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THE COMBINATORICS OF MOTZKIN POLYOMINOES

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Abstract A word $w = w_1 \cdots w_n$ over the set of positive integers is a Motzkin word whenever $w_1 = \mathtt{1}$, $1 \leq w_k \leq w_{k-1} + 1$, and $w_{k-1} \neq w_k$ for $k = 2, \ldots, n$. It can be associated to a n-column Motzkin polyomino whose i-th column contains w_i cells, and all columns are bottom-justified. We provide generating functions with respect to the length, area, semiperimeter, value of the last symbol, and number of interior points of Motzkin polyominoes. We give asymptotics and close expressions for the total of those statistics among all polyominoes of a given length. For details and other results see [2].

1. Introduction

Motzkin words were introduced by Mansour and Ramírez in [5]. Specifically, a word $w = w_1 w_2 \cdots w_n$ of length n over the set of positive integers is called a *Motzkin word* whenever $w_1 = \mathtt{1}$, $1 \leq w_k \leq w_{k-1} + 1$, and $w_{k-1} \neq w_k$ for $k = 2, \ldots, n$. The empty word ϵ is the only one word of length 0. For $n \geq 0$, let $\mathcal{M}_n$ denote the set of Motzkin words of length n. For example, $\mathcal{M}_5 = \{\mathtt{12121}, \mathtt{12123}, \mathtt{12312}, \mathtt{12321}, \mathtt{12323}, \mathtt{12341}, \mathtt{12342}, \mathtt{12343}, \mathtt{12345}\}$.

A Motzkin word $w = w_1 \cdots w_n$ can also be viewed as a polyomino (also called bargraph) whose i-th column contains w_i cells for $1 \leq i \leq n$, and where all columns are bottom-justified. The polyomino associated to a Motzkin word of length n is called a *Motzkin polyomino* of length n.

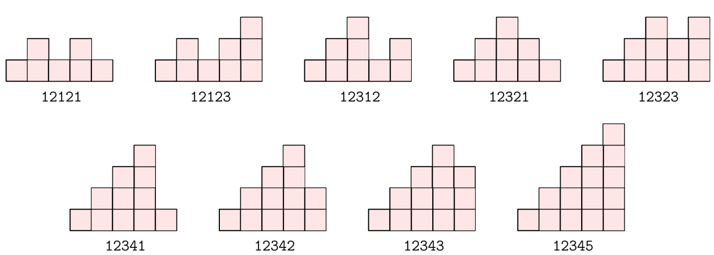

Figure 1. Motzkin words of length 5 and their associated Motzkin polyominoes.

Let w be a Motzkin word and $P(w)$ its associated polyomino. We denote by $\mathtt{area}(w)$ the number of cells (or the *area*) of $P(w)$. The *semiperimeter* of $P(w)$, denoted $\mathtt{sper}(w)$, is half of the *perimeter* of $P(w)$. An *interior-vertex* of $P(w)$ is a point that belongs to exactly four cells of $P(w)$. We denote by $\texttt{inter}(w)$ the number of interior points of $P(w)$. For instance, if $w = \texttt{12341}$, then $\texttt{area}(w) = 11$, $\texttt{sper}(w) = 9$, and $\texttt{inter}(w) = 3$.

2. Area and semiperimeter statistics

For all $1 \leq i \leq n$, let $\mathcal{M}_{n,i}$ be the set of Motzkin words of length n whose last symbol is i. We define the generating functions

$$A_i(x;p,q) := \sum_{n\geq 1} x^n \sum_{w\in\mathcal{M}_{n,i}} p^{\texttt{sper}(w)} q^{\texttt{area}(w)} \text{ and } A(x;p,q;v) := \sum_{i\geq 1} A_i(x;p,q)v^{i-1}.$$

Theorem 2.1. *The generating function $A(x;p,q;v)$ satisfies the functional equation*

$$A(x;p,q;v) = p^2qx + \frac{pqx}{1-qv}A(x;p,q;1) + \left(p^2q^2xv - \frac{pqx}{1-qv}\right)A(x;p,q;qv). \tag{1}$$

Note that when $p = q = 1$, we have $\left(1 - xv + \frac{x}{1-v}\right)A(x;1,1;v) = x + \frac{x}{1-v}A(x;1,1;1)$. By differentiating $A(x;1,1;v)\cdot v$ at $v = 1$, we deduce:

Corollary 2.2. *The generating function for the total sum g_n of the last symbol in all Motzkin words of length n is $\frac{1-x-2x^2-\sqrt{1-2x-3x^2}}{2x^2}$. An asymptotic approximation for the coefficient g_n of x^n is given by $\frac{3\sqrt{3}\left(\frac{1}{n}\right)^{\frac{3}{2}}3^n}{2\sqrt{\pi}}$.*

3. The semiperimeter statistic

By (1) with $q = 1$, we obtain

$$\left(1 - p^2xv + \frac{px}{1-v}\right)A(x;p,1;v) = p^2x + \frac{px}{1-v}A(x;p,1;1). \tag{2}$$

In order to compute $S(x,p) := A(x;p,1;1)$, we use the kernel method (see [1, 6]).

