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On a particular specialization of monomial symmetric functions

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1 Introduction

Let $p_n^{(r)}$ be the symmetric functions defined for any pair of integers (n, r) such that $n \geq r \geq 1$ by:

$$p_n^{(r)} = \sum_{|\lambda|=n,\ l(\lambda)=r} m_\lambda \tag{1.1}$$

where the m_λ are the monomial symmetric functions, the sum being over the integer partitions λ of n, with length $l(\lambda) = r$. The functions $p_n^{(r)}$ are introduced with this notation in exercise 19 p.33 of [10]. In particular $p_n^{(1)}$ is the power symetric function $p_n = m_n$. In [5] we have shown the following theorem for the specialzation given by Equa (1.2) and sometimes called the q-deformation of the exponential.

Theorem 7.2 of [5] : *For $n \geq r \geq 1$ and*

$$E_{xp}(t) = \sum\nolimits_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{n!} \tag{1.2}$$

we have:

$$p_n^{(r)} = (1-q)^{n-r} \frac{q^{\binom{r}{2}}}{r!\,(n-r)!} J_{n,r}(q) \tag{1.3}$$

where $J_{n,r}$ is a monic polynomial with positive integer coefficients, a constant term equal to $(n-r)!$ and which degree is $\binom{n-1}{2} - \binom{r-1}{2}$. Moreover, for all $r \geq 1$ $J_{r,r} = 1$.

When $r = 1$ this gives:

$$m_n = p_n = p_n^{(1)} = \frac{(1-q)^{n-1}}{(n-1)!} J_n \tag{1.4}$$

where $J_n = J_{n,1}$ is the enumerator polynomial of inversions in trees on n vertices, introduced in [11]. More generally, we have seen that $J_{n,r}(q)$ are enumerator polynomials of inversions for sequences of "colored" forests introduced by Stanley and Yan (see [17]), or level statistics enumerators introduced by the author in Section 10 of [5]. Alternatively the reciprocal polynomial of $J_{n,r}$, denoted $\overline{J_{n,r}}$ in [5], is the sum enumerator of generalized parking functions. We refer to [18] and [5] for more details on these combinatorial interpretations.

In this article, for all integer partitions λ and for the specialization (1.2), m_λ is expressed using a polynomial $J_\lambda(q)$ whose coefficients belong to $\mathbb{Z}$ (Corollary 3.3). This is proved by induction with two total orders on the set of integer partitions. The particular case of the partition $\lambda = (n)$ gives again $J_{(n)} = J_n$, thus Corollary 3.3 can be seen as a generalization of (1.4). From the calculations of J_λ for $n = |\lambda| \leq 6$, we conjecture that for any partition λ the coefficients of J_λ are positive and log-concave. One arguments for these conjectures is that it is possible to show the log-concavity of $J_{n,r}$ using Huh's results on the h-vector of the matroid complex of representable matroids [8]. We also prove that the last $n-1$ coefficients of J_λ are proportional to the first $n-1$ coefficients of column $n-r-1$ of Pascal's triangle, r being the length of λ. This gives a third argument for stating the conjectures since it is well known that these columns are log-concave. To conclude, it is underlined the need to strengthen the conjectures by continuing the calculations for larger values of $|\lambda|$. We also show by computing $J_{(3,2,1)}$ that obtaining a complete proof of the log-concavity of J_λ by an approach analogous to that used for $J_{n,r}$ seems not possible.

