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THE IMMERSION POSET ON PARTITIONS

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Abstract We introduce the immersion poset $(\mathcal{P}(n), \leqslant_I)$ on partitions, defined by $\lambda \leqslant_I \mu$ if and only if $s_\mu(x_1,\dots,x_N) - s_\lambda(x_1,\dots,x_N)$ is monomial-positive. Relations in the immersion poset determine when irreducible polynomial representations of $GL_N(\mathbb{C})$ form an immersion pair, as defined by Prasad and Raghunathan [3]. To determine relations and covers in the immersion poset, we develop injections $\mathsf{SSYT}(\lambda,\nu) \hookrightarrow \mathsf{SSYT}(\mu,\nu)$ on semistandard Young tableaux given constraints on the shape of λ and define a refinement of the immersion poset, the standard immersion poset $(\mathcal{P}(n), \leqslant_{std})$. We classify maximal elements of certain shapes in the standard immersion poset using the hook length formula. Finally, we prove Schur-positivity of power sum symmetric functions p_{A_μ} on conjectured lower intervals in the immersion poset, addressing questions posed by Sundaram [5].

1. Introduction

Given two finite-dimensional representations $\pi_1 : G \to GL(W_1)$ and $\pi_2 : G \to GL(W_2)$ of a group G, we say that the representation (W_1, π_1) is *immersed* in the representation (W_2, π_2) if the eigenvalues of $\pi_1(g)$, counting multiplicities, are contained in the eigenvalues of $\pi_2(g)$ for all $g \in G$.

Question 1.1 ([3]). Classify immersion of representations $W_1 \leqslant_I W_2$ for a given group.

Recently, some progress was made on the above problem for symmetric groups [1] and alternating groups [2]. In this paper, we study immersion pairs for representations of the general linear group $GL_N(\mathbb{C})$. Specifically, we focus on the finite-dimensional homogeneous irreducible polynomial representations known as the *Weyl modules* $(W_\lambda(\mathbb{C}^N), \rho_\lambda)$, indexed by integer partitions λ (of size n) with at most N non-zero parts. The corresponding irreducible characters, known as *Schur polynomials* $s_\lambda(x_1,\dots,x_N)$, are homogeneous symmetric polynomials (of degree n) in N variables.

The Schur polynomials $\{s_\lambda \mid \lambda \vdash n\}$ form a basis for the vector space of symmetric polynomials of degree n, Λ_n. The *monomial symmetric polynomials* $\{m_\lambda \mid \lambda \vdash n\}$ also form a basis for Λ_n, where m_λ is defined by $m_\lambda(x_1,\dots,x_N) \coloneqq \sum_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ and the sum is over all distinct permutations α of the parts of the partition λ. A symmetric polynomial $f(x_1,\dots,x_N)$ is *monomial-positive* if

$$f(x_1,\dots,x_N) = \sum_\lambda c_\lambda m_\lambda(x_1,\dots,x_N),$$

where the coefficients c_λ are non-negative integers.

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It is a known fact that (for example, see [4, Chapter 7]) if $g \in GL_N(\mathbb{C})$ has the eigenvalues $x_1, \dots, x_N$, then the eigenvalues of $\rho_\lambda(g)$ are the monomials appearing in the Schur polynomial $s_\lambda(x_1, \dots, x_N)$. Thus, given two partitions λ, μ of n with $\ell(\lambda), \ell(\mu) \leqslant N$, the Weyl module $W_\lambda(\mathbb{C}^N)$ is immersed in $W_\mu(\mathbb{C}^N)$ if and only if $s_\mu(x_1, \dots, x_N) - s_\lambda(x_1, \dots, x_N)$ is monomial-positive.

Let $\mathcal{P}(n)$ denote the set of integer partitions of n. We define a partial order on $\mathcal{P}(n)$ as follows.

Definition 1.2. For $\lambda, \mu \in \mathcal{P}(n)$, we define $\lambda \leqslant_I \mu$ if $s_\mu(x_1, \dots, x_N) - s_\lambda(x_1, \dots, x_N)$ is monomial-positive. We call the poset $(\mathcal{P}(n), \leqslant_I)$ the *immersion poset*.

