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A COLLECTION OF SUMS OVER CATALAN PATHS AND ITS CONNECTION WITH SOME PARKING FUNCTION ENUMERATION PROBLEMS

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Abstract In this talk, we discuss part of our recent work in [6], which investigates a collection of sums over the set $\mathcal{C}_n$ of Catalan paths of length $2n$ that turns out to be connected to some enumeration problems involving parking functions. This extended abstract is organised as follows. In Section 1, we define this collection of sums over $\mathcal{C}_n$ and examine their evaluations. Section 2 focuses on the problem of pattern avoidance in parking functions, where we show that some cases of this problem reduce to evaluating sums of this type. In Section 3, we consider the problem of counting congruence classes of generalised parking functions under two different notions of congruences, and show that this is also equivalent to evaluating the above type of sums.

1. A collection of sums over Catalan paths

Let $\mathcal{C}_n$ be the set of Catalan paths of length $2n$. For each $C \in \mathcal{C}_n$, let $\mathbf{u}(C)$ be the vector recording the length of each block of consecutive up-steps in C, and let $|\mathbf{u}(C)|$ be the number of entries in $\mathbf{u}(C)$. For example, for $C = UUDUDUUUDDDD \in \mathcal{C}_6$, $\mathbf{u}(C) = (2, 1, 3)$ and $|\mathbf{u}(C)| = 3$. In what follows, we consider sums over all $C \in \mathcal{C}_n$, each of whose summand is a product whose terms are expressions of entries of $\mathbf{u}(C)$.

An example of such a sum and its evaluation is the following.

Theorem 1.1.

$$p_n = \sum_{C\in\mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} \mathbf{u}(C)_i = \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} \binom{n+k-1}{2k-1}. \tag{1}$$

This can be proved as follows. For $k \in [n]$, let $\mathcal{C}_{n,k}$ be the set of Catalan paths of length $2n$, which begins with a block of k up-steps, or equivalently, $\mathcal{C}_{n,k} = \{C \in \mathcal{C}_n \mid \mathbf{u}(C)_1 = k\}$. For each $C \in \mathcal{C}_{n,k}$, it is easy to see that C has a unique decomposition of the form $C = U_1 \cdots U_k D_1 C_1 \cdots D_k C_k$, where $U_1, \cdots, U_k$ are the first k consecutive up-steps in C, $D_1, \cdots, D_k$ are down-steps, and $C_1, \cdots, C_k$ are themselves Catalan paths, possibly of length 0. We call this the *canonical decomposition* of C. Moreover, for any $n \geq k \geq 1$ and any Catalan paths $C_1, \cdots, C_k$, possibly of length 0, with total length $2(n-k)$, $C = U_1 \cdots U_k D_1 C_1 \cdots D_k C_k$ is a Catalan path in $\mathcal{C}_{n,k}$. Hence, the map $C \mapsto (C_1, \cdots, C_k)$ defined in this way is a bijection, and it satisfies that $\mathbf{u}(C)$ is obtained by attaching $\mathbf{u}(C_1), \cdots, \mathbf{u}(C_k)$ together in order, and adding an entry of k to the front.

From this, using standard generating function methods, it follows that the generating function $P(x) = \sum_{n\geq 0} p_n x^n$ satisfies

$$P(x) = 1 + \sum_{k\geq 1} k x^k (P(x))^k = 1 + \frac{xP(x)}{(1-xP(x))^2}.$$

The formula for p_n can then be obtained by applying the Lagrange Implicit Function Theorem.

Using the same canonical decomposition, we can also evaluate the following similar sums.

Theorem 1.2.

$$\sum_{C\in\mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} (\mathbf{u}(C)_i)! = \frac{1}{n+1} \sum_{\substack{a_1+\cdots+a_{n+1}=n \\ a_1,\cdots,a_{n+1}\geq 0}} a_1! \cdots a_{n+1}!, \tag{2}$$

$$\sum_{C\in\mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} (1 + m\mathbf{u}(C)_i) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} \binom{3n+1-k}{2n+1} (m-1)^k. \tag{3}$$

With some slight modifications, we can also evaluate the following variants of (3), where either the first or the last term in the product is omitted.

