Sequentially constrained Hamilton Cycles in random graphs

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Choosing a graph at random

Evolution of random graphs

Choosing a graph at random

$G_{n,p}$: Each edge e of the complete graph K_n is included independently with probability $p = p(n)$.

Whp $G_{n,p}$ has $\sim \binom{n}{2}p$ edges, provided $\binom{n}{2}p \to \infty$

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$G_{n,m}$: Vertex set $[n]$ and m random edges.

If $m \sim \binom{n}{2}p$ then $G_{n,p}$ and $G_{n,m}$ have "similar" properties.

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Erdős (1947): **Whp** the maximum size of a clique or independent set in $G_{n,1/2}$ is $\leq 2\log_2 n$.

Therefore

$$R(k,k) \geq 2^{k/2}.$$

I.e. it is possible to color the edges of the complete graph on $2^{k/2}$ vertices so that there is no mono-chromatic clique of size k.

Evolution of random graphs

Random graphs first used to prove existence of graphs with certain properties:

Mantel (1907): There exist triangle free graphs with arbitrarily large chromatic number.
Erdős (1959): There exist graphs of arbitrarily large girth and chromatic number.

$m = cn$, $c > 0$ is a large constant. **Whp** $G_{n,m}$ has $o(n)$ vertices on cycles of length $\leq \log \log n$ and no independent set of size more than $\frac{2 \log c}{c} n$.

So removing the vertices on small cycles gives us a graph with girth $\geq \log \log n$ and chromatic number $\geq \frac{c+o(1)}{2 \log c}$.

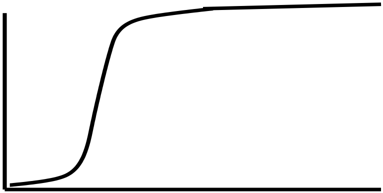
Erdős and Rényi began the study of random graphs in their own right.

On Random Graphs I (1959): $m = \frac{1}{2}n(\log n + c_n)$

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \text{ is connected}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

$$= \lim_{n\to\infty} \mathbf{Pr}(\delta(G_{n,m}) \geq 1)$$

$\mathbf{Pr}(G_{n,m} \text{ is connected})$



m

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The evolution of a random graph, Erdős and Rényi (1960)

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$n^{\frac{k-1}{k}}\log n$	Components are trees of vertex size $1, 2, \ldots, k$.

Evolution of random graphs

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$\frac{1}{2}cn$ $c < 1$	Mainly trees. Some unicyclic components. Maximum component size $O(\log n)$
$\frac{1}{2}n$	Complicated. Maximum component size order $n^{2/3}$. Has subsequently been the subject of moreintensive study e.g. Janson, Knuth, Łuczak and Pittel (1993).
$\frac{1}{2}cn$ $c > 1$	Unique giant component of size $G(c)$n. Remainder almost all trees. Second largest component of size $O(\log n)$

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Only very simple probabilistic tools needed. Mainly first and

Connectivity threshold

$$p = (1+\epsilon)\frac{\log n}{n}$$

X_k = number of k-components, $1 \leq k \leq n/2$.
$X = X_1 + X_2 + \cdots + X_{n/2}$
$G_{n,p}$ is connected iff $X = 0$.

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$$\begin{aligned}
\mathbf{Pr}(X \neq 0) &\le \mathbf{E}(X) \\
&\le \sum_{k=1}^{n/2} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \\
&\le \frac{n}{\log n} \sum_{k=1}^{n/2} \left(\frac{e \log n}{n^{(1+\epsilon)(1-k/n)}} \right)^k \\
&\to 0.
\end{aligned}$$

Evolution of random graphs

Hitting Time: Consider $G_0, G_1, \ldots, G_m, \ldots,$ where G_{i+1} is G_i plus a random edge.
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- **Whp** At time m_2 there are $(\log n)^{n-o(n)}$ distinct Hamilton cycles.
Cooper and Frieze (1989), Glebov and Krivelevich (2013).

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- **Whp** m_k is the "time" when G_m first has $k/2$ edge disjoint Hamilton cycles. $k = O(1)$ Bollobás and Frieze (1985).

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- Recently, results of Krivelevich and Samotij and Knox, Kühn and Osthus proved the much more difficult result, allowing k to grow up to $k \sim n/2$.

Random Digraphs – strong components

In the random digraph $D_{n,p}$ we include each possible directed edge $(u, v) \in [n]^2$ with probability p. For $D_{n,m}$ we choose m random directed edges.

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$n(\log n + \omega)$	Strongly connected.

$$\mathbf{Pr}(D_{n,p} \text{ is Hamiltonian}) \geq \mathbf{Pr}(G_{n,p} \text{ is Hamiltonian}).$$

McDiarmid (1980).
So if $p \geq \frac{\log n + \log\log n + \omega}{n}$ then $D_{n,p}$ is Hamiltonian w.h.p.

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Random Digraphs – Hamiltonicity

Let $e_1, e_2, \ldots, e_N$ be an enumeration of the edges of the complete graph K_n. Each $e_i = \{v_i, w_i\}$ gives rise to two directed edges $\overrightarrow{e_i} = (v_i, w_i)$ and $\overleftarrow{e_i} = (w_i, v_i)$.

