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On the Sparseness of the Downsets of Permutations via Their Number of Separators

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Abstract: Conventionally, a pair (σ_i, σ_{i+1}) is a *bond* in a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ if $\sigma_i - \sigma_{i+1} = \pm 1$. The number of bonds in a permutation $\sigma \in S_n$ has a direct influence on the number of distinct patterns of order $n-1$ contained in σ, affecting the structure of the downset of σ in the containment poset $\bigcup_{n\in\mathbb{N}} S_n$. Thus, to characterize the sparseness of the downset of a permutation $\sigma \in S_n$, we aim not only to find the number of bonds in σ, but also to predict the number of bonds contained in its patterns. To this end, we introduce a new statistic, *separator number*, as a significant factor in measuring the sparseness of this poset. An element σ_j in a permutation $\sigma = \sigma_1\cdots\sigma_n \in S_n$ is defined to be a *separator* of σ if we can obtain a new bond by omitting it from σ. We also present some enumerative and asymptotic results regarding this new statistic.

Keywords: Downset; Pattern; Pattern poset; Permutation; Permutation statistics; Prolific permutation
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1. Introduction

Let S_n be the symmetric group of n elements. Let $\sigma, \pi \in \bigcup_{n\in\mathbb{N}} S_n$. We say that σ *contains* π or that π is a *pattern* of σ if there exists a subsequence of elements of σ that is order-isomorphic to π. As an example, the permutation $\sigma = 3624715$ (written in one-line-notation) contains $\pi = 3142$, as both the subsequences 6275 and 6475 testify. If π is contained in σ, this relation is denoted by the expression $\pi \preceq \sigma$.

The set of all permutations $\cup_{n\in\mathbb{N}} S_n$ is a poset under the containment order. This is called the *permutation pattern poset*.

Omitting one element from a permutation $\sigma = \sigma_1\cdots\sigma_n \in S_n$ and standardizing, we obtain a permutation $\pi \in S_{n-1}$ such that $\pi \preceq \sigma$. The omission of σ_i or σ_j produces the same permutation if and only if (σ_i and σ_j) belong to the same strip, defined as follows.

Definition 1.1. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation and let $i \in \{1, 2, ..., n-1\}$. Then, we say that the pair (σ_i, σ_{i+1}) is a* 2-block *or a* bond *in σ if $\sigma_i - \sigma_{i+1} = \pm 1$. We further define a sequence $(\sigma_i, \sigma_{i+1}, \cdots, \sigma_{i+k-1})$ to be a* strip *of length $k > 1$ if, for each $0 \le j \le k-2$, the pair $(\sigma_{i+j}, \sigma_{i+j+1})$ is a* bond*. We also allow (trivial) strips of length $k = 1$. Note that a strip of a permutation might be ascending or descending. Occasionally, we omit parentheses in bonds or strips.*

Example 1.1. *The permutation $\sigma = 45187623$ has 45, 1, 876 and 23 as its maximal strips.*

Permutations of S_n with zero bonds are connected to the problem of placing n non-attacking kings in an $n \times n$ chessboard. These permutations were counted in [12] (see also [10]), and the structure of their pattern poset was discussed in a recent paper by the authors of the present work [5], where they were called *king permutations*, and the set of such permutations of order n was denoted by K_n.

The distribution of these bonds has been examined previously in [7, 8, 13]. The number of bonds in a permutation $\sigma \in S_n$ affects the structure of the poset of all permutations contained in σ, the downset of σ, because the number of permutations $\pi \in S_{n-1}$ such that $\pi \preceq \sigma$ is $n - \beta(\sigma)$, where $\beta(\sigma)$ is the number of bonds in σ (see Theorem 6 in [7]).

To better understand the structure and the extent of the sparseness of the poset $\bigcup_{n\in\mathbb{N}} S_n$, we would like to obtain information not only about the number of bonds of a given $\sigma \in S_n$, but also on the number of bonds of the permutations contained in σ. Hence, we introduce a new concept. A *separator* in σ is an element in σ, the removal of which produces a **new** bond (see the formal definition below).

This concept can also be seen via an analogy to a chessboard. Note that if we draw a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ on a chessboard, then "σ_i separates σ_k from σ_l" means that knights located at (k, σ_k) and (l, σ_l) can attack each other.

In [1], the authors defined the notion of a *k-prolific permutation*. A permutation $\sigma \in S_n$ is called k-prolific if each $(n-k)$-subset of the elements of $\sigma = \sigma_1\cdots\sigma_n$ forms a unique pattern. By Corollary 7 of [7], a permutation has the maximum number of patterns of order $n-1$ if and only if it has no bonds; hence, $\pi \in K_n$ if and only if it is 1-prolific as a member of S_n.

Moreover, using Theorem 2.25 from [1], one can prove that a permutation is 2-prolific if and only if it has no bonds and no separators.

1.1 Overview and main results

Let $\sigma \in S_n$, and $D_2(\sigma) = \{\pi \in S_{n-2} \mid \pi \prec \sigma\}$ be the $(n-2)$-th level of the downset of σ. The first main result in this paper provides an upper bound on $D_2(\sigma)$ as follows.

$$|D_2(\sigma)| \leq \binom{n-\beta(\sigma)}{2} + \beta(\sigma) - sep^*(\sigma), \tag{1}$$

where $\beta(\sigma)$ is the number of bonds in σ and $sep^*(\sigma)$ is the number of separators in σ not contained in a bond of σ (see Theorem 3.1).

Our second result characterizes the permutations in which each element is a separator. We refer to these as *fully separated permutations*. According to the former result, the downset of a fully separated permutation is sparse in some sense. It is worth noting that the set of fully separated permutations consists of inflations of the permutations 2413 and 3142 (see Definition 4.3). These two permutations are common in the literature on avoiding permutations, e.g., the set $Av(2413, 3142)$, which coincides with the set of separable permutations (See for example [2–4, 6, 11]). In particular, we prove the following.

> In a permutation $\sigma \in S_n$, each element is a separator if and only if $n = 4k$, $k \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_k \in \{3142, 2413\}$ and $\pi \in S_k$ such that $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$ (see Theorem 4.1).

