

Combinatorialization of Sury and McLaughlin Identities and General Linear Recurrences by a Unified Approach

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ABSTRACT: In this article, we provide combinatorial proofs of some prior identities due to Sury and McLaughlin. We show that the solution of a general linear recurrence with constant coefficients can be interpreted as a determinant of a matrix. Moreover, we derive determinantal expressions for the Fibonacci and Lucas numbers. We prove Binet's formula for Fibonacci and Lucas numbers in a purely combinatorial way and, in the course of doing so, find a determinantal identity which we believe to be new.

Keywords: Combinatorial proof; Determinants; Digraphs; Symmetric functions; Words

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1. Introduction

This paper involves finding combinatorial proofs of some prior identities due to Sury and McLaughlin and providing a unified treatment of determinantal formulas of general linear recurrence with constant coefficients. It should be noted that combinatorial proofs are often preferred by enumerative combinatorists to more analytical ones as they frequently yield more insight into “why” a particular result is true rather than just “how” it is true [1, 5, 8, 12].

It is not always possible to find a closed form expression for an arbitrary term of a sequence defined by a recurrence relation with given initial conditions. Many well-known integer sequences, however, such as the Fibonacci and Lucas numbers and their generalizations and relatives, do admit to such a closed form. Identities concerning Fibonacci numbers and their relatives frequently can be afforded a combinatorial explanation in terms of tilings of an n -board [4]. Moreover, there are several known proofs of Binet's formula for the Fibonacci and Lucas numbers [2, 3], some of which are combinatorial. In particular, in [3], a probabilistic proof employing random tilings of an infinite board with squares and dominoes is given for Binet's formula and its generalization to arbitrary initial conditions. See also the related paper [10] where infinite tilings are used to explain some reciprocal Fibonacci number relations. Here, we demonstrate that the solution of a general linear recurrence with constant coefficients can be expressed as a determinant in a purely combinatorial way and particularizing our results yields the Binet formulas for the Fibonacci and Lucas numbers. Our combinatorial approach also implies some identities due to Sury and McLaughlin.

Let us briefly summarize the content of this paper. In Section 2, we give a bijective proof of an identity regarding elementary and homogeneous symmetric polynomials and, as a corollary, one gets bijective proofs of some identities of Sury and McLaughlin. In Section 3, we prove that the solution of a general linear recurrence with constant coefficients can be interpreted as a determinant of some matrix. We prove Binet's formula for Fibonacci numbers in a combinatorial way. In Section 4, we derive a new combinatorial identity and, as a corollary, we get Binet's formula for the Lucas numbers.

2. Combinatorial Explanation of Sury and McLaughlin Identities

In this section, we give combinatorial proofs of some identities due to Sury and McLaughlin. Before starting, let us review some necessary background. For further details, we refer the reader to [6, 9]. Let $A = (a_{ij})$ denote an $n \times n$ matrix. We associate with A a weighted digraph $D(A)$ whose vertex set is $[n] = \{1, 2, \dots, n\}$ such that, for each ordered pair (i, j) , there is an edge directed from i to j with weight a_{ij} . A *linear subdigraph* (LSD) L of $D(A)$ is a spanning collection of pairwise vertex-disjoint cycles. A loop around a single vertex is also considered to be a cycle. The *weight* of a linear subdigraph L , written as $w(L)$, is the product of the weights of all its cycles. The weight of a cycle is the product of the weights of all its edges. The *length* of a cycle is the number of edges present in that cycle. The number of cycles contained in L is denoted by $c(L)$. Now the cycle-decomposition of permutations yields the following description of $\det(A)$, namely, $\det(A) = \sum_L (-1)^{n-c(L)} w(L)$, where the summation runs over all linear subdigraphs L of $D(A)$.

A *partition* λ of a positive integer m is a weakly decreasing finite sequence $(\lambda_1, \dots, \lambda_r)$ of non-negative integers such that $\sum_{i=1}^r \lambda_i = m$. For n variables x_1, \dots, x_n , the elementary symmetric polynomial $e_k(x_1, \dots, x_n)$ (in short, e_k) of degree $k \geq 0$ is defined as

$$e_k(x_1, \dots, x_n) := \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} x_{j_1} x_{j_2} \cdots x_{j_k}, \text{ with } e_0 = 1.$$

The n -variable complete homogeneous symmetric polynomial $h_k(x_1, \dots, x_n)$ (in short, h_k) of degree $k \geq 0$ is the sum of all monomials of total degree k . Formally,

$$h_k(x_1, \dots, x_n) := \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq n} x_{j_1} x_{j_2} \cdots x_{j_k}, \text{ with } h_0 = 1.$$

