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THE STOCHASTIC SANDPILE MODEL ON COMPLETE BIPARTITE GRAPHS

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Abstract The stochastic sandpile model (SSM) generalises the standard Abelian sandpile model (ASM) by making topplings of unstable vertices random. When unstable, a vertex sends one grain to each of its neighbours independently with probability $p \in (0,1)$. We study the SSM on complete bipartite graphs. We characterise recurrent configurations of the model in terms of a simple series of inequalities. This allows us to exhibit a bijection between sorted recurrent configurations and pairs of *compatible* Ferrers diagrams. We also provide a stochastic version of Dhar's burning algorithm to check if a given (stable) configuration is recurrent or not, with linear complexity on sorted configurations.

1. Introduction

The Abelian sandpile model (ASM), originally introduced by Bak, Tang and Wiesenfeld [1, 2], is a random process on a graph, where vertices are assigned a number of grains of sand. At each unit of time, a grain is added to a randomly chosen vertex. If this causes a vertex's number of grains to exceed its degree, the vertex is called *unstable*, and *topples*, sending one grain to each of its neighbours. A special vertex, the *sink*, absorbs grains, and so the process eventually stabilises.

Of central interest in the ASM are the *recurrent configurations* – those which appear infinitely often in the long-time running of the model. A fruitful direction of ASM research has focussed on combinatorial studies of these for graph families with high levels of symmetry, such as complete graphs [5], complete bipartite [10] and multi-partite [4] graphs, complete split graphs [9, 7] (see also [6]), wheel and fan graphs [14], Ferrers graphs [12], permutation graphs [11], and so on.

In the ASM, the only randomness lies in the choice of vertex where grains are added at each time step. After this, the toppling and stabilisation processes are entirely deterministic. In this work, we study a stochastic variant of the ASM, called *stochastic sandpile model* (SSM), as introduced in [3], in which topplings are made according to (biased) random coin flips. The SSM was studied on complete graphs in [13]. We begin by setting some notation and formally defining the model.

As usual, $\mathbb{N}$ denotes the set of strictly positive integers. We let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ denote the set of non-negative integers. For $n \in \mathbb{N}$, we define $[n] := \{1, \ldots, n\}$. For a vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we write $\mathrm{inc}\,(a) = (\tilde{a}_1, \ldots, \tilde{a}_n)$ for the non-decreasing rearrangement of a. In this abstract, we consider the complete bipartite graph $K^0_{m,n}$. This is the graph with vertex set $\{v^t_0, v^t_1, \cdots, v^t_m\} \sqcup \{v^b_1, \cdots, v^b_n\}$ and edge set $\{(v^t_i, v^b_j);\ i \in [m] \cup \{0\}, j \in [n]\}$. We refer to vertices v^t_i, resp. v^b_j, as *top*, resp. *bottom*, vertices in $K^0_{m,n}$. We will use the notation v^*_i to refer to any arbitrary vertex of $K^0_{m,n}$. The vertex v^t_0, called the *sink*, will play a special role in the SSM. Finally, we fix a probability $p \in (0,1)$.

A (sandpile) *configuration* on $K^0_{m,n}$ is a vector $c = (c^t_1, \cdots, c^t_m; c^b_1, \cdots, c^b_n) \in \mathbb{Z}_+^{m+n}$. For simplicity, we write $c = (c^t; c^b)$. We think of c^*_i as the number of grains at vertex v^*_i. We denote by $\mathrm{Config}_{m,n}$ the set of all configurations on $K^0_{m,n}$. A top vertex v^t_i (for $i \in [m]$), resp. bottom vertex v^b_j (for $j \in [n]$), is *stable* if $c^t_i < n$, resp. $c^b_j < m+1$ (i.e. the number of grains at the vertex is less than its degree). A configuration is stable if all of its vertices are stable. The set of stable configurations is denoted $\mathrm{Stable}_{m,n}$. Unstable vertices *topple*. If a top vertex v^t_i is unstable, then for each bottom neighbour v^b_j we draw a Bernoulli random variable B_j with parameter p (the B_j's are independent of each other and of all prior topplings). If $B_j = 1$, then v^b_j receives one grain from v^t_i when it topples, otherwise vertex v^t_i keeps that grain. The process is the same for toppling a bottom vertex v^b_j (here we include the sink v^t_0 in the set of neighbours). The sink never topples, representing the system's exit point.

