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GENERALIZED k CHROMATIC POLYNOMIALS AND MULTIPLICITIES OF FREE ROOT SPACES OF BORCHERDS-KAC-MOODY (BKM) LIE SUPERALGEBRAS

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This talk is based on my paper [2].

Abstract Borcherds-Kac-Moody Lie superalgebras (referred to as BKM superalgebras) represent a natural extension encompassing two important classes of Lie algebras: Borcherds algebras (Generalized Kac-Moody algebras) and Kac-Moody Lie superalgebras. These algebras have found widespread applications in the realm of mathematical physics. Notably, physicists have harnessed the power of BKM superalgebras to describe various phenomena, including supersymmetry, chiral supergravity, and Gauge theory etc. In essence, our exploration revolves around the interplay between combinatorial structures and BKM superalgebras, seeking to unveil the unique properties and relationships that free roots hold within this mathematical framework.

Consider a Borcherds-Kac-Moody Lie superalgebra, denoted as $\mathfrak{g}$, associated with the graph G. This Lie superalgebra is constructed from a free Lie superalgebra by introducing three sets of relations on its generators:

(1) Chevalley relations,
(2) Serre relations, and
(3) the commutation relations derived from the graph G.

The Chevalley relations lead to a triangular decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where each root space $\mathfrak{g}_\alpha$ is contained in either $\mathfrak{n}_+$ or $\mathfrak{n}_-$. Importantly, each $\mathfrak{g}_\alpha$ is determined solely by relations (2) and (3). We focus on the root spaces of $\mathfrak{g}$ that are unaffected by the Serre relations. We refer to these root spaces as "free roots" of $\mathfrak{g}$ [1]. Since these root spaces only involve commutation relations derived from the graph G, we can examine them purely from a combinatorial perspective.

The graph G associated to BKM Lie superalgebra $\mathfrak{g}$ has vertex set V and edge set E. Graph polynomials serve as crucial graph invariants, providing valuable insights into the properties of associated graphs. Among these, chromatic polynomials hold a prominent position. They were first introduced by Birkhoff in his pursuit of solving the four-color conjecture, a famous problem in graph theory. Building on this foundation, previous work ([3, Propositions 1 and 2]) established a link between the characters of integrable representations of Kac-Moody Lie algebras and the coefficients of chromatic polynomials for the corresponding Dynkin diagrams. This connection between algebraic structures and graph theory was further extended in a subsequent study ([4]), wherein the chromatic polynomial of a graph G was expressed in terms of root multiplicities of the associated Kac-Moody Lie algebra. Expanding on these findings, another study ([1]) took this connection to new heights, extending it to Borcherds algebras and the broader class of generalized **k**-chromatic polynomials. This research also had applications, involving the construction of bases for specific root spaces, referred to as "free root spaces".

Our results build upon these foundations by extending the connection between root multiplicities in Borcherds algebras and the chromatic polynomial of the associated quasi-Dynkin diagram. However, this extension goes beyond traditional Lie algebras, encompassing Borcherds-Kac-Moody Lie superalgebras. The goal is to explore the implications and applications of this extended connection in combinatorics.

In summary, our results advances the understanding of algebraic structures and their relationships with graph theory. By extending established connections to Borcherds-Kac-Moody Lie superalgebras, it aims to uncover new insights and applications in combinatorics.

The following theorem gives connection between k Chromatic polynomial of graph G and root multiplicities, mult $\beta(J)$, of free roots. This is our main result whose notations are described after theorem in detail.