2 Prerequisites

We assume the reader to have a certain familiarity with integer partitions and symmetric functions like explained in Chap. 1 of [10]. We set out our notations. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ is a partition of the integer n, $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_r = n$, $l(\lambda) = r$, $\lambda! = \lambda_1!\lambda_2!...\lambda_r!$ and

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda_i'}{2} \tag{2.1}$$

where $\lambda' = (\lambda_1', \lambda_2', ..)$ is the conjugate of the partition λ. If r_i is the numbers of λ equal to $i \in \mathbb{N}^*$, the sequence of multiplicity is $m(\lambda) = (r_1, r_2, ...)$, and we set $|m(\lambda)| = r_1 + r_2 + ... = r$, $m(\lambda)! = r_1!r_2!....$, we also write $\lambda = 1^{r_1}2^{r_2}....$

$\mathcal{P}$ and $\mathcal{P}_n$ are respectively the set of all the integer partitions of $n \in \mathbb{N}$. In $\mathbb{P}_n$ $\preceq$, $\leq$ and $\sqsubseteq$ will respectively designate the reverse lexicographic order, the dominance order and the refinement order (see [10] respectively p.6, p.7 and p.103) It is known ([10] Chap.1) that in $\mathcal{P}_n$,

$$\lambda \sqsubseteq \mu \Rightarrow \lambda \leq \mu \Rightarrow \lambda \preceq \mu \tag{2.2}$$

We generalize some definitions above to any strict composition of n, i.e. any r-multiplet $u = (u_1, u_2, ..., u_r)$ of integers strictly greater than zero such that $u_1 + u_2 + ... + u_r = n$, by setting $|u| = u_1 + u_2 + ... + u_r$, $l(u) = r$. Then $\Lambda(u)$ is the partition of n, which is composed of the u_i arranged in a non-increasing way.

If K is a commutative field, $\mathbf{\Lambda}_K$ is the algebra of symmetric functions in the indeterminates $X = (x_i)_{i\geq 1}$with coefficients in K. Here, K will be $\mathbb{Q}(q)$, the rational fractions in the indeterminate q. For $\lambda \in \mathcal{P}$, $(m_\lambda), (e_\lambda)$ and (p_λ) and are the classical bases of $\mathbf{\Lambda}_K$. Agreeing that $e_0 = 1$ we recall that $E(t) = \sum_{n=0}^{\infty} e_n t^n = \prod_{i\geq 1}(1 - x_i t)$.

Basic knowledge is required on poset, matroid, and Tutte polynomial as, for example, stated in Wikipedia articles on these topics. Occasional references will also be made to [1], [13] for matroids and to [16] for poset. If $n \in \mathbb{N}$, $[n]_q = 1 + q + q^2 + ... + q^{n-1}$ is the q-analog of n. If $P(q)$ is a polynomial in q, $\langle q^m \rangle P(q)$ is the coefficient of q^m in $P(q)$. Finally, if E is a finite set, $|E|$ is the cardinality of E.

3 Expression of m_λ for the specialization $e_n = q^{\binom{n}{2}}/n!$:

Let us define the augmented monomial symmetric functions by

$$\widetilde{m_\lambda} = m(\lambda)!\, m_\lambda. \tag{3.1}$$

Theorem 3.1 *For any partition λ and for the specialization $E_{xp}(t)$, $\widetilde{m_\lambda}$ is given by*

$$\widetilde{m_\lambda} = (1-q)^{|\lambda|-l(\lambda)} M_\lambda(q) \tag{3.2}$$

where $M_\lambda(q)$ belongs to $\mathbb{Q}[q]$ and satisfies:

a) The degree of M_λ is

$$d(\lambda) = \binom{|\lambda|-1}{2} + l(\lambda) - 1 \tag{3.3}$$

and

$$\left\langle q^{d(\lambda)} \right\rangle M_\lambda = \frac{(l(\lambda)-1)!}{(|\lambda|-1)!} \tag{3.4}$$

b) The valuation of M_λ is

$$val(M_\lambda) = n(\lambda) \tag{3.5}$$

and

$$\left\langle q^{n(\lambda)} \right\rangle M_\lambda = \frac{m(\lambda)!}{\lambda'!} \tag{3.6}$$

c) For $\lambda = (n)$ we have

$$M_{(n)} = \frac{J_n}{(n-1)!} \tag{3.7}$$

Note that the highest degree monomial of M_λ only depends on $|\lambda|$ and $l(\lambda)$.

