(u,\alpha) = y \quad \text{iff} \quad u = \int_0^y \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-\alpha^2 t^2)}},$$

where $\alpha \in (0,1)$ is a real number. When $a = 1$, $\mathrm{sn}(u,\alpha)$ becomes the sine function $\sin(u)$. For a multiset M, a binary tree whose nodes are labeled exactly by M such that each child node receives a label weakly greater than its parent is called a *weakly increasing binary trees* on M. According to the work in [11, 10], the *multiset Schett polynomials* $S_M(x,y)$, which extends the Jacobi elliptic function from sets to multisets, can be interpreted as

$$S_M(x,y) = \sum_{T \in \mathcal{B}_M} x^{\mathsf{olc}(T)} y^{\mathsf{orc}(T)},$$

where $\mathcal{B}_M$ is the set of weakly increasing binary trees on M and $\mathsf{olc}(T)$ (resp., $\mathsf{orc}(T)$) denotes the number of left (resp., right) chains of T with odd orders. Note that weakly increasing binary trees on $[n]$ are exactly *increasing binary trees* on $[n]$, while weakly increasing trees on $\{1^n\}$ are in obvious bijection with $\mathcal{B}_n$. For convenience, we write $S_M(x,y)$ as $S_n(x,y)$ (resp., $C_n(x,y)$) when $M = [n]$ (resp., $M = \{1^n\}$). Then, $S_n(x,y)$ are the classical Schett polynomials (see [14, 3, 11]) that specialize to the Jacobi elliptic function $\mathrm{sn}(u,\alpha)$. The first combinatorial interpretation of $S_n(x,y)$, which is in terms of $(\mathsf{lpk}_o, \mathsf{lpk}_e)$ on permutations, was found by Dumont [3]. The bivariate extension of Catalan numbers $C_n(x,y)$ will be named the *Catalan–Schett polynomials*. The first few values of $C_n(x,y)$ are:

$$\begin{aligned}
&C_1(x,y) = xy, \quad C_2(x,y) = x^2 + y^2, \quad C_3(x,y) = x^3y + xy^3 + 3xy,\\
&C_4(x,y) = x^4 + 8y^2x^2 + y^4 + 2x^2 + 2y^2,\\
&C_5(x,y) = x^5y + 5x^3y^3 + xy^5 + 15x^3y + 15xy^3 + 5xy\\
&C_6(x,y) = x^6 + 27x^4y^2 + 27y^4x^2 + y^6 + 8x^4 + 54y^2x^2 + 8y^4 + 3x^2 + 3y^2.
\end{aligned}$$

As

$$\mathcal{X}(T) = \frac{n - \mathsf{olc}(T)}{2} \quad \text{and} \quad \mathcal{Y}(T) = \frac{n - \mathsf{orc}(T)}{2}$$

for any $T \in \mathcal{B}_n$, we have

$$C_n(x,y) = \sum_{T \in \mathcal{B}_n} x^{n-2\mathcal{X}(T)} y^{n-2\mathcal{Y}(T)}.$$

Our first main result provides a new interpretation of $C_n(x,y)$ in terms of stack-sortable permutations.

Theorem 1.3. *There is a bijection $\Upsilon : \pi \mapsto T$ between $\mathfrak{S}_n(231)$ and $\mathcal{B}_n$ such that*

$$(\mathsf{DR}(\pi), \mathsf{AR}(\pi^{-1})) = (\mathsf{LC}(T), \mathsf{RC}(T)). \tag{1.4}$$

Consequently,

$$\sum_{\pi \in \mathfrak{S}_n(231)} x^{\mathsf{mnd}(\pi)} y^{\mathsf{mna}(\pi^{-1})} = \sum_{T \in \mathcal{B}_n} x^{\mathcal{X}(T)} y^{\mathcal{Y}(T)} \tag{1.5}$$

and there follows the permutation interpretation for the Catalan–Schett polynomials

$$C_n(x,y) = \sum_{\pi \in \mathfrak{S}_n(231)} x^{\mathsf{odr}(\pi)} y^{\mathsf{oar}(\pi^{-1})}.$$

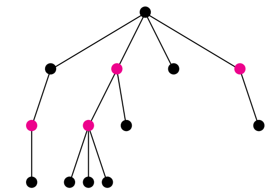


FIGURE 2. A plane tree with four marked nodes (in magenta).