We also characterize the permutations which have no separators. These permutations are counted by the sequence A137774 in the Online Encyclopedia of Integer Sequences (OEIS).

Next, we calculate the expectation of the number of separators.

- The expectation of the number of separators in a randomly chosen n-permutation is $\frac{4(n^3-6n^2+14n-13)}{n(n-1)(n-2)}$.
- The asymptotic value of this expectation is 4 (see Theorem 5.3).

Finally, we use the principle of inclusion-exclusion to construct a generating function that gives the distribution of the number of separators of a certain type over the poset $\bigcup_{n\in\mathbb{N}} S_n$ (see Theorem 6.1).

This paper is organized as follows. In Section 2, we present the formal definition of separators and list some of their fundamental properties. In Section 3, we present the upper bound for the number of patterns of a permutation $\sigma \in S_n$ of order $n-2$. In Section 4, we present a characterization and an enumeration of fully separated permutations. In Section 5, we calculate the expectation of the number of separators in a random permutation and its asymptotic limit using a direct combinatorial method. Section 6 is devoted to the generating function for the number of permutations with a specific number of separators. Finally, in Section 7, we present directions for further research.

2. Separators

We begin with the formal definition of our new concept of separators of a permutation.

Definition 2.1. *For $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ we say that σ_i* separates *σ_{j_1} from σ_{j_2} in σ if by omitting σ_i from σ we obtain a* **new** *2-block. This occurs if and only if one of the following cases holds.*

1. *j_1, i, j_2 are subsequent numbers and $|\sigma_{j_1} - \sigma_{j_2}| = 1$, i.e.,*

$$\sigma = \cdots \mathbf{b} \quad \color{red}{\sigma_\mathbf{i}} \quad \mathbf{b \pm 1} \cdots.$$

In this case, we say that σ_i is a **vertical separator***.*

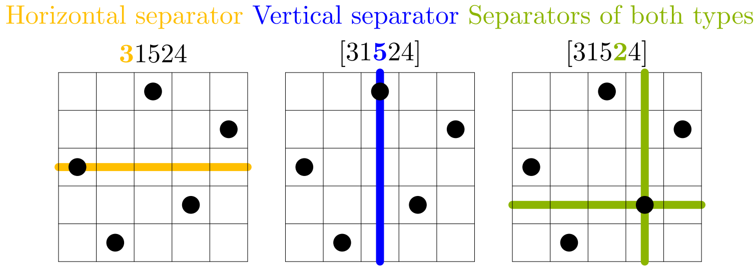
2. *$\sigma_{j_1}\sigma_i\sigma_{j_2}$ are subsequent numbers and $|j_1 - j_2| = 1$, i.e.,*

$$\sigma = \cdots \color{red}{\sigma_\mathbf{i}} \color{black}{\cdots \sigma_\mathbf{i} \pm \mathbf{1} \quad \sigma_\mathbf{i} \mp \mathbf{1} \cdots} \quad \text{or} \quad \sigma = \cdots \sigma_\mathbf{i} \pm \mathbf{1} \quad \sigma_\mathbf{i} \mp \mathbf{1} \cdots \color{red}{\sigma_\mathbf{i}} \color{black}{\cdots}.$$

In this case, we say that σ_i is a **horizontal separator***.*

Example 2.1. *Let $\sigma = 123645$. The element 4 is a vertical separator and the element 5 is a horizontal separator. The element 2 is not a separator of any type, because its removal creates the permutation 12534, containing the bond 12, which already exists in σ.*

The choice of the names of the separators is explained in the following figure, in which $\sigma_1 = 3$ is a horizontal separator (the omitting of which forms the 2-block 23), $\sigma_3 = 5$ is a vertical separator (the omitting of which forms the 2-block 12) and $\sigma_4 = 2$ is both a vertical and a horizontal separator (the omitting of which forms the two 2-blocks 21 and 43).



Definition 2.1. *Let $Sep_V(\pi)$ and $Sep_H(\pi)$ be the sets of vertical and horizontal separators of a permutation π respectively. Let $Sep(\pi) = Sep_V(\pi) \cup Sep_H(\pi)$ and $sep(\pi) = |Sep(\pi)|$.*

Example 2.2. *Let $\sigma = 132465879$. Then, $Sep_V(\sigma) = \{2, 3, 6, 7\}$, and $Sep_H(\sigma) = \{2, 3, 5, 8\}$. Note that 7 is a vertical separator, although 7 is a part of a 2-block, specifically, 87, because by omitting 7 from σ, we obtain a* **new** *2-block, 78.*

Remark 2.1. *Several comments are now in order for a permutation $\sigma = \sigma_1 \cdots \sigma_n$.*

1. *Notice the significance of the word 'new' in Definition 2.1. For example, the identity permutation has an abundance of 2-blocks, although it has no separators.*
2. *The numbers 1 and n can only be vertical separators, while σ_1 and σ_n can only be horizontal separators.*
3. *If σ_i is a vertical separator in σ, then i is a horizontal separator in σ^{-1}. Hence, $Sep_V(\sigma) = Sep_H(\sigma^{-1})$, and thus $Sep(\sigma) = Sep(\sigma^{-1})$.*
4. *For $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$, let $\sigma^r = \sigma_n \cdots \sigma_2\sigma_1$ be the reverse of σ. Then, we have $Sep_V(\sigma) = Sep_V(\sigma^r)$ and $Sep_H(\sigma) = Sep_H(\sigma^r)$.*
5. *A separator can be simultaneously of both vertical and horizontal types. For instance, in example 2.2, the elements 2, 3 are separators of both types.*

3. An upper bound for the number of $(n-2)$-patterns

Let $\sigma \in S_n$. Recall that $D_2(\sigma) = \{\pi \in S_{n-2} \mid \pi \prec \sigma\}$ is the $(n-2)$-th level of the downset of σ. Recall also that $\beta(\sigma)$ is the number of bonds in σ, and let $sep^*(\sigma)$ be the number of separators in σ not contained in a bond of σ. For example, $sep^*(3124) = 1$ because 1 and 2 are separators contained in the bond 12, while 3 is a separator not contained in any bond.