Theorem 1. *Let G be the quasi Dynkin diagram of a BKM superalgebra $\mathfrak{g}$. Assume that $\mathbf{k} = (k_i : i \in I) \in \mathbb{Z}_+^I$ such that $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$. Then*

$$\pi_{\mathbf{k}}^G(q) = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))} \sum_{\mathbf{J} \in L_G(\mathbf{k})} (-1)^{|\mathbf{J}|+|\mathbf{J}_1|} \prod_{J \in \mathbf{J}_0} \binom{q \ \mathit{mult}(\beta(J))}{D(J,\mathbf{J})} \prod_{J \in \mathbf{J}_1} \binom{-q \ \mathit{mult}(\beta(J))}{D(J,\mathbf{J})}.$$

where $L_G(\mathbf{k})$ is the bond lattice of weight $\mathbf{k}$ of the graph G.

The following corollary gives us a recurrence formula for the root multiplicities of free roots of $\mathfrak{g}$.

Corollary 2. *Let $\eta(\mathbf{k}) = \sum_{i \in I} k_i \alpha_i \in \Delta^+$ such that $k_i \leq 1$ for all $i \in I^{re} \sqcup \Psi_0$. Then*

$$\mathrm{mult}(\eta(\mathbf{k})) = \begin{cases} \sum\limits_{\ell | \mathbf{k}} \frac{\mu(\ell)}{\ell} \ |\pi_{\mathbf{k}/\ell}^G(q)[q]|, & \mathit{if} \ \ \eta(\mathbf{k}) \in \Delta_+^0 \\ \sum\limits_{\ell | \mathbf{k}} \frac{(-1)^{\ell+1}\mu(\ell)}{\ell} \ |\pi_{\mathbf{k}/\ell}^G(q)[q]|, & \mathit{if} \ \ \eta(\mathbf{k}) \in \Delta_+^1 \end{cases}$$

where $|\pi_{\mathbf{k}}^G(q)[q]|$ denotes the absolute value of the coefficient of q in $\pi_{\mathbf{k}}^G(q)$ and μ is the Mobius function. If k_i's are relatively prime (in particular if for some $i \in I$, $k_i = 1$), we have,

$$\mathrm{mult}(\eta(\mathbf{k})) = |\pi_{\mathbf{k}}^G(q)[q]| \quad \mathit{for\ any} \ \ \eta(\mathbf{k}) \in \Delta_+.$$

□

We now describe our results in detail.

A **Lie superalgebra** is a $\mathbb{Z}_2$ graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with Lie bracket satisfying

- $[a, b] = ab - (-1)^{\bar{a}\bar{b}} ba$ (super commutativity).
- $[a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}} [[a, c], b]$ (super-Jacobi identity).

for all homogeneous elements $a, b \in \mathfrak{g}$, where $\bar{x}$ denotes the degree of homogeneous element x.

Borcherds Kac-Moody(BKM) super matrix is a generalization of Generalized Cartan Matrix (GCM). Fix a subset $\Psi \subset I$, a complex matrix $(a_{ij})_{i,j\in I}$ together with the choice of Ψ is said to be a BKM-supermatrix if the following are satisfied:

(B1) $a_{ii} = 2$ or $a_{ii} \leq 0$.
(B2) $a_{ij} \leq 0$ if $i \neq j$.
(B3) $a_{ij} = 0$ if and only if $a_{ji} = 0$
(B4) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$.
(B5) $a_{ij} \in 2\mathbb{Z}$ if $a_{ii} = 2$ and $i \in \Psi$.

We are interested only in symmetrizable BKM Lie superalgebras.

An index $i \in I$ is said to be real if $a_{ii} = 2$ and imaginary if $a_{ii} \leq 0$. Denote by $I^{\rm re} = \{i \in I : a_{ii} = 2\}$, $\Psi^{re} = \Psi \cap I^{re}$, and $\Psi_0 = \{i \in \Psi : a_{ii} = 0\}$.