Theorem 1.3.

$$\sum_{C\in\mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} (1 + \mathbf{u}(C)_i) = \frac{\binom{3n+1}{n}}{n+1} - \sum_{k=0}^{n-1} \frac{\binom{3n-3k+1}{n-k}}{2^{k+1}(n-k+1)}, \tag{4}$$

$$\sum_{C\in\mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} (1 + m\mathbf{u}(C)_i) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{3n-k}{2n+1} (m-1)^k. \tag{5}$$

The more general version of (4) with $1+m\mathbf{u}(C)_i$ instead of $1+\mathbf{u}(C)_i$ in each product can also be evaluated, but we omit it here as the its expression is more complicated and it is not relevant to the applications below.

A family of sums that is more difficult to deal with is the following.

Theorem 1.4. *Fix $m \geq 1$ and let*

$$p_n = \sum_{C\in\mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right).$$

Then, $p_n = \sum_{k=1}^{n} p_{n,k}$*, where*

$$p_{n,k} = \begin{cases} 1, & \text{if } k = n, \\ (1+m(n-k)) \displaystyle\sum_{i=n-k}^{n-1} \sum_{j=k+1-n+i}^{i} p_{i,j}, & \text{if } 1 \leq k \leq n-1. \end{cases} \tag{6}$$

It follows that the sequence p_n *satisfies*

$$\frac{x}{1-x} = \sum_{n=1}^{\infty} p_n \frac{x^n(1-x)^n}{\prod_{\ell=1}^{n}(1+m\ell x)}. \tag{7}$$

Note that the terms in each product is now a cumulative sum of the entries in $\mathbf{u}(C)$, unlike the other sums we have seen so far. As such, the canonical decomposition technique we used for those sums no longer apply. Instead, we consider the following alternate decomposition of Catalan paths.

Let $p_{n,k} = \sum_{C \in \mathcal{C}_{n,k}} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right)$, so that $p_n = \sum_{k=1}^{n} p_{n,k}$. Note that $p_{n,n} = 1$ as $\mathcal{C}_{n,n}$ contains only the Catalan path consisting of n up-steps followed by n down-steps. For $n > k \geq 1$ and $C \in \mathcal{C}_{n,k}$, we have $|\mathbf{u}(C)| \geq 2$. Let $n-k \leq i \leq n-1$ be such that the length of the first block of down-steps in C is $n-i$ and let C' be obtained by deleting the last $n-i$ up-steps in the first block of up-steps in C, as well as the first $n-i$ down-steps following it. It is easy to see that $C' \in \mathcal{C}_{i,j}$ for some $k+1-n+i \leq j \leq i$. We say that C' is obtained by *deleting the first peak* of C. It is also straightforward to verify that this process is reversible, from which it follows that $p_{n,k} = (1+m(n-k)) \sum_{i=n-k}^{n-1} \sum_{j=k+1-n+i}^{i} p_{i,j}$.

To obtain (7), we first show, by manipulating the recurrence relation algebraically, that the functions $S_n(x) = \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} p_{i,i-j} x^{i-1}$ satisfy $S_n(x) = \frac{1+mnx}{1-x} S_{n-1}(x) - p_n x^{n-1}$. Solving for p_n and substituting into the right hand side of (7), we see that the sum telescopes and what remains after simplification is precisely $\frac{x}{1-x}$.

It remains to be seen whether a more explicit formula for p_n can be found. Many other sums of this type and their potential applications could be considered in the future, but we now pivot to the connections of the above sums to several parking function enumeration problems.

2. Pattern avoidance in parking functions

A function $f : [n] \to [n]$ is a *parking function* of *size* n if for all $i \in [n]$, $|\{j \mid f(j) \leq i\}| \geq i$. The study of pattern avoidance in parking functions is motivated by the notion of pattern avoidance in permutations. For $m \leq n$ and $\sigma \in S_m$, $\pi \in S_n$, we say that π *contains* σ *as a pattern* if there exists $1 \leq i_1 < \cdots < i_m \leq n$, such that $\pi(i_a) < \pi(i_b)$ if and only if $\sigma(a) < \sigma(b)$ for all $a, b \in [m]$, and we say π *avoids* σ otherwise. For any collection $\sigma_1, \cdots, \sigma_k$ of permutations, we denote by $\mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)$ the set of all permutations in S_n containing none of $\sigma_1, \cdots, \sigma_k$ as a pattern. Depending on how one associates a permutation to each parking function, there are several possible notions of pattern avoidance in parking functions.

One such notion looks at the final parking positions. For a parking function $f : [n] \to [n]$, the *parking permutation* associated to f is defined to be the permutation $\rho_f \in S_n$ satisfying that the i-th spot in the parking lot is occupied by the $\rho_f(i)$-th car. Note that different parking functions could have the same associated parking permutation.