The monomial symmetric expansion of the Schur polynomial s_λ for $\lambda \vdash n$ is given by

$$s_\lambda(x_1, \dots, x_N) = \sum_{\alpha \vdash n} K_{\lambda,\alpha} m_\alpha(x_1, \dots, x_N), \tag{1.1}$$

where $K_{\lambda,\alpha}$ are the *Kostka numbers* which count the number of semistandard Young tableaux of shape λ and content α. Using (1.1), Definition 1.2 can be restated as saying $\lambda \leqslant_I \mu$ if $K_{\lambda,\alpha} \leqslant K_{\mu,\alpha}$ for all $\alpha \in \mathcal{P}(n)$.

In this paper we analyze various properties of the immersion poset in order to improve our understanding of the immersions of $GL_N(\mathbb{C})$ representations. We begin in Section 2 by defining the standard immersion poset. In the standard immersion poset, one only compares the number of standard tableaux of shape λ and μ (instead of semistandard tableaux of all content). We study properties and maximal elements in the standard immersion poset, which are also maximal elements in the immersion poset. In Section 3, we study properties of the immersion poset. In particular, in Section 3.2 we study relations and covers in the immersion poset using explicit injections between sets of semistandard tableaux. Finally, in Section 3.3 we analyze intervals $[(1^n), \mu] := \{\lambda \mid (1^n) \leqslant_I \lambda \leqslant_I \mu\}$ in the immersion poset, for $\mu \vdash n$ and apply this to analyzing Schur-positivity of the following sums of *power sum symmetric polynomials*. Given a subset A_n of partitions of n, consider

$$p_{A_n} := \sum_{\mu \in A_n} p_\mu, \tag{1.2}$$

where $p_\mu := p_{\mu_1} \cdots p_{\mu_n}$ and $p_k = \sum_{i=1}^n x_i^k$.

Question 1.3 ([5])**.** For which choices of A_n is the symmetric polynomial p_{A_n} Schur-positive? In other words, which subsets A_n of columns in the character table of S_n result in non-negative row sums?

One may ask for what choices of μ the symmetric polynomial $p_{[(1^n),\mu]}$ defined in Equation (1.2) is Schur-positive. Assuming Conjectures 3.8 and 3.9, we prove that:

1. $p_{[(1^n),(n-2,1,1)]}$ is Schur-positive;
2. $p_{[(1^n),(n-2,2)]}$ is Schur-positive for $n \neq 7$.

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2. Standard immersion poset

2.1. **Definition of the standard immersion poset.** Following the discussion from the introduction (1.1), we have the following lemma.

Lemma 2.1. *For $\lambda, \mu \in \mathcal{P}(n)$, $\lambda \leqslant_I \mu$ if $K_{\lambda,\alpha} \leqslant K_{\mu,\alpha}$ for all $\alpha \in \mathcal{P}(n)$.*

In particular, Lemma 2.1 implies that a necessary condition for $\lambda \leqslant_I \mu$ is that $K_{\lambda,(1^n)} \leqslant K_{\mu,(1^n)}$, which count the standard Young tableaux of shape λ and μ, respectively. Define $f^\lambda := K_{\lambda,(1^n)}$.

Let $\lambda, \mu \in \mathcal{P}(n)$. Define $\lambda \leqslant_D \mu$ in *dominance order* on partitions by requiring that

$$\sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \mu_i \qquad \text{for all } k \geqslant 1.$$

The Kostka matrix $(K_{\lambda,\alpha})_{\lambda,\alpha \in \mathcal{P}(n)}$ is unit upper-triangular with respect to dominance order, that is, $K_{\lambda,\lambda} = 1$ and $K_{\lambda,\alpha} = 0$ unless $\alpha \leqslant_D \lambda$. This implies another necessary condition for $\lambda \leqslant_I \mu$, namely $\lambda \leqslant_D \mu$. This motivates the definition of the standard immersion poset.

Definition 2.2. On $\mathcal{P}(n)$, define $\lambda \leqslant_{std} \mu$ if $\lambda \leqslant_D \mu$ in dominance order and $f^\lambda \leqslant f^\mu$. We call this poset the *standard immersion poset*.