Sketch of proof: *i) If $\lambda = (n)$ which corresponds to c), then $r = 1$ and according to (1.4): $\widetilde{m}_{(n)} = p_n = (1-q)^{n-1} J_n(q) / (n-1)!$ where J_n is a monic polynomial of degree $\binom{n-1}{2}$, with a valuation equal to 0 and a constant term equal to $(n-1)!$. Taking $M_{(n)} = J_n/(n-1)!$ Equation (3.2) and points a) and b) are verified in this case and so c) also.*

ii) Equation (3.2) and a) is proven in the general case by induction with the total order $\trianglelefteq$ defined on $\mathcal{P}$ by:

$$\lambda \triangleleft \mu \Leftrightarrow \begin{cases} |\lambda| < |\mu| \\ |\lambda| = |\mu| \text{ and } \mu \prec \lambda \end{cases}$$

This gives, with $\lambda = (\lambda_1, ..., \lambda_r)$, $\lambda^* = (\lambda_1, ..., \lambda_{r-1})$ and $\lambda^{(i)} = \Lambda(\lambda_1, ..., \lambda_{i-1}, \lambda_i + \lambda_r, \lambda_{i+1}, ..., \lambda_{r-1})$, the recurence:

$$M_\lambda = M_{(\lambda_r)} M_{\lambda^*} + (q-1) \sum_{i=1}^{r-1} M_{\lambda^{(i)}} \tag{3.8}$$

iii) Let us now prove b) of Theorem 3.1. According to Equation (2.3) p. 20 of [10], for any partition λ:

$$m_\lambda = e_{\lambda'} - \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$$

For $E_{xp}(q)$ it follows from (2.1): $e_{\lambda'} = \prod_{i\geq 1} q^{\binom{\lambda'_i}{2}}/\lambda'_i! = q^{n(\lambda)}/\lambda!$. For $\lambda = 1^n$, $\widetilde{m}_{1^n} = q^{\binom{n}{2}}$, therefore $M_{1^n} = q^{\binom{n}{2}}$ which verifies the equations of b). For $\lambda \neq 1^n$ and $|\lambda| = n$, b) is proven by *induction on the order $\preceq$ in P_n.*

Corollary 3.2 *Let (n, r) be a pair of positive integers, then:*

$$\text{For } n \geq r \geq 1 \quad q^{\binom{r}{2}} \frac{J_{n,r}}{r!\,(n-r)!} = \sum_{|\lambda|=n,\ l(\lambda)=r} \frac{M_\lambda}{m(\lambda)!} \tag{3.9}$$

$$\text{For } n-1 \geq r \geq 1 \quad r! \sum_{|\lambda|=n,\ l(\lambda)=r} \frac{M_\lambda}{m(\lambda)!} = q^{\binom{r}{2}} \sum_{|\mu|=n-r} [r]_q^{l(\mu)} \frac{M_\mu}{m(\mu)!} \tag{3.10}$$

In particular for $r = 1$ and $n \geq 2$

$$\frac{J_n}{(n-1)!} = M_{(n)} = \sum_{|\lambda|=n-1} \frac{M_\lambda}{m(\lambda)!} \tag{3.11}$$

Corollary 3.3 *We have the following generalization of* (1.4) *for any partition λ:*

$$m_\lambda = (1-q)^{|\lambda|-l(\lambda)} \frac{q^{n(\lambda)}}{(|\lambda|-1)!\, m(\lambda)!} J_\lambda \tag{3.12}$$

with

$$J_\lambda(q) = (|\lambda|-1)! M_\lambda(q)\, q^{-n(\lambda)} \tag{3.13}$$

J_λ is a polynomial with coefficients in $\mathbb{Z}$, with zero valuation and degree equal to $\binom{|\lambda|-1}{2} + l(\lambda) - 1 - n(\lambda)$.