The **Borcherds-Kac-Moody Lie superalgebra** (BKM superalgebra in short) associated with a BKM supermatrix (A, Ψ) is the Lie superalgebra $\mathfrak{g}(A, \Psi)$ generated by $e_i, f_i, h_i, i \in I$ with the following defining relations:

(i) $[h_i, h_j] = 0$ for $i, j \in I$,
(ii) $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$ for $i, j \in I$,
(iii) $[e_i, f_j] = \delta_{ij} h_i$ for $i, j \in I$,
(iv) $\deg h_i = 0, i \in I$,
(v) $\deg e_i = 0 = \deg f_i$ if $i \notin \Psi$, $\deg e_i = 1 = \deg f_i$ if $i \in \Psi$,
(vi) $(\text{ad } e_i)^{1-a_{ij}} e_j = 0 = (\text{ad } f_i)^{1-a_{ij}} f_j$ if $i \in I^{re}$ and $i \neq j$,
(vii) $(\text{ad } e_i)^{1-\frac{a_{ij}}{2}} e_j = 0 = (\text{ad } f_i)^{1-\frac{a_{ij}}{2}} f_j$ if $i \in \Psi^{re}$ and $i \neq j$,
(viii) $(\text{ad } e_i)^{1-\frac{a_{ij}}{2}} e_j = 0 = (\text{ad } f_i)^{1-\frac{a_{ij}}{2}} f_j$ if $i \in \Psi_0$ and $i = j$,
(ix) $[e_i, e_j] = 0 = [f_i, f_j]$ if $a_{ij} = 0$.

The relations (vi), ..., (viii) are called the Serre relations of $\mathfrak{g}$.

The formal root lattice Q is defined to be a free abelian group generated by α_i, $i \in I$ with a real valued bilinear form $(\alpha_i, \alpha_j) = a_{ij}$. These α_i, $i \in I$ are called the simple roots. Let Δ be root system of BKM superalgebra. $\Delta_+ := \Delta \cap Q_+$ denotes the set of positive roots. For $\alpha = \sum_{k=1}^{j} \alpha_{i_k} \in Q$, the root space $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_{-\alpha}$) is generated by the elements $[e_{i_j}, [\cdots [e_{i_2}, e_{i_1}]]]$ (resp. $[f_{i_j}, [\cdots [f_{i_2}, f_{i_1}]]]$). If $\mathfrak{g}_\alpha \neq 0$ then the element $\alpha \in Q$ is said to be a root of $\mathfrak{g}$. Such a root α is said to be an odd root if the number of $i_k, 1 \leq k \leq j$ coming from I_1 is odd otherwise it is an even root, denoted as Δ_1 and Δ_0 respectively. So, a root space $\mathfrak{g}_\alpha$ is either contained in the even part $\mathfrak{g}_{\bar{0}}$ or odd part $\mathfrak{g}_{\bar{1}}$ of the BKM superalgebra $\mathfrak{g}$. The dimension of root space $\mathfrak{g}_\alpha$ is called the multiplicity of root α. All root spaces are finite dimensional. Observe that $\dim \mathfrak{g}_{\alpha_i} = 1 = \dim \mathfrak{g}_{-\alpha_i}$, $i \in I$.

Let G be a countable (possibly infinite) simple graph with vertex set $V = \{\alpha_i : i \in I\}$. For a subset $\Psi \subseteq I$, the pair (G, Ψ) is called a **supergraph**. The vertices parameterized by Ψ (resp. $I \backslash \Psi$) are called odd (resp. even) vertices of G. If A is the classical adjacency matrix of the graph G then the pair (A, Ψ) is called the adjacency matrix of the supergraph (G, Ψ).

An element $\alpha = \sum\limits_{i \in I} k_i \alpha_i \in Q^+$, where $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$, is called **a free root**. We define $supp\,(\alpha) = \{j : k_j \neq 0\}$. A root $\alpha \in \Delta$ is called real if and only if $(\alpha, \alpha) > 0$ otherwise we call it an imaginary root. The set of real roots is denoted by Δ^{re} and imaginary roots by $\Delta^{im} = \Delta \setminus \Delta^{re}$.