As argued above, the standard immersion poset is a refinement of the immersion poset, that is, $\lambda \leqslant_I \mu$ implies that $\lambda \leqslant_{std} \mu$. The converse is not always true. For $n \geqslant 12$, there are examples of $\lambda \leqslant_{std} \mu$, which do not satisfy $\lambda \leqslant_I \mu$. For example $(5,3,1,1,1,1)$ covers $(4,2,2,2,1,1)$ in the standard immersion poset for $n = 12$, but not in the immersion poset.

2.2. **Properties of the standard immersion poset.** We now state and prove properties of the standard immersion poset. Our main tool is the *hook length formula* for $\lambda \in \mathcal{P}(n)$

$$f^\lambda = \frac{n!}{\prod_{u \in \lambda} h(u)}, \tag{2.1}$$

where $h(u)$ is the hook length of the cell u in λ, which counts the cells weakly to the right of u and strictly below u (in English notation for partitions).

We write $\lambda \lessdot_{std} \mu$ if μ covers λ in the standard immersion poset. More precisely, $\lambda \lessdot_{std} \mu$ if $\lambda <_{std} \mu$ and there does not exist any ν such that $\lambda <_{std} \nu <_{std} \mu$.

Lemma 2.3. *We have*

1. (1^n) *is the unique minimal element in the standard immersion poset and*
2. $(1^n) \lessdot_{std} (n)$ *for all* n.

Proof. The partition (1^n) is the unique minimal element in dominance order. Furthermore, $f^{(1^n)} = 1 \leqslant f^\lambda$ for all $\lambda \in \mathcal{P}(n)$. This proves claim (1). We have $(1^n) <_D (n)$ and $f^{(1^n)} = f^{(n)} = 1$. There is no other partition λ with $f^\lambda = 1$. This implies $(1^n) \lessdot_{std} (n)$. ☐

Remark 2.4. Let $\lambda <_{std} \mu$. If μ covers λ in dominance order, then μ covers λ with respect to $<_{std}$. The converse is not true. Take $\lambda = (1^n)$ and $\mu = (n)$.

Lemma 2.5. *Let* $\lambda = (2^a, 1^b)$ *and* $\mu = (2^{a+1}, 1^{b-2})$. *Then* $\lambda \lessdot_{std} \mu$ *if and only if* $\frac{b(b-1)}{2} > a$.

Proof. We have $\lambda \lessdot_D \mu$. Hence by Remark 2.4, it suffices to show that $\lambda <_{std} \mu$. By the hook length formula, this is true if $\frac{f^\lambda}{f^\mu} = \frac{(b+1)(a+1)}{(b-1)(a+b+1)} \leqslant 1$, which is equivalent to the condition $\frac{b(b-1)}{2} > a$. ☐

2.3. **Classifying maximal elements.** Since the standard immersion poset is a refinement of the immersion poset, all maximal partitions in the standard immersion poset are also maximal in the immersion poset. To show that λ is a maximal element, our general proof strategy involves proving that $\frac{f^\mu}{f^\lambda} < 1$ for all partitions μ such that $\lambda <_D \mu$.

Proposition 2.6. *The partition* $(a+b, a)$ *is a maximal element in the standard immersion poset if and only if* $\frac{b(b+3)}{2} \geqslant a$.

Proposition 2.7. *Let* $\lambda = (a+b, a, 1)$ *where* $a \geqslant 2$. *Then* λ *is maximal in the standard immersion poset if and only if* $a \leqslant \frac{(b+1)(b+2)}{2}$.

Proposition 2.8. *Let $\lambda = (a+b, a, 2)$ where $a \geqslant 3$. Then λ is maximal in the standard immersion poset if and only if $a \leqslant \frac{(b+1)(b+2)}{2}$.*

We conclude this section with a conjecture about more general maximal elements in the standard immersion poset.

Conjecture 2.9. *Suppose $\lambda = (\sum_{i=1}^{\ell} a_i, \sum_{i=1}^{\ell-1} a_i, \dots, a_2 + a_1, a_1)$ for $\ell > 2$. If*

$$\binom{a_j + 2}{2} \geqslant \sum_{i=1}^{j-1} a_i + j - 2$$

is satisfied for all $2 \leqslant j \leqslant \ell$, then λ is maximal in the standard immersion poset.