Proof. For $\lambda = (n)$ Equation (3.12) gives (1.4) with $J_{(n)} = J_n$, which shows that (3.12) is a generalization of (1.4).Equations (3.12) and (3.13) follow easily from (3.1) and (3.2). The nullity of the valuation and the value of the degree of J_λ come respectively from (3.5) and (3.3). The substitution of (3.13) in (3.8) gives for all partitions $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$:

$$q^{(r-1)\lambda_r} J_\lambda = (|\lambda|-1)! \binom{|\lambda|-2}{\lambda_r - 1} J_{\lambda_r} J_{\lambda^*} + (q-1) \sum_{i=1}^{r-1} q^{(i-1)\lambda_r} J_{\lambda^{(i)}} \tag{3.14}$$

whith λ^* and $\lambda^{(i)}$ defined above. Equation (3.14) is a recurrence whose coefficients are polynomials in q with coefficients in $\mathbb{Z}$. It makes it possible to calculate all the polynomials J_λ from the polynomials J_n which are themselves with coefficients in $\mathbb{N}$. So the coefficients of J_λ are in $\mathbb{Z}$. ■

We give below J_λ for $|\lambda| \leq 4$ ($J_{n,r}$ have already been given in [5])
$|\lambda| = 1$ $J_1 = 1$
$|\lambda| = 2$ $J_2 = 1$, $J_{1^2} = 1$
$|\lambda| = 3$ $J_3 = 2 + q$, $J_{(2,1)} = 1 + q$, $J_{1^3} = 2$
$|\lambda| = 4$ $J_4 = 6 + 6q + 3q^2 + q^3$, $J_{(3,1)} = 3 + 3q + 2q^2 + q^3$, $J_{(2,2)} = 3 + 2q + q^2$, $J_{(2,1,1)=1^2 2^1} = 2 + 2q + 2q^2$, $J_{1^4} = 6$

With (3.9) and (3.10) it is also possible to calculate the following particular cases:

$$\text{For } n \geq 1 \quad J_{1^n} = (n-1)! J_{n,n} = (n-1)! \tag{3.15}$$

$$\text{For } n \geq 2 \quad J_{1^{n-2}2} = (n-2)! J_{n,n-1} = (n-2)! [n-1]_q \tag{3.16}$$

4 Conjectures about J_λ

Three heuristic arguments lead us to state the following conjectures.

Conjecture 1. *For any partition λ the coefficients of J_λ are strictly positive.*

Conjecture 2. *For any partition λ, J_λ is log-concave.*

If these two conjectures are true then J_λ is also unimodal (see Lemma 7.1.1 of [3]), thus we can also state:

Conjectures 3. *For any partition λ, J_λ is unimodal.*

Let us note that it is equivalent to formulate these conjectures for the polynomials J_λ or M_λ. The first argument for stating these conjectures results from the calculations already made. The conjectures are obviously true for J_{1^n} and $J_{1^{n-2}2}$ given by (3.15) and (3.16). And we checked both conjectures for $|\lambda| \leq 6$. It is certain that it would be desirable to continue the calculations for values of $|\lambda|$ much bigger, which we intend to do.

The second argument in favor of these conjectures comes from the fact that the coefficients of polynomials $J_{n,r}$ are strictly positive and log-concave, which is a consequence of Huh's famous work. This is actually true, for the largest class of polynomials, denoted $I_m^{(a,b)}$ in [18]. We have seen in [5] that these polynomials are linked to $J_{n,r}$ by $J_{n,r} = I_{n-r}^{(r,1)}$. Polynomials $I_m^{(a,b)}$ and their reciprocal have been the subject of much research, of which one will find a summary in [18]. It is likely that the following properties are known to specialists but as we have not seen them in the literature (except the case $I_{n-1}^{(1,1)} = J_n$), we state them in the following proposition.