Denominator identity of BKM superalgebras. Let Ω be the set of all $\gamma \in Q_+$ such that

(1) $\gamma = \sum_{j=1}^{r} \alpha_{i_j} + \sum_{k=1}^{s} l_{i_k} \beta_{i_k}$ where the α_{i_j} (resp. β_{i_k}) are distinct even (resp. odd) imaginary simple roots,
(2) $(\alpha_{i_j}, \alpha_{i_k}) = (\beta_{i_j}, \beta_{i_k}) = 0$ for $j \neq k$; $(\alpha_{i_j}, \beta_{i_k}) = 0$ for all j, k;
(3) if $l_{i_k} \geq 2$, then $(\beta_{i_k}, \beta_{i_k}) = 0$.

The following denominator identity of BKM superalgebras is proved in [5, Section 2.6]:

$$U := \sum_{w \in W} \sum_{\gamma \in \Omega} \epsilon(w)\epsilon(\gamma) e^{w(\rho - \gamma) - \rho} \;=\; \frac{\prod_{\alpha \in \Delta_+^0} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}}{\prod_{\alpha \in \Delta_+^1} (1 + e^{-\alpha})^{\mathrm{mult}(\alpha)}} \tag{0.1}$$

where $\mathrm{mult}(\alpha) = \dim \mathfrak{g}_\alpha$, $\epsilon(w) = (-1)^{l(w)}$ and $\epsilon(\gamma) = (-1)^{\mathrm{ht}\,\gamma}$.

Notations:

- For $w \in W$, fix a reduced word $w = \mathbf{s}_{i_1} \cdots \mathbf{s}_{i_k}$ and let $I(w) = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$.
- For $\gamma = \sum\limits_{i \in I} m_i \alpha_i \in \Omega$, we set $I_m(\gamma)$ is the multiset $\{\underbrace{\alpha_i, \ldots, \alpha_i}_{m_i \text{ times}} : i \in I\}$ and $I(\gamma)$ is the underlying set of $I_m(\gamma)$.
- $\Psi_0(\gamma) := I(\gamma) \cap \Psi_0$ and $\mathcal{J}(\gamma) := \{w \in W \backslash \{e\} : I(w) \cup I(\gamma) \text{ is an independent set}\}$.

Multicoloring and the k-chromatic polynomial of G. For any finite set S, let $\mathcal{P}(S)$ be the power set of S. For a tuple of non–negative integers $\mathbf{k} = (k_i : i \in I)$, we have $\mathrm{supp}(\mathbf{k}) = \{i \in I : k_i \neq 0\}$.

Definition 3. Let G be a graph with vertex set I and the edge set $E(G)$. Let $\mathbf{k} \in \mathbb{Z}_+^I$. We call a map $\tau : I \to \mathcal{P}\big(\{1, \ldots, q\}\big)$ a proper vertex $\mathbf{k}$-multicoloring of G if the following conditions are satisfied:

(i) For all $i \in I$ we have $|\tau(i)| = k_i$,
(ii) For all $i, j \in I$ such that $(i, j) \in E(G)$ we have $\tau(i) \cap \tau(j) = \emptyset$.

The case $k_i = 1$ for $i \in I$ corresponds to the classical graph coloring of graph G.

Definition 4. The number of ways a graph G can be $\mathbf{k}$–multicolored using q colors is a polynomial in q, called the generalized $\mathbf{k}$-chromatic polynomial ($\mathbf{k}$-chromatic polynomial in short) and denoted by $\pi_{\mathbf{k}}^G(q)$. The $\mathbf{k}$-chromatic polynomial has the following well–known description. We denote by $P_k(\mathbf{k}, G)$ the set of all ordered k–tuples $(P_1, \ldots, P_k)$ such that:

(1) each P_i is a non–empty independent subset of I, i.e. no two vertices have an edge between them; and
(2) For all $i \in I$, α_i occurs exactly k_i times in total in the disjoint union $P_1 \dot\cup \cdots \dot\cup P_k$.