Combining Υ with several known bijections, we can prove another interpretation of the Catalan–Schett polynomials in terms of 321-avoiding permutations, with the role of descents replaced by excedances.

There is another interpretation of the number (1.3) in terms of plane trees found by Callan [2]. Recall that a *plane tree* is a rooted tree in which the children of each node are linearly ordered. An internal node (other than the root) in a plane tree is *marked* if it has a leaf as its child. See Fig. 2 for a plane tree with four marked nodes. Let $\mathcal{T}_n$ be the set of plane trees with n edges and let $\mathsf{mark}(T)$ be the number of marked nodes in a tree $T \in \mathcal{T}_n$. Callan [2] proved that the number (1.3) counts plane trees $T \in \mathcal{T}_n$ with $\mathsf{mark}(T) = k$. In order to provide the second proof of Theorem 1.1 (ii), Zhang and Kitaev [8] constructed a recursive bijection, involving five different cases, between $\{T \in \mathcal{T}_n : \mathsf{mark}(T) = k\}$ and $\{\pi \in \mathfrak{S}_n(231) : \mathsf{mnd}(\pi) = k\}$. Here we provide a case-free recursive bijection which requires extra decompositions of plane trees.

Theorem 1.4. *There is a bijection $\vartheta : \mathcal{T}_n \to \mathfrak{S}_n(231)$ such that $\mathsf{mark}(T) = \mathsf{mnd}(\vartheta(T))$ for any $T \in \mathcal{T}_n$.*

Various bijections and equidistributions between $\mathfrak{S}_n(231)$ and $\mathfrak{S}_n(321)$ have already been investigated in the literature; see Kitaev's monograph [7, Chap. 4] for a survey. Inspired by Dumont's permutation interpretation for the Schett polynomials, we find the following unexpected equidistribution.

Theorem 1.5. *There is a bijection $\Psi : \mathfrak{S}_n(321) \to \mathfrak{S}_n(231)$ preserving the set-valued statistic "*LPK*". Consequently,*

$$\sum_{\pi\in\mathfrak{S}_n(321)} x^{\mathsf{lpk}_e(\pi)} y^{\mathsf{lpk}_o(\pi)} = \sum_{\pi\in\mathfrak{S}_n(231)} x^{\mathsf{lpk}_e(\pi)} y^{\mathsf{lpk}_o(\pi)}. \tag{1.6}$$

Note that $\mathrm{LPK}(\pi) = \mathrm{DES}(\pi)$ for any $\pi \in \mathfrak{S}_n(321)$. Using the bijection Ψ and a bijection of Krattenthaler [9] between 321-avoiding permutations and Dyck paths, we prove the following two refinements of Barnabei et al.'s generating function formula (1.1).

Theorem 1.6. *Let*

$$G(t,x,y) = \sum_{n\geqslant 1} t^n \sum_{\pi\in\mathfrak{S}_n(321)} x^{\mathsf{oar}(\pi)} y^{\mathsf{ear}(\pi)} \quad \text{and} \quad M(t,x,y) = \sum_{n\geqslant 1} t^n \sum_{\pi\in\mathfrak{S}_n(321)} x^{\mathsf{lpk}_e(\pi)} y^{\mathsf{lpk}_o(\pi)}.$$

Then both generating functions G and M are algebraic.