Theorem 3.1. *The generating function for the number of nonempty Motzkin polyominoes according to the length and the semiperimeter is given by*

$$S(x,p) = \frac{1 - p^2x - \sqrt{1 - 2p^2x - 4p^3x^2 + p^4x^2}}{2px}.$$

$$S(x,p) = p^2x + p^4x^2 + (p^5 + p^6)x^3 + (3p^7 + p^8)x^4 + (2p^8 + \boldsymbol{6p^9} + p^{10})x^5 + O(x^6).$$

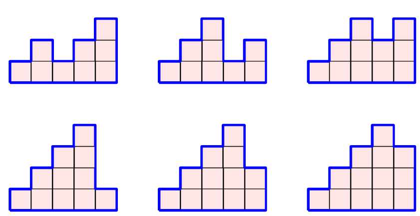

Figure 2. Motzkin polyominoes of length 5 and semiperimeter 9.

Corollary 3.2. *The generating function of the sequence $s(n)$ is given by $\frac{1+x^2-(1+x)\sqrt{1-2x-3x^2}}{2x\sqrt{1-2x-3x^2}}$. An asymptotic approximation of $s(n)$ is given by $\frac{5\sqrt{3}\sqrt{\frac{1}{n}}\,3^n}{6\sqrt{\pi}}$. Moreover, the total semiperimeter over all Motzkin polyominoes of length n is given by $s(n) = T_n + 2T_{n-1} - m_{n-1}$, where m_n is the n-th Motzkin number, and T_n is the central Trinomial coefficient.*

4. The area statistic

By (1) with $p = 1$ we obtain

$$A(x;1,q;v) = qx + \frac{qx}{1-qv}A(x;1,q;1) - \frac{qx(1-qv+q^2v^2)}{1-qv}A(x;1,q;qv). \tag{3}$$

By iterating this equation an infinite number of times (here we assume $|x|< 1$ or $|q|< 1$), we obtain

$$A(x;1,q;v) = \sum_{j\geq 1}(-1)^{j-1}q^jx^j\left(1+\frac{A(x;1,q;1)}{1-q^jv}\right)\prod_{i=1}^{j-1}\frac{1-q^iv+q^{2i}v^2}{1-q^iv}.$$

Theorem 4.1. *The generating function $U(x,q) := A(x;1,q;1)$ for the number of nonempty Motzkin polyominoes according to the length and the area is given by*

$$U(x,q) = \frac{\sum_{j\geq 1}(-1)^{j-1}q^jx^j\prod_{i=1}^{j-1}\frac{1-q^i+q^{2i}}{1-q^i}}{1-\sum_{j\geq 1}(-1)^{j-1}\frac{q^jx^j}{1-q^jv}\prod_{i=1}^{j-1}\frac{1-q^i+q^{2i}}{1-q^i}}.$$

Theorem 4.2. *The total area over all Motzkin polyominoes of length n is given by $u(n) = \frac{1}{2}\left(3^n - T_n\right) = \frac{1}{2}\left(3^n - \sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}\right)$, and an asymptotic is $3^n/2$.*

The sequence $u(n)$ (A055217) corresponds to the sum of first n coefficients of $(1+x+x^2)^n$. Enigmatically, the monomials from the first part of the expansion of $(1+x+x^2)^n$ can be literally written on the cells of all Motzkin polyominoes of size n in a simple one-to-one manner. Term x^k goes onto a cell of height $n-k$.

Let $T(n,i)$ be the i-th coefficient in the expansion of $(1+x+x^2)^n$. One has that

$$T(n,i) = T(n-1,i) + T(n-1,i-1) + T(n-1,i-2), \quad 0 \leq i \leq n-2, \tag{4}$$

and $T(n,n-1) = T_{n-1} + T(n-1,n-2) + T(n-1,n-3)$, where T_n is the central trinomial coefficient, that is $T_n = T(n,n)$.

Let w be a Motzkin word. We denote by $\mathtt{h}_i(w)$ the number of cells of height i in the Motzkin polyomino associated with w. We introduce the following generating functions

$$H_i(x,q) := 1 + \sum_{w\in\mathcal{M}} x^{|w|}q^{\mathtt{h}_i(w)} \text{ and } B_i(x) := \left.\frac{\partial H_i(x,q)}{\partial q}\right|_{q=1}.$$

Theorem 4.3. *For $i \geq 2$, we have*

$$H_i(x,q) = \frac{1+x}{1-(H_{i-1}(x,q)-x)},$$

and $H_1(x,q) = 1 + qxM(xq)$, where $M(x)$ is the generating function of the Motzkin numbers.