The following result provides an upper bound on $|D_2(\sigma)|$.

Theorem 3.1.

$$|D_2(\sigma)| \leq \binom{n-\beta(\sigma)}{2} + \beta(\sigma) - sep^*(\sigma). \tag{2}$$

Proof. Let $\sigma = \sigma_1 \cdots \sigma_n$. Every pattern $\pi \in D_2(\sigma)$ is obtained by omitting a pair (σ_i, σ_j); hence, we have $|D_2(\sigma)| \leq \binom{n}{2}$. Moreover, if (σ_i, σ_j) is a bond, then for every $\sigma_k \notin \{\sigma_i, \sigma_j\}$, the pattern obtained by omitting σ_i and then σ_k is identical to the pattern obtained by omitting σ_j and then σ_k (assuming without loss of generality that $\sigma_i, \sigma_j > \sigma_k$). This means that each pair constituting a bond in σ can be considered as a single element of σ; thus, the maximal number of patterns $\pi \in D_2(\sigma)$ obtained by omitting pairs of entries which do not form a bond is not more than $\binom{n-\beta(\sigma)}{2}$.

However, to obtain a pattern in $D_2(\sigma)$, we can also omit pairs of the form (σ_i, σ_j) which do constitute a bond in σ. Therefore, we add $\beta(\sigma)$ to the sum.

Note that if we happen to omit from σ a separator σ_i, then we obtain $\tau \in S_{n-1}$ such that $\tau \prec \sigma$ and τ contains a bond (u, v). Omitting σ_i and then u is identical to omitting σ_i and then v; we, therefore, subtract the number of separators of σ to prevent double counting. Note that because we already dealt with entries σ_i which are part of a bond, here we have only to consider the separators $sep^*(\sigma)$ which are not part of a bond. ☐

Remark 3.1. *A slightly tighter upper bound can be achieved if we add $s(\sigma)$ instead of $\beta(\sigma)$ to the binomial in Equation (2), where $s(\sigma)$ is the number of nontrivial strips in σ. The new upper bound is as follows.*

$$|D_2(\sigma)| \leq \binom{n-\beta(\sigma)}{2} + s(\sigma) - sep^*(\sigma). \tag{3}$$

Example 3.1. *Let $\sigma = 641235$. Then, $\beta(\sigma) = 2, s(\sigma) = 1, sep^*(\sigma) = 2$; thus, we have $|D_2(\sigma)| \leq 5$. Indeed, $D_2(\sigma) = \{1234, 3124, 4123, 4213, 4312\}$.*

Remark 3.2. *Note that a separator in σ which is part of a bond is necessarily a part of a triple of the form $(a, a+2, a+1)$ or it is reverse. In this triple, there is one bond and two separators, both of which are parts of the bond $(a+2, a+1)$. Omitting every two entries from this triple produces a single element (a); thus, in this subpermutation π we have $n = 3$, , $s(\pi) = 1$, $sep^*(\pi) = 0$; therefore, $\binom{n-\beta(\pi)}{2} + s(\pi) - sep^*(\pi) = 2$, which is not equal to $|D_2(\pi)| = 1$. We can correct this by subtracting 1 for each such triple, or in other words, by subtracting half of the number of separators that are parts of a bond. The new upper bound is then given by the following expression.*

$$|D_2(\sigma)| \leq \binom{n-\beta(\sigma)}{2} + s(\sigma) - sep^*(\sigma) - \frac{1}{2}sep^{**}(\sigma), \tag{4}$$

where $sep^{**}(\sigma)$ is the number of separators in σ which are contained in a bond of σ.

Note that even the least upper bound is not tight, as the next example testifies.

Example 3.2. *Let $\sigma = 35142$. Then $\beta(\sigma) = 0$, $s(\sigma) = 0$, $sep^*(\sigma) = 3$, $sep^{**}(\sigma) = 0$ so according to Equation 4, we must have $|D_2(\sigma)| \leq 7$. However, we know that $D_2(\sigma) \subseteq S_3$, and thus $|D_2(\sigma)| \neq 7$.*

4. Fully separated permutations

We first deal with the permutations of S_n that have no separators of any type. The table below presents these permutations for $1 \leq n \leq 4$.

$n = 1$	1
$n = 2$	12, 21
$n = 3$	123, 321,
$n = 4$	1234, 4321, 2143, 3412, 4123, 3214, 2341, 1432

The permutations in S_n that have no separators of any type are counted by the sequence A137774 of the OEIS. They correspond to the number of ways to place n non-attacking empresses (a chess piece that moves like a rook and a knight) on an $n \times n$ chessboard. It may be easily observed that this set is closed to reverse and inverse.

Theorem 4.1 below deals with the opposite case, i.e., the number of permutations, all the elements of which are separators. First, we introduce some definitions. A comprehensive survey of the concepts described here can be found in [9].

Definition 4.1. *Let $\pi = \pi_1 \cdots \pi_n \in S_n$. A* block *(or* interval*) of π is a nonempty contiguous sequence of entries $\pi_i\pi_{i+1}\cdots\pi_{i+k}$ whose values also form a contiguous sequence of integers.*

Example 4.1. *If $\pi = 2647513$, then 6475 is a block, but 64751 is not.*

Each permutation can be decomposed into singleton blocks, and also forms a single block by itself; these are the *trivial blocks* of the permutation. All other blocks are called *proper.*

Definition 4.2. *A* block decomposition *of a permutation is a partition of it into disjoint blocks.*

For example, the permutation $\sigma = 67183524$ can be decomposed as 67 1 8 3524. In this example, the relative order between the blocks forms the permutation 3142, i.e., if we take for each block one of its elements as a representative, then the sequence of representatives is order-isomorphic to 3142. Moreover, the block 67 is order-isomorphic to 12, and the block 3524 is order-isomorphic to 2413. These are instances of the concept of *inflation*, defined as follows.