One can show (see [3, Theorem 2.2]) that, starting from an unstable configuration c and successively toppling unstable vertices, we eventually reach a (random) stable configuration c'. Moreover, the configuration c' reached does not depend on the order in which vertices are toppled. We write $c' = \mathrm{Stab}(c)$ and call it the *stabilisation* of c. Figure 1 shows an example of the stabilisation process for the configuration $c = (2,1;0,2)$. Here, blue vertices are unstable, green edges represent grains being sent from an unstable vertex to a neighbour, while red edges represent no movement of grain.

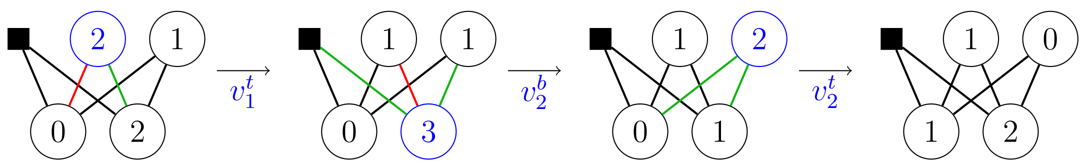

FIGURE 1. Illustrating a possible stabilisation for $c = (2,1;0,2) \in \mathrm{Config}_{2,2}$. Vertices under the arrows represent the vertex being toppled in that phase.

We define a Markov chain on the set $\mathrm{Stable}_{m,n}$. At each step, we add a grain to a non-sink vertex of $K^0_{m,n}$, chosen uniformly at random, and stabilise the resulting configuration. A configuration c is called *recurrent* if it appears infinitely often in the long-time running of this Markov chain. We denote by $\mathrm{Rec}_{m,n}$ the set of recurrent configurations on $K^0_{m,n}$. The following is a consequence of [13, Theorem 2.6] in the complete bipartite graph case.

Theorem 1.1. *Let $c = (c^t; c^b) \in \mathrm{Stable}_{m,n}$ be a stable configuration on $K^0_{m,n}$. Then $c \in \mathrm{Rec}_{m,n}$ if, and only if, for all subsets $A \subseteq [m], B \subseteq [n]$, we have:*

$$\sum_{i \in A} c^t_i + \sum_{j \in B} c^b_j \geq |A| \cdot |B|. \tag{1}$$

If A, B do not *satisfy Inequality* (1)*, we say that (A,B) is a* forbidden subconfiguration*.*

The symmetries of $K^0_{m,n}$ make it natural to study $\mathrm{Rec}_{m,n}$ up to re-ordering in each part. We therefore say that a configuration $c = (c^t; c^b) \in \mathrm{Config}_{m,n}$ is *sorted* if c^t and c^b are both weakly increasing. We denote by $\mathrm{SortedRec}_{m,n}$ the set of sorted recurrent configurations.

A popular statistic on recurrent configurations $c \in \mathrm{Rec}_{m,n}$ is their *level*, defined by:

$$\mathrm{level}(c) := \sum_{i \in [m]} c^t_i + \sum_{j \in [n]} c^b_j - m \cdot n, \tag{2}$$

which satisfies $0 \leq \mathrm{level}(c) \leq m(n-1)$ (see e.g. [13, Equation (8)]).

2. Our results

This section states our main results for the SSM on complete bipartite graphs. We sketch proofs for brevity; full proofs will appear in an upcoming companion paper [15].

2.1. Characterisation of $\mathrm{Rec}_{m,n}$.

We start by exhibiting a necessary and sufficient condition under which a (stable) configuration on $K^0_{m,n}$ is recurrent.