For a collection $\sigma_1, \cdots, \sigma_k$ of permutations, let $\mathrm{Pk}_n(\sigma_1, \cdots, \sigma_k)$ be the set of parking function $f : [n] \to [n]$ such that its associated parking permutation ρ_f contains none of $\sigma_1, \cdots, \sigma_k$ as a pattern, or equivalently, $\mathrm{Pk}_n(\sigma_1, \cdots, \sigma_k) = \{f : [n] \to [n] \mid f \text{ is a parking function and } \rho_f \in \mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)\}$. Let $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k) = |\mathrm{Pf}_n(\sigma_1, \cdots, \sigma_k)|$.

We have the following key lemma, a proof of which can be found in [2], and its immediate corollary.

Lemma 2.1. *For any $\rho \in S_n$ and $i \in [n]$, let $\ell(i, \rho) = \max\{\ell \mid \rho(j) \le \rho(i) \text{ for all } i - \ell + 1 \le j \le i\}$, and let $\ell(\rho) = \prod_{i=1}^n \ell(i, \rho)$. Then, the number of parking function $f : [n] \to [n]$ with $\rho_f = \rho$ is $\ell(\rho)$.*

Corollary 2.2.

$$\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k) = \sum_{\rho \in \mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)} \ell(\rho) = \sum_{\rho \in \mathrm{Av}_n(\sigma_1, \cdots, \sigma_k)} \prod_{i=1}^n \ell(i, \rho).$$

In [6], using the results above, we systematically compute the values of $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k)$ for all collections $\sigma_1, \cdots, \sigma_k$ of permutations in S_3. Most cases are relatively straightforward, but of interest to us here are the computations of $\mathrm{pk}_n(\sigma)$, with $\sigma \in \{123, 213, 312, 321\}$. In [4], Krattenthaler defined a bijection between $\mathrm{Av}_n(123)$ and $\mathcal{C}_n$. We can also define similar bijections between $\mathrm{Av}_n(\sigma)$ and $\mathcal{C}_n$ for each $\sigma \in \{213, 312, 321\}$. It turns out that these four bijections further satisfy the property that if $\rho \in \mathrm{Av}_n(\sigma)$ corresponds to $C \in \mathcal{C}_n$, then $\ell(\rho) = \prod_{i=1}^n \ell(i, \rho)$ can be rewritten as a product whose terms are expressions of entries of $\mathbf{u}(C)$. Specifically, we have the following results connecting $\mathrm{pk}_n(\sigma)$ to sums considered earlier in Section 1.

Theorem 2.3.

$$\mathrm{pk}_n(123) = \sum_{\rho \in \mathrm{Av}_n(123)} \ell(\rho) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} \mathbf{u}(C)_i. \tag{8}$$

$$\mathrm{pk}_n(213) = \sum_{\rho \in \mathrm{Av}_n(213)} \ell(\rho) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|} (\mathbf{u}(C)_i)!. \tag{9}$$

$$\mathrm{pk}_n(312) = \sum_{\rho \in \mathrm{Av}_n(312)} \ell(\rho) = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right). \tag{10}$$

$$\mathrm{pk}_n(321) = \sum_{\rho \in \mathrm{Av}_n(321)} \ell(\rho) = \sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right) \prod_{i=1}^{|\mathbf{u}(C)|} (\mathbf{u}(C)_i - 1)!. \tag{11}$$

Note that the right-most sums in (8) and (9) are exactly the ones evaluated in (1) and (2), while the right-most sum in (10) is the $m = 1$ special case of the sums considered in Theorem 1.4, and thus satisfies the corresponding versions of (6) and (7). The right-most sum in (11) is quite complicated, though a recurrence formula analogous to (6) can be obtained as well.

We remark that our computations of $\mathrm{pk}_n(\sigma_1, \cdots, \sigma_k)$ for all collections $\sigma_1, \cdots, \sigma_k$ of permutations in S_3 is motivated by the work of Adeniran and Pudwell in [1], where they obtained analogous results using a different notion of pattern avoidance in parking functions, and denoted the corresponding values by $\mathrm{pf}_n(\sigma_1, \cdots, \sigma_k)$. One of their result is the following.

Theorem 2.4.

$$\mathrm{pf}_n(312, 321) = \sum_{C \in \mathcal{C}_n} \prod_{i=1}^{|\mathbf{u}(C)|-1} (1 + \mathbf{u}(C)_i). \tag{12}$$

Note that the sum on the right hand side of (12) is exactly evaluated by our result in (4). As such, we obtain a more explicit formula for $\mathrm{pf}_n(312, 321)$.