Definition 4.3. *Let $n_1, \cdots, n_k$ be positive integers with $n_1 + \cdots + n_k = n$. The* inflation *of a permutation $\pi \in S_k$ by the permutations $\alpha_i \in S_{n_i}$ $(1 \leq i \leq k)$ is the permutation $\pi[\alpha_1, \cdots, \alpha_k] \in S_n$ obtained by replacing the i-th entry of π with a block which is order-isomorphic to the permutation α_i on the numbers $\{s_i+1, \cdots, s_i+n_i\}$ instead of $\{1, \cdots, n_i\}$, where $s_i = n_1 + \cdots + n_{i-1}$ $(1 \leq i \leq k)$.*

Example 4.2. *The inflation of* 2413 *by* 213, 21, 132 *and* 1 *is*

$$2413[213, 21, 132, 1] = 546\ 98\ 132\ 7.$$

We are interested in the structure of all permutations in S_n in which every element is a separator.

Recall that K_n is the set of *king permutations* of size n, i.e., the set of permutations $\sigma \in S_n$ such that for each $1 < i \leq n$, we have $|\sigma_i - \sigma_{i-1}| > 1$.

In Theorem 3.19 of [5], we proved that in a permutation $\sigma \in K_n$, each element of σ is a separator if and only if $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$, where $\alpha_1, \ldots, \alpha_k \in \{3142, 2413\}$ and $\pi \in S_k$. In the following theorem, we extend this result and show that this structure also captures all the permutations in S_n in which each element is a separator.

Theorem 4.1. *In a permutation* $\sigma \in S_n$*, each element is a separator if and only if* $n = 4k$, $k \in \mathbb{N}$ *and there are* $\alpha_1, \ldots, \alpha_k \in \{3142, 2413\}$ *and* $\pi \in S_k$ *such that* $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$.

Proof. As the "only if" part of the statement is obvious, we prove only the "if" part. Let $\sigma \in S_n$ be a permutation such that each element of σ is a separator. If we show that σ lacks 2-blocks, i.e., $\sigma \in K_n$, then by Theorem 3.18 in [5], this completes the proof. We assume to the contrary that σ contains a block of the form $a, a+1$, and show that not all the elements of σ are separators.

We distinguish two different cases according to the type of separation of $a+1$.

- $a+1$ is a vertical separator. In this case, σ contains the sub-sequence $\cdots a, a+1, a-1 \cdots$. The element $a-1$ is also a separator, so we distinguish between two subcases according to the type of separation of $a-1$.

  1. If $a-1$ is a vertical separator then $\sigma = \cdots a \quad a+1 \quad a-1 \quad a+2 \cdots$. The element $a+2$ must be a separator, but not a horizontal one (otherwise, $a+3$ and $a+1$ should have been adjacent to each other), so $a+2$ is vertical, and we obtain the following:
  $\sigma = \cdots a \quad a+1 \quad a-1 \quad a+2 \quad a-2 \quad \cdots$. By the same argument, $a-2$ must be a vertical separator, and thus $\sigma = \cdots a \quad a+1 \quad a-1 \quad a+2 \quad a-2 \quad a+3 \cdots$. This process continues until we reach σ_n, which cannot be a horizontal separator; however, by Remark 2.1.2, it also cannot be a vertical separator.
  2. If $a-1$ is a horizontal separator then $\sigma = \cdots a-2 \quad a \quad a+1 \quad a-1 \cdots$. Hence, $a-2$ must be a horizontal separator, such that $\sigma = \cdots a-2 \quad a \quad a+1 \quad a-1 \quad a-3 \cdots$. By the same argument, $a-3$ must be a horizontal separator, and thus $\sigma = \cdots a-4 \quad a-2 \quad a \quad a+1 \quad a-1 \quad a-3 \cdots$. This process continues until we reach $a-k = 1$, which cannot be a vertical separator; however, by Remark 2.1.2, it also cannot be a horizontal separator.

- $a+1$ is a horizontal separator; in this case, we consider σ^{-1} in which $(\sigma^{-1})_{a+1}$ is a vertical separator by Remark 2.1.3. Because $a, a+1$ is a block in σ, σ^{-1} contains a block in locations $a, a+1$, which means that $(\sigma^{-1})_a = b, (\sigma^{-1})_{a+1} = b+1$ constitutes a bond with $b+1$ as a vertical separator, returning to the previous case.

We conclude that $\sigma \in K_n$ where K_n is the set of king permutations of order n, and as mentioned before, the proof is completed by Theorem 3.18 of [5]. ☐

Figure 1 illustrates the structure of such permutations in which each of their elements is a separator, according to the above theorem.

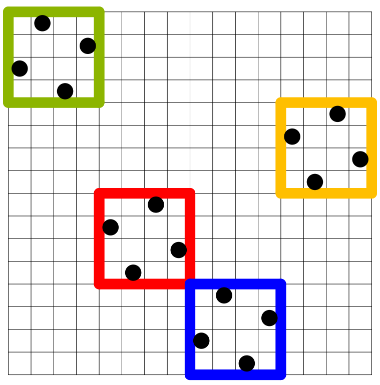


Figure 1: The plot of 14 16 13 15 7 5 8 6 2 4 1 3 11 9 12 10

According to Theorem 4.1, we can now enumerate these permutations.

Corollary 4.1. *The number of permutations in S_n which have exactly n different separators is given by the following expression.*

$$\begin{cases} 2^k k! & if \quad n = 4k \\ 0 & otherwise. \end{cases}$$

Those permutations for $k = 1$ and $k = 2$ are:

$k = 1$ $(n = 4)$	$2413, 3142$
$k = 2$ $(n = 8)$	$24136857, 24137586, 31426857, 31427586, 68572413, 75862413, 68573142, 75863142$

5. The expectation of the number of separators

In this section, we calculate the expectation of the number of separators in a randomly chosen permutation of S_n. To do so, let us first calculate the expectation of the number of vertical separators. Consider the sample space of all $n!$ permutations, with a uniform probability, and for each $\pi \in S_n$, let $X(\pi)$ count the number of vertical separators in π. For each $1 \leq i \leq n$, let X_i be the characteristic random variable of entry i as a vertical separator. Obviously, $X = \sum_{i=1}^n X_i$ counts the number of vertical separators for each $\pi \in S_n$.