Theorem 2.1. *Let $c = (c^t; c^b) \in \mathrm{Stable}_{m,n}$ be a stable configuration on $K^0_{m,n}$. For $j \in [n]$, define $k_j := |\{i \in [m];\ c^t_i < j\}|$. Then $c \in \mathrm{Rec}_{m,n}$ if, and only if,*

$$\forall j \in [n],\ \tilde{c}^b_1 + \cdots + \tilde{c}^b_j \geq k_1 + \cdots + k_j, \tag{3}$$

where $\mathrm{inc}\left(c^b\right) := \left(\tilde{c}^b_1, \cdots, \tilde{c}^b_n\right)$ is the non-decreasing re-arrangement of c^b. Moreover, if c is recurrent, we have $\mathrm{level}\,(c) = c^b_1 + \cdots + c^b_n - (k_1 + \cdots + k_n)$.

Proof sketch. It is sufficient to show the result when c is sorted, in which case, for $j \geq 0$, we have $c^t_i = j$ if, and only if, $k_j < i \leq k_{j+1}$ (with the convention $k_0 = 0$). Fix $j \in [n]$. We have:

$$\sum_{i=1}^{k_j} c^t_i = 0\cdot(k_1 - k_0) + 1\cdot(k_2 - k_1) + \cdots + (j-1)\cdot(k_j - k_{j-1}) = j\cdot k_j - (k_1 + \cdots + k_j). \tag{4}$$

It follows that Inequality (3) is equivalent to Inequality (1) when we take $A = [k_j], B = [j]$. It therefore suffices to show that if there exists a forbidden subconfiguration (A, B) for c, then there exists j such that $([k_j], [j])$ is a forbidden subconfiguration for c.

For this, take $B \subseteq [n]$ to be a minimal subset such that there exists $A \subseteq [m]$ with (A, B) forbidden. Let $q := \max B$. We claim that for any $q' < q$, we have $c^b_{q'} < |A|$. Otherwise one can check that $(A, [q'] \cap B)$ would be a forbidden subconfiguration (using that $c^b_j \geq c^b_{q'} \geq |A|$ for $j \in (q', q]$), contradicting the minimality of B. This implies that $(A, [q])$ is also forbidden. By a similar argument on A, we can find a forbidden subconfiguration $([p], [q])$. It is then straightforward to check that $([k_q], [q])$ is also forbidden (distinguishing cases $p > k_q$ and $p < k_q$). The level formula follows from the definition (Equation (2)) and Equation (4) with $j = n$, noting that $k_n = m$. ☐

2.2. A stochastic burning algorithm for complete bipartite graphs.

In this part, we exhibit an algorithm to check if a given stable configuration $c \in \mathrm{Stable}_{m,n}$ is recurrent or not, in two steps.