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Thus Γ_0 is just $G_{n,p}$ with each edge $\{v, w\}$ replaced by a pair of directed edges $(v, w), (w, v)$ and $\Gamma_N = \mathbb{D}_{n,p}$. McDiarmid's result follows from

$$\mathbf{Pr}(\Gamma_i \text{ is Hamiltonian}) \geq \mathbf{Pr}(\Gamma_{i-1} \text{ is Hamiltonian}).$$

To prove this we condition on the existence or otherwise of directed edges associated with $e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_N$. Let $\mathcal{C}$ denote this conditioning.

Random Digraphs – Hamiltonicity

Either

(a) $\mathcal{C}$ gives us a Hamilton cycle without arcs associated with e_i, or

(b) not (a) and there exists a Hamilton cycle if at least one of $\overrightarrow{e_i}, \overleftarrow{e_i}$ is present, or

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In Γ_{i-1} (b) happens with probability p. In Γ_i either (i) exactly one of $\overrightarrow{e_i}, \overleftarrow{e_i}$ yields Hamiltonicity and in this case the conditional probability is p or (ii) either of $\overrightarrow{e_i}, \overleftarrow{e_i}$ yields Hamiltonicity and in this case the conditional probability is $1-(1-p)^2 > p$.

Edge colored graphs – rainbow structures

A set of edges S is said to be rainbow colored if each edge has a different color.

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We consider the graph process, $e_1, e_2, \dots, e_m$ where we randomly color each edge independently from a set of k colors.

$$\tau_C = \min_t : n-1 \text{ distinct colors are used on } e_1, \dots, e_t.$$
$$\tau_T = \min_t : e_1, \dots, e_t \text{ contains a spanning tree.}$$
$$\tau_{RT} = \min_t : e_1, \dots, e_t \text{ contains a rainbow spanning tree.}$$

Frieze and McKay (1994). $\tau_{RT} = \max\{\tau_C, \tau_T\}$ w.h.p.

Relies on Edmund's matroid intersection theorem.

Hamilton cycles

$$\tau_c = \min_t : n \text{ distinct colors are used on } e_1, \ldots, e_t.$$

$$\tau_H = \min_t : e_1, \ldots, e_t \text{ contains a Hamilton cycle.}$$

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It is known that if we have $n + o(n)$ colors and $(1 + o(1))n \log n$ random edges then w.h.p. there is a rainbow Hamilton cycle. Frieze and Loh (2014), Ferber and Krivelevich (2015).

Edge colored graphs – patterns

Let $G_{n,p;\alpha}$ denote $G_{n,p}$ where each edge is independently given a random color i from the *palette* $[k]$ with probability α_i.

A color pattern will be a sequence $\mathbf{c} = (c_1, c_2, \dots, c_n)$.

Given a sequence $\mathbf{c}$ we say that the Hamilton cycle $H = (x_1, x_2, \dots, x_n, x_1)$ (as a sequence of vertices) is $\mathbf{c}$-colored if $c(\{x_i, x_{i+1}\}) = c_i$ for $i = 1, 2, \dots, n$.

Theorem

Let $\mathbf{c}$ *be an arbitrary sequence of colors. Let* $p = (\log n + \log\log n + \omega)/n$ *where* $\omega \to \infty$*. Then w.h.p.* $G_{n,\beta p;\alpha}$ *contains a* $\mathbf{c}$*-colored Hamilton cycle, where* $\beta = 1/\alpha_{\min}$*.*

Edge colored graphs – patterns

Previous works Espig,F,Krivelevich or Anastos,F dealt with sequences that were repetitions of a small fixed sequence e.g. Black/White/Black/White/... "zebraic".

Previous works proved hitting time results, not so here.

In our result, an *adversary* chooses the sequence **c** and then we generate $G_{n,p;\alpha}$. It would be much harder if things were done in reverse order.

We do not claim that w.h.p. $G_{n,p;\alpha}$ simultaneously contains a cycle of every pattern.

Edge colored graphs – patterns

Let $N = \binom{n}{2}$ and consider the following sequence of (partially) edge colored graphs $\Gamma_m, m = 0, 1, \ldots, N$.

Let $e_1, e_2, \ldots, e_N$ be an enumeration of the edges of K_n.

To construct Γ_t we include $e_1, e_2, \ldots, e_t$ independently with probability kp and give each included edge a random color using distribution α.

Then for $i > t$ we include each edge independently with probability p.

Thus Γ_0 is a copy of $G_{n,p}$ and Γ_N is a copy of $G_{n,kp;\alpha}$

Edge colored graphs

A Hamilton cycle $H = (e_{\pi(i)}, i = 1, 2, \ldots, n)$ (as a sequence of edges) of Γ_t is $(\mathbf{c}, t)$-*proper* if $c(e_{\pi(j)}) = c_j$ for $\pi(j) \leq t$.

Let $\mathcal{G}_t$ denote the set of graphs containing a $(\mathbf{c}, t)$ -proper Hamilton cycle.