To calculate $E[X] = \sum_{i=1}^n E[X_i]$, let us first calculate $E[X_i]$ for each $1 \leq i \leq n$. The entry i is a vertical separator in a permutation $\pi \in S_n$ if π contains a consecutive sequence of the form $a, i, a+1$ or its reverse. If $i \in \{1, n\}$, the entry a can be chosen in $n-2$ ways. After choosing a, we have $(n-2)!$ ways to arrange the rest of the permutation, such that $E[X_1] = E[X_n] = \frac{2(n-2)(n-2)!}{n!}$. Now, for each $1 < i < n$, the same consideration applies, but now we have only $(n-3)$ ways to choose a. Thus, we can observe that $E[X_i] = \frac{2(n-3)(n-2)!}{n!}$. From the above arguments, we obtain

$$E[X] = \sum_{i=1}^{n} E[X_i] = 2E[X_1] + (n-2)E[X_2].$$

A simple calculation now yields the following.

Theorem 5.1. *The expectation of the number of vertical separators in a randomly chosen n-permutation is $\frac{2(n-2)}{n}$. The asymptotic value of the expectation is 2.*

We calculate now the expectation of the number of separators that are both vertical and horizontal. Let Y be a random variable counting the number of entries of a permutation π which are separators of both types and for $1 < i < n$, denote by Y_i the appropriate characteristic variable. Note that the entries 1 and n cannot be separators of both types.

Let us calculate $E[Y_i]$, where $1 < i < n$ is a separator of both types. Because entry i is vertical, we have $\pi = \cdots a \quad i \quad a+1 \cdots$ or its reverse. Moreover, i, being also horizontal, the pair $i-1, i+1$ must be adjacent in π.

Depending on the structure of π, we now obtain one of the following cases.

1. If $a = i+1$ or $a+1 = i-1$ then $\pi = \cdots i-1 \quad i+1 \quad \mathbf{i} \quad i+2 \cdots$ or $\pi = \cdots i-2 \quad \mathbf{i} \quad i-1 \quad i+1 \cdots$ respectively. For $i = 2$, only the first case or its reverse can hold, whereas for $i = n-1$, only the second case or its reverse can hold. For $2 < i < n-1$, both cases and their reverses can hold. For each of these subcases, there are exactly $(n-3)!$ ways to arrange the rest of the permutation.

2. Else, the permutation π contains two different sub-sequences of the forms $a, i, a+1$ and $i-1, i+1$ or their reverses.

   If $i \in \{2, n-1\}$, then there are $n-4$ ways to choose a, whereas if $2 < i < n-1$, then there are $n-5$ ways to choose a. Each choice of such two subsequences leaves $(n-3)!$ ways to arrange the remainder of the permutation.

Hence, we have

$$E[Y_i] = \begin{cases} 0 & i \in \{1, n\} \\ \frac{2\cdot(n-3)!+2\cdot2\cdot(n-3)!(n-4)}{n!} & i \in \{2, n-1\} \\ \frac{4\cdot(n-3)!+2\cdot2\cdot(n-3)!(n-5)}{n!} & 2 < i < n-1 \end{cases}$$

This allows us to observe that

$$E[Y] = \sum_{i=1}^{n} E[Y_i] = 2E[Y_2] + (n-4)E[Y_3].$$

A simple calculation now yields the following.

Theorem 5.2. *The expectation of the number of separators of both types, vertical and horizontal, in a randomly chosen n-permutation is*

$$\frac{4(n-3)^2}{n(n-1)(n-2)},$$

and the asymptotic value of the expectation is 0.

Let Z be a random variable which counts the total number of separators (regardless of type). By Remark 2.1.3, the number of vertical separators has the same distribution as the number of horizontal separators; therefore, $E(Z) = 2E(X) - E(Y)$. Thus, we have the following.

Theorem 5.3. *The expectation of the number of separators in a randomly chosen n-permutation is*

$$\frac{4(n^3-6n^2+14n-13)}{n(n-1)(n-2)}.$$

The asymptotic value of the expectation is 4.

6. A generating function for vertical or horizontal separators

In this section we present a generating function for the number of vertical separators. For each $n, m \in \mathbb{N}$, let $s_{n,m}$ be the number of permutations $\pi \in S_n$ with exactly m vertical separators. We want to calculate the generating function $h(z,u) = \sum_{n\geq 0}\sum_{m=0}^{n} s_{n,m} z^n u^m$. According to remark 2.1.3, this generating function is the same for horizontal separators.

To derive the function $h(z,u)$, we enlarge the set of elements to contain marked permutations (to be defined below). We then use the principle of inclusion-exclusion together with a method of splitting permutations into two parts to attain the generating function for the number of vertical separators.

6.1 Counting permutation with marked bonds

A *marked permutation* is a permutation in which each bond can be chosen to be marked or not. The marked bonds are denoted by bars above the corresponding part of the permutation. If several adjacent bonds are marked, then we mark a long bar above the corresponding strip. An entry that is not contained in a marked bond is considered to be a strip of length 1.

Example 6.1. *Let* $\pi = 613452879$. *Here are some permutations with marked bonds, made out of* π*:* $61\overline{345}2\overline{87}9$, $61\overline{34}52879$, *and* $61345\overline{28}\overline{7}9$.

To count the marked permutations, we introduce some alternative notation.

Recall that for $n \in \mathbb{N}$, a *composition* with m non-zero parts of n is a vector $(a_1, \ldots, a_m)$ such that $a_i \in \mathbb{N}$ and $\sum_{i=1}^{m} a_i = n$. We define an *arrowed composition* of n to be a composition in which, after every part which is greater than 1, there exists one of the signs $\uparrow$ or $\downarrow$. For example, $(2\uparrow, 1, 7\downarrow, 2\uparrow)$ is an arrowed composition of $n = 12$.

Now, each marked permutation $\pi \in S_n$ can be uniquely presented as an arrowed composition $(a_1, \ldots, a_m)$ of n, together with a permutation $\sigma \in S_m$. This idea is best clarified by the following example.