Algorithm 1 Pre-processing: calculating the vector $k = (k_1, \cdots, k_n)$

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Require: c^t = (c^t_1, ··· , c^t_m) ∈ {0, ··· , n − 1}^m
  Initialise: k' = (k'_0, ··· , k'_{n−1}) = (0, ··· , 0); k = (k_1, ··· , k_n) = (0, ··· , 0); sum = 0
  for i from 1 to m do
      k'_{c^t_i} ← k'_{c^t_i} + 1                    ▷ Calculate k'_j := |{i ∈ [m]; c_i = j}|
  end for
  for j from 1 to n do
      sum ← sum + k'_{j−1}; k_j ← sum
  end for
  return k = (k_1, ··· , k_n)
```

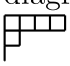
We now describe the stochastic burning algorithm. The terminology *burning algorithm* refers to Dhar's process for the deterministic ASM (see [8, Section 6.2]).

Theorem 2.2. *Algorithm 2 returns True if, and only if, the input (stable) configuration c is recurrent. Moreover, the algorithm runs in $O(m + n\log(n))$ time on unsorted configurations, and $O(m + n)$ time if the bottom part c^b of c is sorted.*

Algorithm 2 Stochastic burning algorithm for complete bipartite graphs

Require: $c = (c^t; c^b) \in \mathrm{Stable}_{m,n}$
  $c^b \leftarrow \mathrm{inc}\left(c^b\right)$ ▷ Sort bottom part c^b of configuration c
  Pre-process: calculate vector $k = (k_1, \cdots, k_n)$ by Algorithm 1
  Initialise: sumK $= 0$; sumC $= 0$
  for j from 1 to n **do**
    sumK $\leftarrow$ sumK $+ k_j$; sumC $\leftarrow$ sumC $+ c^b_j$
    if sumC $<$ sumK **then**
      return False
    end if
  end for
  return True

Proof. The recurrence check follows from Theorem 2.1. The pre-processing of k has complexity $O(m+n)$, sorting c^b has complexity $O(n\log(n))$, and the rest of Algorithm 2 runs in $O(n)$ time. ☐

2.3. **Recurrent configurations as pairs of Ferrers diagrams.** We now present a combinatorial interpretation of $\mathrm{Rec}_{m,n}$ in terms of *Ferrers diagrams.* A Ferrers diagram is a left-aligned collection of cells such that the number of cells in each row is weakly increasing from bottom to top (some rows may be empty). We denote by $\mathrm{Ferrers}_{m,n}$, resp. $\mathrm{Ferrers}_{\leq m,n}$, the set of Ferrers diagrams with m columns, resp. at most m columns, and n rows. The *area* $\mathrm{Area}(F)$ of a Ferrers diagram F is its number of cells. Given a weakly increasing sequence $s = (s_1, \cdots, s_n) \in \mathbb{Z}^n_+$, we denote $F(s) \in \mathrm{Ferrers}_{s_n,n}$ the Ferrers diagram with s_i cells in row i (rows are ordered from bottom to top). For example, $F(0,1,4) :=$  is an element of $\mathrm{Ferrers}_{4,3}$ with area 5.

We consider the following two operations on Ferrers diagrams:

(1) Shift which shifts a cell of the diagram in a given row to some row below.
(2) Add which adds a cell to the right of a given row.

These operations are called *legal* if they still result in a Ferrers diagram (possibly with a different number of columns). Figure 2 illustrates these operations.

Definition 2.3. We say that an ordered pair (F, F') of Ferrers diagrams is *compatible* if F' can be obtained from F through a sequence of legal Shift and Add operations.

We now return to our study of $\mathrm{Rec}_{m,n}$. Note that the vectors $k := (k_1, \cdots, k_n)$ and $\mathrm{inc}\left(c^b\right) = (\tilde{c}_1, \cdots, \tilde{c}_n)$ which appear in Inequality (3) are both weakly increasing. Moreover, we have $k_n = m$ and $\tilde{c}_n \leq m$ (the latter is the stability condition). This yields the following.

