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ON HAMILTONIAN REGULAR GRAPHS AND TWO CONJECTURES OF HAYTHORPE

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1. Introduction

Motivated by Sheehan's conjecture stating that every Hamiltonian 4-regular graph contains at least two Hamiltonian cycles [13], Fleischner's surprising result that there exist graphs in which every vertex has degree 4 or 14 and which contain exactly one Hamiltonian cycle [4], as well as recent work of Haythorpe [10], in this talk we investigate bounds for the minimum number of Hamiltonian cycles occurring in Hamiltonian graphs, typically under the assumption that the graphs we treat are regular. For further results treating Sheehan's conjecture and its variations we refer to recent articles and preprints on the subject, see [5, 6, 20, 1, 12].

It is natural to treat a more general form of Sheehan's conjecture, where "4" is replaced by some integer $d \geq 3$. Thomason, via his "lollipop" method, showed that for every *odd* integer $d \geq 3$, every Hamiltonian d-regular graph has at least three pairwise distinct Hamiltonian cycles [14]. Thomassen observed that by Thomason's result, Sheehan's conjecture implies that every Hamiltonian regular graph that is not a cycle contains at least two Hamiltonian cycles. Using an ingenious proof, the statement "every Hamiltonian d-regular graph has a second Hamiltonian cycle" was verified by Thomassen [17] for all $d > 72$ and subsequently by Haxell, Seamone, and Verstraëte [9] for all $d > 22$ using a refinement of Thomassen's method.

In 2019, Girão, Kittipassorn, and Narayanan [5] proved that if an n-vertex graph with minimum degree at least 3 has a Hamiltonian cycle, then it contains another cycle of length at least $n - cn^{4/5}$, where $c > 0$ is an absolute constant. Already in 1980, Entringer and Swart [3] had described an infinite family of graphs containing exactly one Hamiltonian cycle, i.e. *uniquely Hamiltonian* graphs, in which exactly two vertices are 4-valent and all other vertices are cubic. In the same paper, they asked whether Sheehan's conjecture extends to graphs of minimum degree 4. The answer turns out to be negative, as proven by Fleischner [4]. He showed that there exist infinitely many uniquely Hamiltonian graphs in which every vertex has degree 4 or 14. Very recently, Brinkmann [1] showed that for any set $M = \{4, d_1, d_2, \ldots, d_k\}$ with $10 \leq d_1 < d_2 < \ldots < d_k$ and $k \geq 1$ there are infinitely many uniquely

Hamiltonian graphs G with $M = \{\deg(v) : v \in V(G)\}$. We now turn to the main subject of our talk, namely two recent conjectures of Michael Haythorpe.

2. Haythorpe's conjectures

In 2018, Haythorpe [10] made two conjectures about the minimum number of Hamiltonian cycles in Hamiltonian regular graphs.

Conjecture 1 (Conjecture 4.2 in Haythorpe's article [10], 2018)**.** *For $n \geq 8$, all Hamiltonian 4-regular graphs of order n have at least $9 \cdot 2^{\frac{n+2}{6}}$ Hamiltonian cycles.*

It turns out that Conjecture 1 is not true in a strong sense, neither for 4-connected graphs (of large genus) nor planar 3-connected graphs:

Theorem 1 (Thomassen and Zamfirescu [18], 2021)**.** *There exists a constant $c > 0$ such that there are infinitely many 4-regular 4-connected graphs, each containing exactly c Hamiltonian cycles.*

Theorem 2 (Zamfirescu [20], 2022)**.** *There exists a constant $c > 0$ such that there are infinitely many 4-regular 3-connected planar graphs, each containing exactly c Hamiltonian cycles.*

However, we note that if in Theorem 2 we increase the connectedness to 4, the behaviour changes, as proven via the Counting Base Lemma and tools such as Tutte cycles and circuit graphs:

Theorem 3 (Brinkmann and Van Cleemput [2], 2021)**.** *Let $\mathcal{G}$ be the family of all planar 4-connected graphs. Then there is a constant $c > 0$ such that each $G \in \mathcal{G}$ has at least $c \cdot |V(G)|$ Hamiltonian cycles.*

In [10] Haythorpe also published the following conjecture.