For example, for $i=3$ we have the expression

$$H_3(x,q) = 1 + x + x^2 + (1+q)x^3 + (1+2q+q^2)x^4 + (\mathbf{1+3q+3q^2+2q^3})x^5 + O(x^6).$$

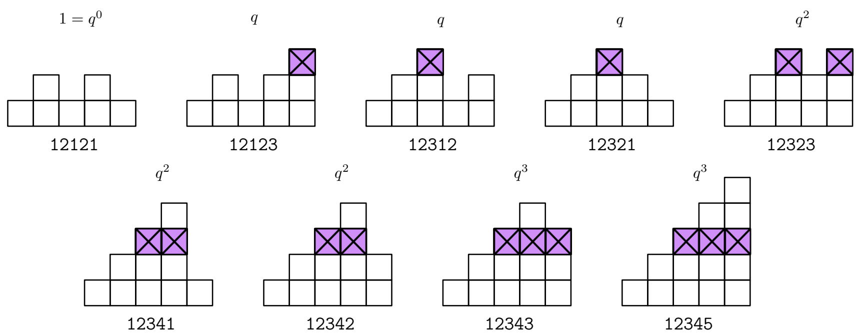

FIGURE 3. Motzkin polyominoes of length 5 and the cells of height 3.

Let $G_i(x) = \sum_{n\geq 0} g(n,i)x^n$ be the generating function for the number of *grand Motzkin paths* of length n ending at height i, that is, $g(n,i) = g(n-1,i-1) + g(n-1,i) + g(n-1,i+1)$.

Theorem 4.4. *For all $i \geq 1$, $G_i(x) = B_i(x) = \frac{x^i M^i(x)}{1-x-2x^2M(x)}$. Moreover, For $n>0, i>1$, we have*

$$h(n,i) = h(n-1,i-1) + h(n-1,i) + h(n-1,i+1),$$

$h(n,1) = n \cdot [x^{n-1}]M(x)$.

From Theorem 4.4 and (4), it is possible to verify that the sequence $T(n,i)$ and $h(n,n-i)$ satisfy the same recurrence relation with the same initial values, therefore they are the same.

Theorem 4.5. *The number of cells of height $n-i$ $(0 \leq i < n)$ in all Motzkin polyominoes of length n is given by the trinomial coefficient $T(n,i)$.*

5. The interior points statistic

We define the generating functions

$$A_i(x;q) := \sum_{n\geq 1} x^n \sum_{w\in\mathcal{M}_{n,i}} q^{\texttt{inter}(w)} \text{ and } A(x;q;v) := \sum_{i\geq 1} A_i(x;q)v^{i-1}.$$

Theorem 5.1. *The generating function $A(x;q;v)$ is given by*

$$A(x;q;v) = \sum_{j\geq 1} x^j \left(1 + A(x;q;1)\frac{1}{1-q^jv}\right) \prod_{i=1}^{j-1} \left(xq^{i-1}v - \frac{x}{1-q^iv}\right).$$

By setting $v=1$ in Theorem 5.1, and solving for $A(x;q;1)$ we can state the following result.

Corollary 5.2. *The generating function $H(x,q) := A(x;q;1)$ for the number of nonempty Motzkin polyominoes according to the length and the number of interior points is given by*

$$H(x,q) = \frac{\sum_{j\geq 1} x^j \prod_{i=1}^{j-1}\left(q^{i-1} - \frac{1}{1-q^i}\right)}{1 - \sum_{j\geq 1}\frac{x^j}{1-q^j}\prod_{i=1}^{j-1}\left(q^{i-1} - \frac{1}{1-q^i}\right)}.$$

$$H(x,q) = x + x^2 + (1+q)x^3 + (1+q+q^2+q^3)x^4 + (\mathbf{1+2q+q^2+2q^3+q^4+q^5+q^6})x^5 + O(x^6).$$

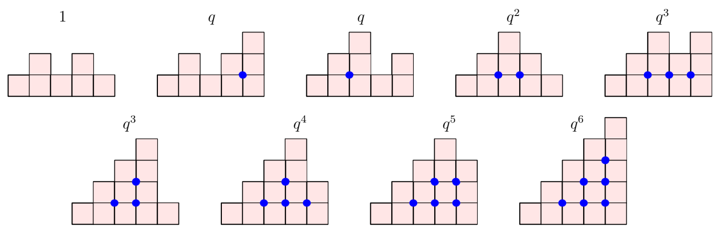

FIGURE 4. Motzkin polyominoes of length 5 and their weighted interior points.

Corollary 5.3. *The generating function of the number of interior points over all Motzkin polyominoes of length n is $\frac{2-3x-5x^2-(2-x-2x^2)\sqrt{1-2x-3x^2}}{2x(1+x)(1-3x)}$, and an asymptotic for the n-th coefficient of the series expansion is $3^n/2$. The expected value of the number of interior points is $\sqrt{\frac{\pi}{3}}n^{3/2}$. Moreover, the number of interior points over all Motzkin polyominoes of length n is $\frac{1}{2}(3^n - 3T_n) - 2T_{n-1} + 2m_{n-1}$.*

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