We claim that

$$\mathbf{Pr}(\Gamma_t \in \mathcal{G}_t) \leq \mathbf{Pr}(\Gamma_{t+1} \in \mathcal{G}_{t+1}) \text{ for } t \geq 0.$$

We modify McDiarmid's argument.

Vertex colored graphs – patterns

Suppose now that each vertex of $G_{n,p}$ is given one of k colors. Let V_i denote the vertices of color i and assume that $|V_i| = \alpha_i n$ for $i \in [k]$.

Given a sequence $\mathbf{c}$ we now say that the Hamilton cycle $H = (x_1, x_2, \ldots, x_n, x_1)$ (as a sequence of vertices) is $\mathbf{c}$-colored if $c(x_i) = c_i$ for $i = 1, 2, \ldots, n$.

Theorem

Let $\mathbf{c}$ be an arbitrary sequence of colors where each color j appears exactly $\alpha_j n$ times. Let $p = K \log n/n$ where $K = K(k)$ is sufficiently large. Then w.h.p. $G_{n,p}^{[k]}$ contains a $\mathbf{c}$-colored Hamilton cycle.

Vertex colored graphs

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A hypergraph $\mathcal{H}$ is r-bounded if $e \in \mathcal{H}$ implies that $|e| \leq r$.

For a set $S \subseteq X = V(\mathcal{H})$ we let $\langle S \rangle = \{T : S \subseteq T \subseteq X\}$ denote the subsets of X that contain S. We say that $\mathcal{H}$ is κ-spread if we have the following bound on the number of edges of $\mathcal{H}$ that contain a particular set S:

$$|\mathcal{H} \cap \langle S \rangle| \leq \frac{|\mathcal{H}|}{\kappa^{|S|}}, \quad \forall S \subseteq X.$$

Vertex colored graphs

Let X_p denote a subset of X where each $x \in X$ is included independently in X_p with probability p.

Theorem

Let $\mathcal{H}$ be an r-bounded, κ-spread hypergraph and let $X = V(\mathcal{H})$. There is an absolute constant $C > 0$ such that if

$$p \geq \frac{C \log r}{\kappa}$$

then w.h.p. X_p contains an edge of $\mathcal{H}$. Here w.h.p. assumes that $r \to \infty$.

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We let $X = \binom{[n]}{2}$. Each $x = \{u, v\} \in X$ will have colored endpoints $\{c(u), c(v)\}$. Our hypergraph $\mathcal{H}$ consists of sets of n edges with colored endpoints that together make up a **c**-colored Hamilton cycle. We find that $\mathcal{H}$ has spread $\kappa = \Omega(n)$.

A fixed order for a subset of vertices

We have a fixed set $S_0 \subseteq [n]$ and a fixed ordering of the vertices in S_0.

Theorem

Let $p = (\log n + \log\log n + \omega)/n$, $\omega = o(\log\log n)$ and $S_0 \subseteq [n]$, $|S_0| = s_0 = \omega_1 n/\log n$ where $\omega_1 = o(\log\log\log n)$. Then w.h.p. $G_{n,p}$ contains a Hamilton cycle in which the vertices S_0 appear in natural order.

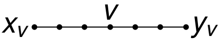
The natural constraint on ω_1 should be $o(\log\log n)$.

This is related to work of Robinson and Wormald who consider random regular graphs and Hamilton cycles that contain a given set of $o(n^{2/5})$ edges to be contained in order in the cycle.

A fixed order for a subset of vertices

We begin by partitioning $G_{n,p}$ into $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

We use Γ_1, Γ_2 to "bury" each $v \in S_0$ inside a short path P_v with endpoints x_v, y_v.
The internal vertices of the P_v's play no further role.



We then use Γ_1 to find vertex disjoint paths Q_v from y_v to x_{v+1} for $v \in S_0$: let $P^* = (P_1, Q_1, P_2, \ldots, Q_{s_0-1}, P_{s_0})$

We then contract P^* to an edge e^* and use fairly standard ideas to find a Hamilton cycle containing the edge e^*.

Bounds on the number of inversions

We place a restiction on the number $\iota(H)$ of invertions in the permutation of $[n]$ that defined by the Hamilton cycle H.

Theorem

Suppose that $M = \Omega(n \log n)$*. If* $p \geq \frac{Kn \log n}{M}$ *then w.h.p.* $G_{n,p}$ *contains a Hamilton cycle* H *with* $\iota(H) \leq M$*. If* $p \leq (1-\epsilon) \min\left\{\frac{\log n}{n}, \frac{n}{eM}\right\}$ *then w.h.p.* $G_{n,p}$ *does not.*

In addition

Theorem

If $M \leq Kn^2 / \log^2 n$ *and* $p \geq \frac{10 \max\{K,1\} n}{M}$ *then w.h.p.* $G_{n,p}$ *contains a Hamilton cycle* H *with* $\iota(H) \leq M$*.*

The first theorem is non-constructive, relying on spread, while the second relies on the analysis of a simple greedy algorithm.

THANK YOU