Proposition 4.1 *The sequence of the coefficients of the polynomials $I_m^{(a,b)}$ (and their reciprocal) are strictly positive and log-concave, hence unimodal.*

Proof. Let

$$I_m^{(a,b)}(q) = \sum_{i=c}^{d} a_i q^i$$

From the various properties of the polynomials $I_m^{(a,b)}$ and their reciprocals (see [17]) it is easy to see that $c = 0$, $d = ma + b\binom{m}{2}$ and

$$a_0 = m! \quad a_d = 1 \tag{4.1}$$

In [17] p.662, it is shown that

$$I_m^{(a,b)}(1+t) = \sum_{G'} t^{e(G')-m} \tag{4.2}$$

where the sum is over the multicolor graphs G' whose the set of vertices is $V = \{0, 1, 2, ..., m\}$, without loop. The edges of G' between two vertices $i, j \neq 0$ are to be taken among b colored edges $\overline{0}$, $\overline{1}$, ..., $\overline{b-1}$ and those between 0 and $i \neq 0$ are to be taken among a colored edges $\overline{0}$, $\overline{1}$,..., $\overline{a-1}$. $e(G')$ is the number of edges of G'. Let us consider now the graph introduced in [14] p. 3115 with the notation $K_{m+1}^{b,a}$. This graph is defined as a complete graph on vertices $V = \{0, 1, 2, ..., m\}$ with the edges (i, j), $i, j \neq 0$ of multiplicity b and the edges $(0, i)$, $i \neq 0$ of multiplicity a. It is clear that there is a bijection between the graphs G' and the connected spanning graphs of $K_{m+1}^{b,a}$. The Tutte polynomial of $K_{m+1}^{b,a}$ is:

$$T(x, y) = \sum_A (x-1)^{c(A)-1} (y-1)^{c(A)+e(A)-(m+1)} \tag{4.3}$$

where the sum is over the spanning graphs A of $K_{m+1}^{b,a}$, $c(A)$ being the number of connected components of A, and $e(A)$ its number of edges. The comparison of (4.2) and (4.3) shows that

$$I_m^{(a,b)}(q) = T(1, q) \tag{4.4}$$

Let us consider now the matroid $\mathcal{M}$ associated to $K_{m+1}^{b,a}$. This matroid is representable over any field so its dual $\mathcal{M}^*$also (Corollary 2.2.9 of [13]). By noting T^* the Tutte polynomial of $\mathcal{M}^*$, we have $T^*(y, x) = T(x, y)$, therefore $I_m^{(a,b)}(q) = T^*(q, 1)$. Let ρ be the rank of $\mathcal{M}^*$ and $(h_0, h_1, ..., h\rho)$ be the h-vector of the matroid complex $IN(\mathcal{M}^*)$. It is known that (see [1] p.142):

$$\sum_{k=0}^{\rho} h_k q^{\rho-k} = T^*(q, 1)$$

From Theorem 3 of [8] it results that $I_m^{(a,b)}$ are log-concave. Moreover, by the same theorem, we know that the coefficients have non internal zeros. But these coefficients are positive or zero (this comes for example, from their combinatorial definition). Therefore all the coefficients are strictly positive and unimodality follows from Lemma 7.1.1 of [3]. ■

The polynomials $J_{n,r} = I$ are therefore strictly positive and log-concave. We deduce that for the particular case $\lambda = (n)$, $J_{(n)} = J_n = J_{n,1}$ satisfies the two conjectures. This gives the second argument to state the conjectures.

We will now see the third argument.