Then we have

$$\pi_{\mathbf{k}}^G(q) = \sum_{k \geq 0} |P_k(\mathbf{k}, G)| \binom{q}{k}. \tag{0.2}$$

We have the following relation between the ordinary chromatic polynomials and the $\mathbf{k}$-chromatic polynomials. We have

$$\pi_{\mathbf{k}}^G(q) = \frac{1}{\mathbf{k}!} \pi_{\mathbf{1}}^{G(\mathbf{k})}(q) \tag{0.3}$$

where $\pi_{\mathbf{1}}^{G(\mathbf{k})}(q)$ is the chromatic polynomial of the graph $G(\mathbf{k})$ and $\mathbf{k}! = \prod_{i \in I} k_i!$. The graph $G(\mathbf{k})$ (the join of G with respect to $\mathbf{k}$) is constructed as follows: For each $j \in \operatorname{supp}(\mathbf{k})$, take a clique (complete graph) of size k_j with vertex set $\{j^1, \ldots, j^{k_j}\}$ and join all vertices of the r–th and s–th cliques if $(r, s) \in E(G)$.

For the rest of this paper, we fix an element $\mathbf{k} \in \mathbb{Z}_+^I$ satisfying $k_i \leq 1$ for $i \in I^{re} \sqcup \Psi_0$, where Ψ_0 is the set of odd roots of zero norm.

Bond lattice and an isomorphism of lattices.

Definition 5. Let $L_G(\mathbf{k})$ be the weighted bond lattice of G, which is the set of $\mathbf{J} = \{J_1, \ldots, J_k\}$ satisfying the following properties:

(i) $\mathbf{J}$ is a multiset, i.e. we allow $J_i = J_j$ for $i \neq j$
(ii) each J_i is a multiset and the subgraph spanned by the underlying set of J_i is a connected subgraph of G for each $1 \leq i \leq k$ and
(iii) For all $i \in I$, α_i occurs exactly k_i times in total in the disjoint union $J_1 \dot\cup \cdots \dot\cup J_k$.

For $\mathbf{J} \in L_G(\mathbf{k})$ we denote by $D(J_i, \mathbf{J})$ the multiplicity of J_i in $\mathbf{J}$ and set $\operatorname{mult}(\beta(J_i)) = \dim \mathfrak{g}_{\beta(J_i)}$, where $\beta(J_i) = \sum_{\alpha \in J_i} \alpha$. We define $\mathbf{J}_0 = \{J_i \in \mathbf{J} : \beta(J_i) \in \Delta_+^0\}$ and $\mathbf{J}_1 = \mathbf{J} \backslash \mathbf{J}_0$.

Lemma 6. [5, Proposition 2.40] *Let $i \in I^{im}$ and $\alpha \in \Delta_+ \backslash \{\alpha_i\}$ such that $\alpha(h_i) < 0$. Then $\alpha + j\alpha_i \in \Delta_+$ for all $j \in \mathbb{Z}_+$.*

Lemma 7. [1, Lemma 3.4] *Let $\mathcal{P}$ be the collection of multisets $\gamma = \{\beta_1, \ldots, \beta_r\}$ (we allow $\beta_i = \beta_j$ for $i \neq j$) such that each $\beta_i \in \Delta_+$ and $\beta_1 + \cdots + \beta_r = \eta(\mathbf{k})$. The map $\psi : L_G(\mathbf{k}) \to \mathcal{P}$ defined by $\{J_1, \ldots, J_k\} \mapsto \{\beta(J_1), \ldots, \beta(J_k)\}$ is a bijection.*

The following lemma is a generalization of [4, Lemma 2.3] (for Kac-Moody Lie algebras) and [1, Lemma 3.6] (for Borcherds algebras) to the setting of BKM superalgebras. Since the proof of this lemma is similar to the proof of the Borcherds algebras case, we omit the proof here. Recall that $\mathbf{k} = (k_i : i \in I)$ satisfies $k_i \leq 1$ for $i \in I^{\text{re}} \sqcup \Psi_0$.