Theorem 2.4. *For $c = (c^t; c^b) \in \mathrm{Config}_{m,n}$, let $k = (k_1, \cdots, k_n)$ be as in Theorem 2.1. Define $\Psi(c) := \Big(F(k), F\left(c^b\right)\Big) \in \mathrm{Ferrers}_{m,n} \times \mathrm{Ferrers}_{\leq m,n}$. Then Ψ is a bijection from the set $\mathrm{SortedRec}_{m,n}$ of sorted recurrent configurations on $K^0_{m,n}$ to the set of compatible pairs $(F, F') \in \mathrm{Ferrers}_{m,n} \times \mathrm{Ferrers}_{\leq m,n}$. Moreover, we have $\mathrm{level}\,(c) = \mathrm{Area}\Big(F\left(c^b\right)\Big) - \mathrm{Area}\,(F(k))$.*

Proof sketch. By preceding remarks, if $c \in \mathrm{Rec}_{m,n}$, then $F(k) \in \mathrm{Ferrers}_{m,n}$ and $F\left(c^b\right) \in \mathrm{Ferrers}_{\leq m,n}$. It is reasonably straightforward to see that Inequality (3) implies that $\Big(F(k), F\left(c^b\right)\Big)$ is compatible, and conversely that any compatible pair of Ferrers diagrams satisfies Inequality (3). To show that Ψ is a bijection, we note that the vector k uniquely defines the non-decreasing re-arrangement $\mathrm{inc}\,(c^t)$ of the top part of c. The level formula follows from the level formula in Theorem 2.1. ☐

Example 2.5. Consider the sorted configuration $c = (0, 2, 2; 2, 2, 2)$. We have $k = (1, 1, 3)$, so c is recurrent by Theorem 2.1. Figure 2 illustrates a possible legal sequence of Shift and Add operations to go from the k-diagram (left) to the c^b-diagram (right). Note that this sequence is not unique: we could instead first add a cell in the middle row, then shift a cell from the top to the bottom row.

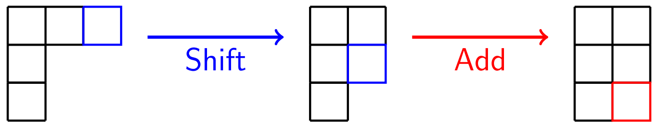


FIGURE 2. Illustrating a legal sequence from $F(1,1,3)$ to $F(2,2,2)$, showing that the configuration $c = (0,2,2;2,2,2)$ is recurrent.

Remark 2.6. The top configuration $\mathrm{inc}\,(c^t)$ can be recovered from the Ferrers diagram $F(k)$ by taking the heights of horizontal steps along the South-East border of the diagram, from the bottom-left corner to the top-right. For example, consider the Ferrers diagrams $F(1,1,3)$ as on the left of Figure 2. The South-East border can be written as $HVVHHV$, with H denoting a horizontal step, and V a vertical one. Then we have $c^t = (0,2,2)$, corresponding to the heights of the H steps.

Remark 2.7. Theorem 2.4 gives a bijective representation of *sorted* recurrent configurations. For the unsorted case, we label the Ferrers diagrams as follows. For the k-diagram $F(k)$, we assign bijectively to each column an element of $[m]$ such that columns of the same height are labelled in increasing order from left to right. Similarly, we label the rows of the c^b-diagram with elements of $[n]$ such that rows of the same length are labelled in increasing order from bottom to top. This yields a bijection from all recurrent configurations to the set of compatible labelled Ferrers diagrams.

Finally, we propose a representation of the compatibility notion through a directed acyclic graph (DAG). The vertices of the graph are the Ferrers diagrams $F \in \mathrm{Ferrers}_{\leq m,n}$ satisfying $\mathrm{Area}(F) \geq m$. For every pair (F, F'), we put an edge from F to F' if $F' = \mathsf{Shift}(F)$ or $F' = \mathsf{Add}(F)$. As in Figure 2, Shift edges are coloured blue, and Add edges are red. We denote $\mathrm{DAG}_{m,n}$ the DAG thus obtained. Note that $\mathrm{DAG}_{m,n}$ is *bipolar*: it has a unique source $F(0,\cdots,0,m)$ and a unique sink $F(m,\cdots,m)$. With this representation, Ψ is a bijection from $\mathrm{SortedRec}_{m,n}$ to pairs of vertices (F,F') of $\mathrm{DAG}_{m,n}$ such that F has m columns and there is a (directed) path from F to F' in $\mathrm{DAG}_{m,n}$. The level of the configuration equals the number of red edges in such a path. Figure 3 illustrates this construction.

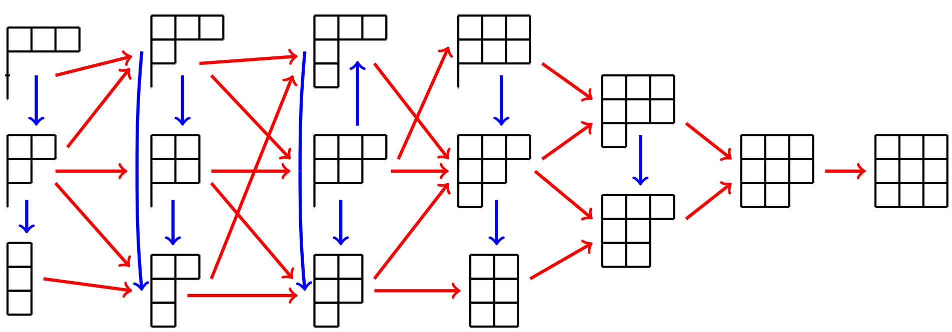


FIGURE 3. The graph $\mathrm{DAG}_{3,3}$.

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