Conjecture 2 (Conjecture 3.1 in Haythorpe's paper [10], 2018)**.** *For $d \geq 5$ and $n \geq d+3$, all Hamiltonian d-regular graphs of order n have at least*

$$f(n,d) := (d-1)^2[(d-2)!]^{\frac{n}{d+1}}$$

Hamiltonian cycles.

In [20], Zamfirescu showed that Conjecture 2 does not hold for $d \in \{5,6,7\}$. By extending the approach from [20], a computer verification for polynomials of small degree, and Stirling's formula, we were able to fully disprove Haythorpe's conjecture, i.e. that it holds for *no* integer $d \geq 5$. Let $h_n(d)$ be the minimum number of Hamiltonian cycles in any Hamiltonian d-regular graph of order n.

Theorem 4 (Goedgebeur, Jooken, Lo, Seamone, and Zamfirescu [7], 2024+)**.** *For any integers $k \geq 0$ and $d \geq 5$, we have*

$$h_n(d) \leq 2[(d-1)!]^{d-2}[(d-2)!]^k < f(n,d),$$

where $n := d^2 + d - 4 + (d+1)k$.

In [7] we also answer questions of Thomassen [17] and Haxell, Seamone, and Verstraëte [9] on the limitations of Thomassen's so-called "independent dominating set" method.

For fixed r and regarding n as free parameter, all upper bounds which appeared in the literature up to this point were asymptotically $O(\sqrt[d+1]{((d-2)!)^n})$. Jooken [12] recently described graphs that provide exponential improvements to this asymptotic upper bound, i.e. these graphs yield a smaller base for the exponent. This shows that Conjecture 2 does not only involve an incorrect constant factor, but is also asymptotically incorrect.

Theorem 5 (Jooken [12], 2024+)**.** *For every integer $d \geq 5$, the minimum number of Hamiltonian cycles in an n-vertex d-regular graph is asymptotically*

$$O\left(\sqrt[3d+1]{(2d-8)((d-4)!)^2)((d-1)!)^n}\right).$$

Coming back to Sheehan's conjecture, Goedgebeur, Meersman, and Zamfirescu [6] determined computationally for every $n \in \{5, \ldots, 21\}$ the minimum number h of Hamiltonian cycles occurring in a Hamiltonian 4-regular graph on n vertices:

n	5	6	7	8	9, 10	11	12	13, 14, 15, 16	17	18	19, 20, 21
h	12	16	23	29	36	48	60	72	96	108	144

These numbers suggest that the minimum number of Hamiltonian cycles continues to increase when one increases the order of the graphs in question. This is however not the case:

Theorem 6 (Zamfirescu [20], 2022)**.** *For every integer $n \geq 19$, there exists a 4-regular graph on n vertices with exactly 144 Hamiltonian cycles.*

I believe 144 to be optimal for sufficiently large orders. Let us conclude with further open questions.

3. Open problems

Problem (Thomassen and Zamfirescu [18]): *Is there an infinite family of Hamiltonian 5-regular 5-connected graphs with a bounded number of Hamiltonian cycles?*

Conjecture (Girão, Kittipassorn, and Narayanan [5]): *If an n-vertex graph G with minimum degree at least 3 contains a Hamiltonian cycle, then G contains another cycle of length at least $n - K$, where $K > 0$ is an absolute constant.*

An old conjecture of Thomassen [15] is equivalent to the above conjecture for $K = 1$. The next conjecture, due to Cantoni, is also long-standing:

Conjecture (Cantoni [19]): *Every planar 3-regular graph with exactly three Hamiltonian cycles contains a triangle.*

Problem (Zamfirescu [20]): *Are there 3-regular 2-connected triangle-free graphs containing a unique longest cycle?*

It has been verified computationally that no counterexample to Cantoni's Conjecture has fewer than 50 vertices, and that there are no graphs of order less than 30 giving an affirmative answer to the last question.

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