Example 6.2. *Let* $\pi = 2\overline{45}61\overline{987}3$. *We write* π *as a pair consisting of an arrowed composition of* $m = 6$ *parts* λ*, and a permutation* $\sigma \in S_6$*. First, we express* π *as a sequence of strips.* $b_1 = 2, b_2 = \overline{45}, b_3 = 6, b_4 = 1, b_5 = \overline{987}, b_6 = 3$. *Each strip contributes its length to the composition. Then, for each part, we add the sign* $\uparrow$ *if the corresponding strip is increasing, the sign* $\downarrow$ *if the strip is decreasing, and no arrow if the strip is of length* 1. *In our case, we obtain* $\lambda = (1, 2\uparrow, 1, 1, 3\downarrow, 1)$. *Now,* $\sigma \in S_6$ *is the permutation induced by the order of the blocks. In this case, we have* $\sigma = 245163$. *The marked permutation* π *is now uniquely defined by the pair* (λ, σ).

In other words, if we replace each $j\uparrow$ with the ascending permutation $123\cdots j$ and each $j\downarrow$ with the descending permutation $j\cdots 321$, we can see that this defines inflation. In the previous example, we can write $\pi = 245163[1, 12, 1, 1, 321, 1]$. For convenience, we denote this inflation by $\sigma[\lambda]$.

In [7] (during the proof of Theorem 10), C. Homberger calculated the generating function, counting the number of permutations having a specific number of bonds. This was done by calculating the generating function of marked bonds, and by using the inclusion-exclusion principle. If we denote by $a_{n,m}$ the number of permutations of S_n with m marked bonds and set $A(z,u) = \sum_{n\geq 1}\sum_{m\geq 0} a_{n,m} z^n u^m$, then the identity permutation contributes z, and for each $j \geq 2$, a strip of order j can be either $123\cdots j$ or $j\cdots 321$, each of which has $j-1$ bonds and thus contributes u^{j-1}. Thus, the contribution is $2z^j u^{j-1}$. It may be easily observed from the above that

$$A(z,u) = \sum_{m\geq 0} m!(z + 2z^2u + 2z^3u^2 + 2z^4u^3 + \cdots)^m = \sum_{m\geq 0} m! z^m \left(\frac{1+zu}{1-zu}\right)^m. \tag{5}$$

(Here m denotes the number of strips).

Remark 6.1. *The coefficients of $z^n u^k$ in the above formula can be easily calculated as follows.*

$$\begin{aligned} A(z,u) &= 1 + \sum_{m\geq 0} (m+1)! z^{m+1}(1+zu)^{m+1}/(1-zu)^{m+1} \\ &= 1 + \sum_{m\geq 0} \sum_{j=0}^{m+1} \sum_{i\geq 0} (m+1)! \binom{m+1}{j} \binom{m+i}{i} z^{m+1+j+i} u^{j+i}. \end{aligned}$$

Thus, $a_{n,k} = [z^n u^k] A(z,u) = \sum_{j=0}^{n-k} (n-k)! \binom{n-k}{j} \binom{n-1-j}{k-j}$.

6.2 Comb decomposition and marked separators

Coming back to our counting of permutations with respect to the number of vertical separators, we now demonstrate a method to reduce this problem to the problem of counting marked permutations with respect to the number of bonds. We begin with a definition of what we refer to as *comb permutations*, as follows.

Definition 6.1. *Let $\sigma = (\sigma_1, \ldots, \sigma_k), \tau = (\tau_1, \ldots, \tau_k)$ be two sequences such that $\{\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_k\} = \{1, 2, \ldots, 2k\}$. Define the comb permutation $\pi = \sigma \odot \tau$ by $\pi = \sigma_1\tau_1\sigma_2\tau_2 \cdots \sigma_k\tau_k \in S_{2k}$. Similarly, let $\sigma = (\sigma_1, \ldots, \sigma_{k+1}), \tau = (\tau_1, \ldots, \tau_k)$ be two sequences such that $\{\sigma_1, \ldots, \sigma_{k+1}, \tau_1, \ldots, \tau_k\} = \{1, 2, \ldots, 2k+1\}$. We define the comb permutation $\pi = \sigma \odot \tau$ as $\pi = \sigma_1\tau_1\sigma_2\tau_2 \ldots \sigma_k\tau_k\sigma_{k+1} \in S_{2k+1}$. When $\pi = \sigma \odot \tau$, we denote σ as π^{odd} and τ as π^{even}.*

Now, let $\pi = \pi^{odd} \odot \pi^{even}$ where π^{odd} and π^{even} are sequences with marked bonds. Note that if $(\pi_i^{odd}\pi_{i+1}^{odd})$ is a marked bond of π^{odd}, then the element of π^{even} which lies between π_i^{odd} and π_{i+1}^{odd} in π is a vertical separator, we call it a *marked separator* and denote it by marking a hat over it. For example, if $\pi^{odd} = (\overline{12}4)$ and $\pi^{even} = (53)$, then $\overline{12}$ is a marked bond in π^{odd}, and thus 5 is a marked separator in $\pi^{odd} \odot \pi^{even} = 1\hat{5}234$. Similarly, we define marked separators for marked bonds in π^{even}. Therefore, we obtain the following.

Observation 6.1. *Let π^{odd}, π^{even} be defined as in Definition 6.1, and let $\pi = \pi^{odd} \odot \pi^{even}$. Then, the number of (marked) vertical separators in π is equal to the total number of (marked) bonds in π^{odd} and π^{even}.*

Let $n = 2k$. Given two arrowed compositions of k, λ_o of size m_o and λ_e of size m_e, and given a permutation $\sigma \in S_{m_o+m_e}$, we take the inflation $\alpha = \sigma[\lambda_o, \lambda_e]$, where λ_o, λ_e denotes the concatenation of λ_o and λ_e (it is a permutation with marked bonds). We construct a permutation π as follows. Denote the first k elements of α by π^{odd} and the last k elements of α by π^{even}. Now, $\pi = \pi^{odd} \odot \pi^{even}$. The case where $n = 2k+1$ is similar.