3. Congruence classes in generalised parking functions

For a function $f : [n] \to [N]$, the *evaluation* of f is the sequence $\mathrm{ev}(f)$ of length N, whose j-th entry is the number of $i \in [n]$ with $f(i) = j$. The *packed evaluation* of f is the sequence $\mathrm{pev}(f)$ obtained by removing all the zero entries from $\mathrm{ev}(f)$.

For $m \geq 1$, an *m-multiparking function* of *size mn* is a function $f : [mn] \to [n]$ such that there exists an ordinary parking function $\overline{f}$ satisfying $\mathrm{ev}(f) = m\,\mathrm{ev}(\overline{f})$, and an *$m$-parking function* of *size n* is a function $f : [n] \to [1 + m(n-1)]$ satisfying $f(i) \leq 1 + m(i-1)$ for all $i \in [n]$.

In [5], Novelli and Thibon studied Hopf algebras of generalised parking functions. One way to obtain a new Hopf algebra is to take the quotient under certain congruences of generalised parking functions. Therefore, the number of congruence classes of generalised parking functions, which is the dimension of the Hopf algebra obtained in this way, is of great interest.

In what follows, we consider what is called the hyposylvester congruence and the metasylvester congruence. For their precise definitions, we refer the audience to [5]. It is clear from these definitions that functions in the same hyposylvester class or the same metasylvester class have the same evaluation and thus the same packed evaluation. Moreover, Novelli and Thibon [5] showed that

Lemma 3.1. *For every possible evaluation α of an m-multiparking function or an m-parking function, the number of hyposylvester classes that contains m-multiparking functions or m-parking functions with common evaluation α depends only on their common packed evaluation β, and is equal to $\prod_{i=2}^{|\beta|}(1 + \beta_i)$.*

For every possible evaluation α of an m-multiparking function or an m-parking function, the number of metasylvester classes that contains m-multiparking functions or m-parking functions with common evaluation α depends only on their common packed evaluation β, and is equal to $\prod_{i=2}^{|\beta|}(1 + \sum_{j=i}^{|\beta|} \beta_j)$.

Further, from Garsia and Haiman [3], the set of increasing ordinary parking functions of size n are in bijection with $\mathcal{C}_n$. This can be generalised to a bijection between increasing m-parking functions of size n with the set $\mathcal{C}_n^{(m)}$ of m-Catalan paths of length $(m+1)n$. We also have the property that if an increasing m-parking function f corresponds to $C \in \mathcal{C}_n^{(m)}$ under this bijection, then $\mathrm{pev}(f) = \mathbf{u}(C)$. Finally, putting all of these together, and noting that permuting the entries of a function f does not change $\mathrm{ev}(f), \mathrm{pev}(f)$, or whether f is an m-multiparking function or an m-parking function, we have

Theorem 3.2. *The number of hyposylvester classes of m-multiparking functions is*

$$\sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m\mathbf{u}(C)_i\right). \tag{13}$$

The number of metasylvester classes of m-multiparking functions is

$$\sum_{C \in \mathcal{C}_n} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + m \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right). \tag{14}$$

With $\mathbf{u}(C)$ defined analogously for all $C \in \mathcal{C}_n^{(m)}$, the number of hyposylvester classes of m-parking functions is

$$\sum_{C \in \mathcal{C}_n^{(m)}} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + \mathbf{u}(C)_i\right). \tag{15}$$

The number of metasylvester classes of m-parking functions is

$$\sum_{C \in \mathcal{C}_n^{(m)}} \prod_{i=2}^{|\mathbf{u}(C)|} \left(1 + \sum_{j=i}^{|\mathbf{u}(C)|} \mathbf{u}(C)_j\right). \tag{16}$$

Note that the sum (13) is exaclty evaluated in (5), while the sum (14) is considered in Theorem 1.4 and thus satisfies (6) and (7). Using analogous version of the canonical decomposition for $C \in \mathcal{C}_n^{(m)}$, we can also evaluate the sum (15) to be $\frac{1}{2mn+1}\binom{(2m+1)n}{n}$. Unfortunately, the removing the first peak method used in Theorem 1.4 does not translate well to the setting of $\mathcal{C}_n^{(m)}$, so the sum (16) remains tricky to evaluate.

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