5 Pascalian part of M_λ

Theorem 5.1 *Let λ be a partition integer with $|\lambda| = n$, $l(\lambda) = r$ and set*

$$v(\lambda) = d(\lambda) - n + 2 = \binom{n-2}{2} + r - 1 \tag{5.1}$$

For $\lambda \neq 1^n$ the part of M_λ of degree $\geq v(\lambda)$ is given by the following polynomial which only depends on n and r:

$$P_{n,r}(q) = \frac{(r-1)!}{(n-1)!} \sum_{i=0}^{n-2} \binom{n-r-1+i}{n-r-1} q^{d(\lambda)-i} = \frac{(r-1)!}{(n-1)!} \sum_{j=0}^{n-2} \binom{2n-r-3-j}{n-2-j} q^{v(\lambda)+j} \tag{5.2}$$

$v(\lambda)$ is the valuation of $P_{n,r}$. Theorem 5.1 shows that the $n-1$ coefficients of $((n-1)!/(r-1!))\, M_\lambda$ with highest degree are given by the $n-1$ first coefficients of the column $n-r-1$ of Pascal's triangle. $P_{n,r}$ and it coefficients will be called Pascalian part, resp. Pascalian coefficients, of M_λ (likewise for the parts and coefficients, corresponding to J_λ).

Sketch of the proof: *We prove it by a double induction on n and r.*

The third announced argument is the following corollary.

Corollary 5.2 *For any partition λ such that $\lambda \neq 1^{|\lambda|}$, the $|\lambda| - 1$ last coefficients of J_λ are strictly positive and log-concave, then unimodal. If $l(\lambda) < |\lambda| - 1$ these coefficient are in fact strictly log-concave.*

6 Further Developments

The first thing to do is to continue the calculations using recurrence (3.8) -or (3.14) if one prefers to work with integers- to verify conjectures numerically up to a value of $n = |\lambda|$ as high as possible. If these calculations do not contradict the conjectures, two ways can be considered to prove them.

The first way is to generalize the proof by induction of Theorem 5.1. The ideal would be to find an explicit formula, generalizing that one given by Theorem 5.1, to all coefficients. But it seems to me rather unlikely that such a formula can be found.

The second way, combinatorial in nature, would be to generalize to all J_λ the method which made it possible to deduce log-concavity of $J_{n,r}$. It would therefore be a question of associating to each J_λ a combinatorial object, itself associated to a matroid $\mathcal{M}_\lambda$ whose Tutte polynomial T_λ satisfies the following analogue of (4.4), $J_\lambda(q) = T_\lambda(1, q)$. This way would allow to prove the three conjectures at the same time. Let us note in this regard that the log-concavity of the h-vector of the matroid complex has recently been demonstrated for any matroid in [2]. Thus the potentiel matroid $\mathcal{M}_\lambda$, is not limited to graphic matroids. However there is an obstacle for this way. Indeed we have calculated that:

$$J_{(3,2,1)}(q) = 10 + 30q + 35q^2 + 35q^3 + 30q^4 + 20q^5 + 12q^6 + 6q^7 + 2q^8$$

2 and 35 being coprime, $J_{(3,2,1)}$ cannot be reduced to a monic polynomial with integer coefficients. But if we find a matroid $\mathcal{M}$ of which the Tutte polynomial T verifies $J_{(3,2,1)}(q) = T(1,q)$. We would have with (1.23) p.138 of [1]: $T(1,q) = \sum (q-1)^{|A|-\rho}$ where the sum is over all the subsets A of the ground set E of $\mathcal{M}$, which rank is equal to ρ. The leading monomial of $T(1,q)$ is thus clearly $q^{|E|-\rho}$, therefore $T(1,q)$ would be monic, which leads to a contradiction.There are then two possibilities:

* The obstacle noted belove can be circumvented by remaining within the framework of matroids and theorems on the log-concavity attached to them.

* The obstacle is inherent to the matroids and it will become necessary to associate to J_λ an algebraic or geometric structure distinct from the matroids. One thinks, for example, of some (graded) poset (see chap.3 of [16]) or convex polytopes, interval greedoids, etc.., for which some log-concavity results have been proven or conjectured (see [3], [4], [6], [9] and [15]).

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