This conjecture has been verified with SageMath [7] for $|\lambda| \leqslant 30$.

3. Immersion poset

3.1. **Properties of the immersion poset.** Analogously to Lemma 2.3, we prove the following.

Lemma 3.1. *We have*

(1) (1^n) *is the unique minimal element in the immersion poset* $(\mathcal{P}(n), \leqslant_I)$ *and*
(2) $(1^n) \lessdot_I (n)$ *for all n.*

3.2. **Explicit injections.** Recall from Lemma 2.1 that $\lambda \leqslant_I \mu$ if and only if $K_{\lambda,\nu} \leqslant K_{\mu,\nu}$ for all $\nu \in \mathcal{P}(n)$. The Kostka number $K_{\lambda,\nu}$ is the cardinality of the set of semistandard Young tableaux $\mathsf{SSYT}(\lambda,\nu)$ of shape λ and content ν. Hence we can analyze the order relations $\lambda \leqslant_I \mu$ by constructing explicit injections $\varphi\colon \mathsf{SSYT}(\lambda,\nu) \to \mathsf{SSYT}(\mu,\nu)$ for all $\nu \in \mathcal{P}(n)$.

To this end, we present one such injection, where μ differs from λ by moving a single cell from the c-th column to the $(c+1)$-th column, and λ has a bound on the relative size of the two columns. Upon establishing this first injection, we refine it to obtain more precise bounds on the relative size of the columns (excluded in the abstract). Let

$$\begin{aligned} \lambda &= (\lambda_1, \dots, \lambda_\alpha, c^\beta, \lambda_{\alpha+\beta+1}, \dots), \\ \mu &= (\lambda_1, \dots, \lambda_\alpha, c+1, c^{\beta-2}, c-1, \lambda_{\alpha+\beta+1}, \dots), \end{aligned} \tag{3.1}$$

such that either $\alpha > 0$ and $\lambda_{\beta+\alpha+1} < c < \lambda_\alpha$, or $\alpha = 0$ and $\lambda_{\beta+\alpha+1} < c$. In particular, $\lambda_{\beta+\alpha+1}$ can be 0. We define a map

$$\varphi_0\colon \mathsf{SSYT}(\lambda,\nu) \to \mathsf{YT}(\mu,\nu),$$

where $\mathsf{YT}(\mu,\nu)$ is the set of all tableaux of shape μ and content ν, not necessarily semistandard. We will show in Proposition 3.2 that when $\beta \geqslant \alpha+2$, the image of φ_0 is contained in $\mathsf{SSYT}(\mu,\nu)$.

For $T \in \mathsf{SSYT}(\lambda,\nu)$, we define $\varphi_0(T)$ as follows. Suppose the entries in the c-th column of T in increasing order are $x_{\beta+\alpha}, x_{\beta+\alpha-1}, \dots, x_1$ and the entries in the $(c+1)$-th column of T in increasing order are $y_\alpha, y_{\alpha-1}, \dots, y_1$. Let i be the smallest index such that $x_i > y_i$. If no such index exists, let $i = \alpha+1$. Then construct the tableau $\varphi_0(T)$ by swapping content $x_i, \cdots, x_1$ in the c-th column of T with the content $y_{i-1}, \cdots, y_1$ in the $(c+1)$-th column of T, and all other content in T remains fixed in $\varphi_0(T)$. Note that this swap of content moves one box in the c-th column of T to the $(c+1)$-th column, therefore creating $\varphi_0(T) \in \mathsf{YT}(\mu,\nu)$.

Proposition 3.2. *Let λ, μ be as in (3.1) with $\beta \geqslant \alpha+2$. Then φ_0 as defined above is an injection*

$$\varphi_0\colon \mathsf{SSYT}(\lambda,\nu) \to \mathsf{SSYT}(\mu,\nu).$$

As a corollary, the injection describes a class of cover relations in the immersion poset. As a specific example, it can partially address the two column case, which was completely addressed by Lemma 2.5 for the standard immersion poset.