Example 6.3. *For $k = 4$, let $\lambda_o = (1, 3\downarrow)$, $\lambda_e = (1, 1, 2\uparrow)$ and let $\sigma = 34215$. Then,* $\alpha = 34215[1, 321, 1, 1, 12] = 3\overline{654}21\overline{78}$. *Thus, $\pi^{odd} = (3\overline{654})$ and $\pi^{even} = (21\overline{78})$; therefore, $\pi = (3\overline{654}) \odot (21\overline{78}) = 326\hat{1}5\hat{7}\hat{4}8$.*

On the other hand, given a permutation $\pi \in S_n$ with marked separators, let π^{odd} be the sequence of the odd entries of π and π^{even} be the sequence of the even entries of π, and mark the relevant bonds, i.e., if π_i is a marked separator in π, then $(\pi_{i-1}\pi_{i+1})$ is a marked bond in π^{odd} or in π^{even}. We denote by α the permutation with marked bonds obtained by π^{odd}, followed by π^{even}. We know that α can be uniquely presented as an arrowed composition of n, together with a permutation of the number of parts m.

Example 6.4. $\pi = 27\hat{1}86\hat{3}549$. *We apply the sequences $\pi^{odd} = (21\overline{659})$ and $\pi^{even} = (\overline{78}34)$ to produce $\alpha = 21\overline{659}\overline{78}34$. This permutation can be presented as $\alpha = \sigma[\lambda_o, \lambda_e]$, where $\lambda_o = (1, 1, 2\downarrow, 1)$ and $\lambda_e = (2\uparrow, 1, 1)$ with $\sigma = 2157634$.*

6.3 Calculating the generating function for vertical separators

Recall that our goal is to find the function $h(z,u)$, which is the generating function for the number of vertical separators. To do so, we first calculate the generating function for the number of **marked** vertical separators. Denote by $b_{n,m}$ the number of permutations of S_n with m marked vertical separators, and let $g(z,u) = \sum_{n\geq 1} \sum_{m\geq 0} b_{n,m} z^n u^m$.

As demonstrated above, there is a correspondence between the number of marked bonds and the number of marked vertical separators; thus, we construct the generating function $g(z,u)$ by separately calculating the generating functions for the marked bonds of the odd and even parts of each permutation. The requirement that the odd and even parts of a permutation must have (almost) the same size will be met by using the well-known *Hadamard product* (element-wise) of polynomials and series.

Definition 6.2. *Let R be a ring and let $f(x)=\sum_{n\in\mathbb{N}} a_n x^n$, $g(x)=\sum_{n\in\mathbb{N}} b_n x^n \in R[[x]]$ be two power series in x. The Hadamard product of $f(x), g(x)$ is $f(x)*g(x)=\sum_{n=0}^{\infty} a_n b_n x^n$.*

Example 6.5. $(2+3x-4x^2)*(5+x+7x^2)=10+3x-28x^2$.

In order to form the generating function of the marked separators, $g(z,u)$, let us observe the permutations $\pi\in S_n$ for a fixed n. We would like to find the monomial contribution of each $\pi=\pi^{odd}\odot\pi^{even}\in S_n$ using the monomial corresponding to the marked bonds of π^{odd} and π^{even}.

Let $n=2k$; in this case, π^{odd},π^{even} are sequences of order k; we therefore use the Hadamard product to combine the two monomials. Each $\pi\in S_{2k}$ contributes to $g(z,u)$ a monomial of the form $u^m z^{2k}$, where the monomials corresponding to the marked bonds of π^{odd} and π^{even} are $f_o(z,u)=u^{m_1}z^k$ and $f_e(z,u)=u^{m_2}z^k$, respectively, where $m=m_1+m_2$. If we view these monomials as functions of the variable z, then we may easily note that the coefficient of z^{2k} in $g(z,u)$ should be the product of the coefficients of z^k of the odd and even parts. Therefore, it is necessary to substitute z^2 instead of z in the monomoials of π^{odd} and π^{even} to obtain

$$f_o(z^2,u)*f_e(z^2,u)=u^{m_1}(z^2)^k*u^{m_2}(z^2)^k=u^m z^{2k}.$$

Example 6.6. *Returning to Example 6.3, the monomial corresponding to the marked bonds of $\pi^{odd}=(3\overline{654})$ is $f_o(z,u)=z^4u^2$. Similarly, the monomial corresponding to the marked bonds of $\pi^{even}=(21\overline{78})$ is $f_e(z,u)=z^4u$. Thus $f_o(z,u)*f_e(z,u)=z^4u^3$. However, the monomial corresponding to the marked separators of $\pi=(3\overline{654})\odot(21\overline{78})=326\hat{1}5\hat{7}\hat{4}8$ should be z^8u^3. Note that if we take $f_o(z^2,u)*f_e(z^2,u)$, we obtain exactly the necessary formulation.*

Similarly, if $n=2k+1$, then each $\pi=\pi^{odd}\odot\pi^{even}\in S_n$ contributes a monomial of the form $u^m z^{2k+1}$ where the monomial of π^{odd} is $f_o(z,u)=u^{m_1}z^{k+1}$ and the monomial of π^{even} is $f_e(z,u)=u^{m_2}z^k$, $m=m_1+m_2$. Thus,

$$\left(\frac{1}{z}f_o(z^2,u)\right)*\left(z\,f_e(z^2,u)\right)=\left(\frac{1}{z}u^{m_1}(z^2)^{k+1}\right)*\left(z\,u^{m_2}(z^2)^k\right)=u^m z^{2k+1}.$$

Example 6.7. *Let $\pi=2\hat{5}\hat{3}4176$. Then, $\pi^{odd}=(\overline{23}16)$ and $\pi^{even}=(\overline{54}7)$. Then $f_o(z,u)=uz^4$ and $f_e(z,u)=uz^3$. Thus,*

$$\left(\frac{1}{z}f_o(z^2,u)\right)*\left(zf_e(z^2,u)\right)=\left(\frac{1}{z}u(z^2)^4\right)*\left(zu(z^2)^3\right)=u^2z^7,$$

as required.