Lemma 8. *Let $w \in W$ and $\gamma = \sum\limits_{i \in I \backslash \Psi_0} \alpha_i + \sum\limits_{i \in \Psi_0} m_i \alpha_i \in \Omega$. We write $\rho - w(\rho) + w(\gamma) = \sum_{\alpha \in \Pi} b_\alpha(w, \gamma) \alpha$. Then we have*

(i) *$b_\alpha(w, \gamma) \in \mathbb{Z}_+$ for all $\alpha \in \Pi$ and $b_\alpha(w, \gamma) = 0$ if $\alpha \notin I(w) \cup I(\gamma)$.*
(ii) *$b_\alpha(w, \gamma) \geq 1$ for all $\alpha \in I(w)$.*
(iii) *$b_\alpha(w, \gamma) = 1$ if $\alpha \in I(\gamma) \backslash \Psi_0(\gamma)$ and $b_\alpha(w, \gamma) = m_\alpha$ if $\alpha \in \Psi_0(\gamma)$.*

(iv) *If* $w \in \mathcal{J}(\gamma)$, *then* $b_\alpha(w,\gamma) = 1$ *for all* $\alpha \in I(w) \cup (I(\gamma)\backslash\Psi_0(\gamma))$, $b_\alpha(w,\gamma) = m_\alpha$ *for all* $\alpha \in \Psi_0(\gamma)$.

(v) *If* $w \notin \mathcal{J}(\gamma) \cup \{e\}$, *then there exists* $\alpha \in I(w) \subseteq \Pi^{\mathrm{re}}$ *such that* $b_\alpha(w,\gamma) > 1$.

The following proposition is an easy consequence of the above lemma and essential to prove Theorem 1. Let U be the sum-side of the denominator identity (Equation (0.1)).

Proposition 9. *Let* $q \in \mathbb{Z}$. *We have*

$$U^q[e^{-\eta(\mathbf{k})}] = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\ \pi^G_{\mathbf{k}}(q),$$

where $U^q[e^{-\eta(\mathbf{k})}]$ *denotes the coefficient of* $e^{-\eta(\mathbf{k})}$ *in* U^q.

Proof of Theorem 1 and Corollary 2

Proof of Theorem 1. Now, we can prove Theorem 1 using the product side of the denominator identity (0.1). Proposition 9 and Equation (0.1) together imply that the $\mathbf{k}$-chromatic polynomial $\pi^G_{\mathbf{k}}(q)$ is given by the coefficient of $e^{-\eta(\mathbf{k})}$ in

$$(-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\frac{\prod_{\alpha\in\Delta^0_+}(1-e^{-\alpha})^{q\,\mathrm{mult}(\alpha)}}{\prod_{\alpha\in\Delta^1_+}(1+e^{-\alpha})^{q\,\mathrm{mult}(\alpha)}} = (-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\prod_{\alpha\in\Delta_+}(1-\epsilon(\alpha)e^{-\alpha})^{\epsilon(\alpha)q\,\mathrm{mult}(\alpha)}. \tag{0.4}$$

where $\epsilon(\alpha) = 1$ if $\alpha \in \Delta^0_+$ and -1 if $\alpha \in \Delta^1_+$. Now,

$$\prod_{\alpha\in\Delta_+}(1-\epsilon(\alpha)e^{-\alpha})^{\epsilon(\alpha)q\,\mathrm{mult}(\alpha)} = \prod_{\alpha\in\Delta_+}\Big(\sum_{k\geq 0}(-\epsilon(\alpha))^k\binom{\epsilon(\alpha)q\,\mathrm{mult}\,\alpha}{k}e^{-k\alpha}\Big).$$

A direct calculation of the coefficient of $e^{-\eta(\mathbf{k})}$ in the right-hand side of the above equation completes the proof of Theorem 1. □

We consider the algebra of formal power series $\mathcal{A} := \mathbb{C}[[X_i : i \in I]]$. For a formal power series $\zeta \in \mathcal{A}$ with constant term 1, its logarithm $\log(\zeta) = -\sum_{k\geq 1}\frac{(1-\zeta)^k}{k}$ is well–defined.