Corollary 3.3. *The partitions λ and μ as in (3.1) with $\beta \geqslant \alpha + 2$ form a cover in the immersion poset. In particular, $\lambda = (2^\alpha, 1^\beta)$ and $\mu = (2^{\alpha+1}, 1^{\beta-2})$ form a cover.*

Proof. The partition μ covers λ in dominance order, and the injection shows that μ is greater than λ in the immersion poset, so μ must also cover λ in the immersion poset. □

The injection also gives a few conditions on which partitions cannot be maximal.

Corollary 3.4. *If $\lambda = (a^\beta, b, \dots)$, where $a > b$, and $\beta \geqslant 2$, then λ is not maximal.*

Corollary 3.5. *If $\lambda = (a, b^\beta, c, \dots)$, where $a > b > c$, and $\beta \geqslant 3$, then λ is not maximal. In particular, $\lambda = (a, 1^\beta)$ is not maximal for $a \geqslant 2$, $\beta \geqslant 3$.*

Corollary 3.6. *If $\lambda = (a, b, c, d)$ is maximal in the immersion poset, then it has no more than two identical non-zero parts.*

Subsequent modifications to the map φ_0 strengthen the results of the above corollaries.

3.3. **Lower intervals and Schur-positivity of interval power sums.** In this section, we study certain lower intervals $A_\mu := \{\lambda \,|\, (1^n) \leqslant_I \lambda \leqslant_I \mu\}$ in the immersion poset. Determining intervals will

(1) enhance our understanding of the immersion of $GL_N(\mathbb{C})$ polynomial representations and
(2) allow us to investigate when p_{A_μ} of Equation (1.2) is Schur-positive, as asked in Question 1.3. We call p_{A_μ} an *interval power sum*.

Sundaram's interest in Question 1.3 stems from representation theory of the symmetric group, where p_{A_n} is a restricted row sum of the character table of S_n that ignores the columns not in $A_n \subseteq \mathcal{P}(n)$. Sundaram conjectured that all intervals $[(1^n), \mu]$ in reverse lexicographic order make (1.2) Schur-positive [5, Conjecture 1], and has proven the conjecture for certain intervals [6].

Question 3.7. Which intervals A_μ in the immersion poset give rise to Schur-positivity of p_{A_μ}?

When $n \geqslant 5$, it appears that the immersion poset always contains some interval(s) which do not give rise to Schur-positivity. For example, $p_{A_{(n-1,1)}} = p_{(1^n)} + p_{(2,1^{n-2})} + p_{(n-1,1)}$ contains $-s_{(1^n)}$ when n is odd. However, using SageMath [7] we observed that at least 73% of intervals give Schur-positive interval power sums for $n \leqslant 18$.

Conjecture 3.8. *For $n = 5$ and $n \geqslant 9$, the interval $A_{(n-2,2)} = \{\lambda \,|\, (1^n) \leqslant_I \lambda \leqslant_I (n-2,2)\}$ is exactly*

$$(1^n) \lessdot_I (2, 1^{n-2}) \lessdot_I (2, 2, 1^{n-4}) \lessdot_I (n-2, 2).$$

Conjecture 3.9. *For $n \geqslant 9$, the interval $A_{(n-2,1,1)} = \{\lambda \,|\, (1^n) \leqslant_I \lambda \leqslant_I (n-2,1,1)\}$ is exactly*

$$(1^n) \lessdot_I (2, 1^{n-2}) \lessdot_I (2, 2, 1^{n-4}) \lessdot_I (3, 1^{n-3}) \lessdot_I (n-2, 1, 1).$$

We prove that p_{A_μ} is Schur-positive for the conjectured intervals by employing the combinatorial Murnaghan–Nakayama rule involving ribbon tableaux (see for example [4, Chapter 7.17]). We prove that all of the partial sums are Schur-positive.

Proposition 3.10.

(1) *For $n < 7$ and $n = 8$, $p_{A_{(n-2,2)}}$ is Schur-positive.*
(2) *For $n \geqslant 9$, $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(n-2,2)}$ is Schur-positive.*

Proposition 3.11.

(1) *For $n < 9$, $p_{A_{(n-2,1,1)}}$ is Schur-positive.*
(2) *For $n \geqslant 9$, $p_{(1^n)} + p_{(2,1^{n-2})} + p_{(2,2,1^{n-4})} + p_{(3,1^{n-3})} + p_{(n-2,1,1)}$ is Schur-positive.*

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