(i) G satisfies an algebraic equation of degree 4. In particular, if we set $A = G(t,x,1)$, then A satisfies

$$\begin{aligned}(t^5 + 4t^4x + 4t^3)A^4 =& (8t^3x^2 + 8t^2x - 4t^5 - 14t^4x - 16t^3)A^3 - (6t^5 + 2t^4x^3 \\ &+ 18t^4x + 22t^3 + 5t^2x^3 + 5tx^2 - 13t^3x^2 - 17t^2x - t)A^2\end{aligned}$$

$$- (4t^5 + 4t^4x^3 + 10t^4x + 12t^3 + 3tx^2 + x - 2t^3x^4 - 2t^3x^2$$
$$- 8t^2x - tx^4 - 2t - x^3)A - (t^5 + 2t^4x^3 + 2t^4x + t^3x^6$$
$$+ 3t^3x^2 + 2t^3 + 2t^2x^5 + t^2x + tx^4 - 2t^3x^4 - 3t^2x^3 - tx^2),$$

which solves Problem 1.2 in view of (1.2).

(ii) M satisfies an algebraic equation of degree 6.

References

[1] M. Barnabei, F. Bonetti and M. Silimbani, The descent statistic on 123-avoiding permutations, Sém. Lothar. Combin. 63 (2010), Art. B63a.

[2] D. Callan, Some bijections and identities for the Catalan and fine numbers, Sém. Lothar. Combin. 53 (2006), Article B53e.

[3] D. Dumont, A combinatorial interpretation for the schett recurrence on the Jacobian elliptic functions, Math. Comp. 33 (1979), 1293–1297.

[4] S. Elizalde and I. Pak, Bijections for refined restricted permutations, J. Combin. Theory Ser. A 105 (2004), 207–219.

[5] D.E. Knuth, The Art of Computer Programming, vol. 1, Addison-Wesley Publishing CO., Reading, Mass, 1968.

[6] S. Kitaev, Partially ordered generalized patterns, Discrete Math. 298 (2005), 212–229.

[7] S. Kitaev, Patterns in Permutations and Words, Springer Science & Business Media, 2011.

[8] S. Kitaev and P.B. Zhang, Non-overlapping descents and ascents in stack-sortable permutations, Discrete Appl. Math. 344 (2024), 112–119.

[9] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, Adv. in Appl. Math. 27 (2001), 510–530.

[10] Z. Lin, J. Liu, S. Wang and W.J.T. Zang, More bijective combinatorics of weakly increasing trees, preprint.

[11] Z. Lin and J. Ma, A symmetry on weakly increasing trees and multiset Schett polynomials, arXiv:2104.10539.

[12] L.-F. Préville-Ratelle and X. Viennot, The enumeration of generalized Tamari intervals, Trans. Amer. Math. Soc. 369 (2017), 5219–5239.

[13] A. Robertson, D. Saracino and D. Zeilberger, Refined restricted permutations, Ann. Comb. 6 (2002), 427–444.

[14] A. Schett, Properties of the Taylor series expansion coefficients of the Jacobian elliptic functions, Math. Comp. 30 (1976), 143–147.

[15] R.P. Stanley, *Enumerative Combinatorics, Vol. 1*, second ed., Cambridge University Press, Cambridge, 2012.

[16] Y. Sun, A simple bijection between binary trees and colored ternary trees, Electron. J. Combin. 17 (2010), #N20.

(Zhicong Lin) Research Center for Mathematics and Interdisciplinary Sciences, Shandong University & Frontiers Science Center for Nonlinear Expectations, Ministry of Education, Qingdao 266237, P.R. China

Email address: linz@sdu.edu.cn

(Jing Liu) Research Center for Mathematics and Interdisciplinary Sciences, Shandong University & Frontiers Science Center for Nonlinear Expectations, Ministry of Education, Qingdao 266237, P.R. China

Email address: lsweet@mail.sdu.edu.cn

(Sherry H.F. Yan) Department of Mathematics, Zhejiang Normal University, Jinhua 321004, P.R. China

Email address: hfy@zjnu.cn