Proof of Corollary 2: (Formula for multiplicities of free roots) We consider U as an element of $\mathbb{C}[[e^{-\alpha_i} : i \in I]]$ where $X_i = e^{-\alpha_i}$ [c.f. Lemma 8]. From the proof of Proposition 9 we obtain that the coefficient of $e^{-\eta(\mathbf{k})}$ in $-\log U$ is equal to

$$(-1)^{\mathrm{ht}(\eta(\mathbf{k}))}\sum_{k\geq 1}\frac{(-1)^k}{k}|P_k(\mathbf{k},G)|$$

which by Equation (0.2) is equal to $|\pi^G_{\mathbf{k}}(q)[q]|$. Now applying $-\log$ to the right hand side of the denominator identity (0.1) gives

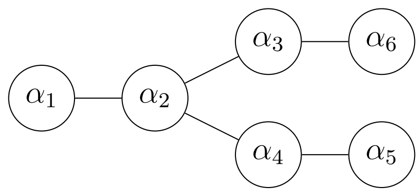
$$|\pi^G_{\mathbf{k}}(q)[q]| = \begin{cases} \sum_{\substack{\ell\in\mathbb{N}\\ \ell|\mathbf{k}}}\frac{1}{\ell}\,\mathrm{mult}\big(\eta\,(\mathbf{k}/\ell)\big), & \text{if } \beta(\mathbf{k}) \in \Delta^+_0\\ \sum_{\substack{\ell\in\mathbb{N}\\ \ell|\mathbf{k}}}\frac{(-1)^{l+1}}{\ell}\,\mathrm{mult}\big(\eta\,(\mathbf{k}/\ell)\big), & \text{if } \beta(\mathbf{k}) \in \Delta^+_1\end{cases}$$

The statement of the corollary is now an easy consequence of the following Mobius inversion formula: $g(d) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n}\mu(\frac{n}{d})g(d)$ where μ is the Mobius function. □

Example 10. Let $I = \{1,2,3,4,5,6\}, \Psi = \{3,5\}$. Consider the BKM supermatrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -3 & -4 & -1 & 0 & 0 \\ 0 & -4 & -4 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & -1 & 0 & 0 & -3 \end{bmatrix}.$$

The quasi-Dynkin diagram G of $\mathfrak{L}$ is as follows:



Let $\alpha = 3\alpha_3 + 3\alpha_6 \in \Delta^1_+$, i.e. $\mathbf{k} = (0,0,3,0,0,3)$ then $\pi^G_{\mathbf{k}}(q) = \binom{q}{3}\binom{q-3}{3}$.

The $\mathbf{k}$-chromatic polynomial of the quasi Dynkin diagram G of $\mathfrak{g}$ is equal to

$$\pi^G_{\mathbf{k}}(q) = \binom{q}{3}\binom{q-3}{3} = \frac{1}{3!3!}q(q-1)(q-2)(q-3)(q-4)(q-5).$$

By Corollary 2, since $\eta(\mathbf{k})$ is odd,

$$\begin{aligned} \operatorname{mult}(\eta(\mathbf{k})) &= \sum_{\ell | \mathbf{k}} \frac{(-1)^{l+1}\mu(\ell)}{\ell} \ |\pi^G_{\mathbf{k}/\ell}(q)[q]| \\ &= |\pi^G_{\mathbf{k}}(q)[q]| + \frac{\mu(3)}{3}|\pi^G_{\mathbf{k}'}(q)[q]| \text{ where } \mathbf{k}' = (0,0,1,0,0,1) \\ &= \frac{10}{3} - \frac{1}{3} = 3 \end{aligned}$$

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