Using the above explanations, and the generating function version for the inclusion-exclusion principle, we obtain the following calculation of the generating function of vertical separators, $h(z,u)$.

Theorem 6.1.

$$\begin{aligned} h(z,u)=&\sum_{m_o,m_e\geq 0}(m_o+m_e)!\left(\left(z^{2m_o}\left(\frac{1+z^2(u-1)}{1-z^2(u-1)}\right)^{m_o}\right)*\left(z^{2m_e}\left(\frac{1+z^2(u-1)}{1-z^2(u-1)}\right)^{m_e}\right)\right)\\ &+\sum_{m_o,m_e\geq 0}(m_o+m_e)!\left(\left(z^{2m_o}\left(\frac{1+z^2(u-1)}{1-z^2(u-1)}\right)^{m_o}\frac{1}{z}\right)*\left(z^{2m_e}\left(\frac{1+z^2(u-1)}{1-z^2(u-1)}\right)^{m_e}z\right)\right),\end{aligned}$$

where $*$ is the Hadamard product in $\mathbb{Q}[[u]][[z]]$.

Proof. Let us denote for each $m\in\mathbb{N}$

$$p_m(z,v)=(z+2z^2v+2z^3v^2+\cdots)^m=z^m\left(\frac{1+zv}{1-zv}\right)^m.$$

Then $p_m(z,v)$ counts the number of possible procedures to construct an arrowed composition λ of m parts. We relate to n even and n odd separately. To construct a permutation of S_{2k} with marked separators, we must choose two arrowed compositions of k, including λ_o of size m_o, and λ_e of size m_e. We also choose a permutation $\sigma\in S_{m_o+m_e}$. It is easy to see that this contributes to the function $(m_o+m_e)!p_{m_o}(z^2,v)*p_{m_e}(z^2,v)$. For S_{2k+1}, we obtain $(m_o + m_e)!\left[\frac{1}{z}p_{m_o}(z^2, v))\right] * \left[z\, p_{m_e}(z^2, v)\right]$. Going over all the values for m_o and m_e for both n even and n odd, we obtain the generating function of the marked vertical separators.

$$g(z,v) = \sum_{m_o,m_e \geq 0} (m_o + m_e)! \left(\left(z^{2m_o} \left(\frac{1+z^2v}{1-z^2v} \right)^{m_o} \right) * \left(z^{2m_e} \left(\frac{1+z^2v}{1-z^2v} \right)^{m_e} \right) \right) + \sum_{m_o,m_e \geq 0} (m_o + m_e)! \left(\left(z^{2m_o} \left(\frac{1+z^2v}{1-z^2v} \right)^{m_o} \frac{1}{z} \right) * \left(z^{2m_e} \left(\frac{1+z^2v}{1-z^2v} \right)^{m_e} z \right) \right).$$

Now, we can use this generating function to obtain $h(z, u)$. The variable v represents the **marked** vertical separators, whereas u represents all vertical separators . Because every vertical separator can either be marked or unmarked, it follows that by replacing $v + 1$ with u, we obtain that the generating function of the vertical separators is $h(z, u) = g(z, u - 1)$. □

7. A direction for further research

In Section 6.1, we introduced a generating function for vertical separators. An interesting direction for continuing this research may be to present for each $n \in N$ a three-variable generating function as follows.

$$F(z,u,v) = \sum_{k>0}\sum_{m>0}\sum_{l>0} a_{k,m,l} z^k u^m v^l,$$

where $a_{k,m,l}$ is the number of permutations of order n with k vertical separators, m horizontal separators, and l separators of both types. One interesting result for such a function might be the distribution of the total number of separators in a permutation of order n. The function $G(z) = F(z, z, \frac{1}{z})$ calculates this distribution. This distribution affects, among other things, the structure of the poset of permutations under containment, because each separator of a permutation $\pi \in S_n$, decreases the number of permutations $\sigma \prec \pi$ of order $n - 2$ by 1.

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References

[1] D. Bevan, C. Homberger, and B. E. Tenner, *Prolific permutations and permuted packings: downsets containing many large patterns*, J. Combin. Theory, Ser. A 153 (2018), 98–121.

[2] J. Bloom and A. Burstein, *Egge triples and unbalanced Wilf-equivalence*, Australasian J. Combin. 64(1) (2016), 232–251.

[3] P. Bose, J.F. Buss, and A. Lubiw, *Pattern matching for permutations*, Information Processing Letters 65 (1998), 277–283.

[4] A. Burstein and J. Pantone, *Two examples of unbalanced Wilf-equivalence*, Journal of Combinatorics 6:1-2 (2015), 55–67.

[5] E. Bagno, E. Eisenberg, S. Reches, and M. Sigron, *On the poset of King-Non-Attacking permutations*, Europ. J. Combin. 87 (2020), Article 103119.

[6] S. Fu, Z. Lin, and J. Zeng, *Two new unimodal descent polynomials*, Discrete Math. 341:9 (2018), 2616–2626.

[7] C. Homberger, *Counting Fixed-Length Permutation Patterns*, Online Journal of Analytic Combinatorics 7 (2012), Article 3.

[8] I. Kaplansky, *The asymptotic distribution of runs of consecutive elements*, Ann. Math. Stat. 16 (1945), 200–203.

[9] S. Kitaev, *Patterns in Permutations and Words*, Springer-Verlag Berlin Heidelberg, 2011.

[10] V́. Kotêsovec, *Non-attacking chess pieces (chess and mathematics)*, 2010; `http://problem64.beda.cz/silo/kotesovecnonattackingchesspieces2010.pdf`.

[11] J. Pantone, *The enumeration of permutations avoiding 3124 and 4312*, 2013, arXiv:1309.0832.

[12] D. P. Robins, *The probability that neighbors remain neighbors after random rearrangements*, The American Mathematical Monthly 87:2 (1980), 122–124.

[13] J. Wolfowitz, *Note on runs of consecutive elements*, Ann. Math. Stat. 15 (1944), 97–98.