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition and $\delta = (\ell - 1, \ell - 2, \dots, 1, 0)$, where $\lambda_1 \geq \dots \geq \lambda_\ell$ and each λ_j is a non-negative integer. Then the function

$$a_{\lambda+\delta}(x_1, \dots, x_\ell) = \det \begin{pmatrix} x_1^{\lambda_1+\ell-1} & x_2^{\lambda_1+\ell-1} & \cdots & x_\ell^{\lambda_1+\ell-1} \\ x_1^{\lambda_2+\ell-2} & x_2^{\lambda_2+\ell-2} & \cdots & x_\ell^{\lambda_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_\ell} & x_2^{\lambda_\ell} & \cdots & x_\ell^{\lambda_\ell} \end{pmatrix}$$

is an alternating polynomial. The Schur polynomial, denoted by $s_\lambda(x_1, \dots, x_\ell)$, is defined as the ratio

$$\frac{a_{\lambda+\delta}(x_1, \dots, x_\ell)}{a_\delta(x_1, \dots, x_\ell)}.$$

Let e_1, \dots, e_m be the elementary symmetric polynomials in the variables x_1, \dots, x_n . We consider the following $m \times m$ matrix

$$E(e_1, \dots, e_m) = \begin{pmatrix} e_1 & e_2 & e_3 & \cdots & e_{m-1} & e_m \\ 1 & e_1 & e_2 & \cdots & e_{m-2} & e_{m-1} \\ 0 & 1 & e_1 & \cdots & e_{m-3} & e_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & e_2 \\ 0 & 0 & 0 & \cdots & 1 & e_1 \end{pmatrix}. \tag{1}$$

Theorem 2.1 (Theorem 4.5.1 [9]). *Let e_1, \dots, e_m be the elementary symmetric polynomials in the variables x_1, \dots, x_n . Let $E(e_1, \dots, e_m)$ be the $m \times m$ matrix defined as (1). Then $\det(E(e_1, \dots, e_m)) = h_m(x_1, \dots, x_n)$.*

Proof. We prove the theorem in the case $m = 5, n = 3$ (similarly, one can show the general case). Regard $W(x_1, x_2, x_3) = \{x_1, x_2, x_3\}$ as the set of letters. The *free monoid* $W(x_1, x_2, x_3)^*$ is the set of all finite sequences (including the empty sequence, denoted by 1) of elements of $W(x_1, x_2, x_3)$, usually called *linear words*, with the operation of concatenation. Construct an algebra from $W(x_1, x_2, x_3)^*$ by taking formal sums of elements of

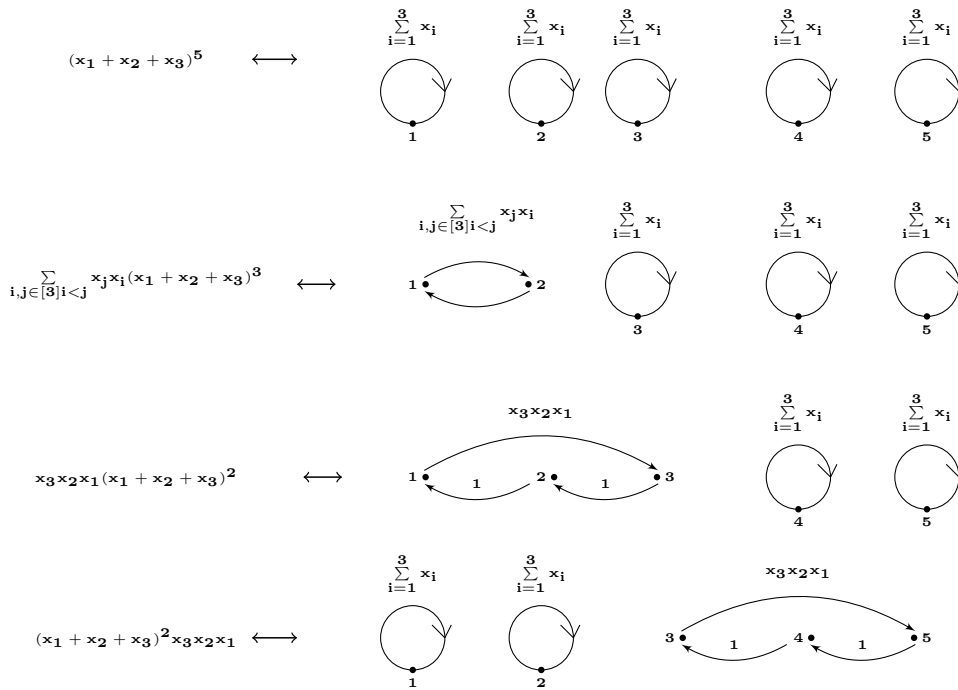


Figure 1: The numbers appearing along the edges in the above diagram are the weights of the corresponding edges. The left-hand side of the figure contains some terms of the PIE expression.

$W(x_1, x_2, x_3)$ with coefficient in \mathbb{Z} and extending the multiplication by the usual distributive law. For example, in this algebra, $(x_1 + x_2 + x_3)(x_1 + x_2 + x_3) = x_1x_1 + x_1x_2 + x_1x_3 + x_2x_1 + x_2x_2 + x_2x_3 + x_3x_1 + x_3x_2 + x_3x_3$ and $(1 + x_3)x_2 = x_2 + x_3x_2$. Note that $(x_1 + x_2 + x_3)(x_1 + x_2 + x_3)$ is usually written as $(x_1 + x_2 + x_3)^2$, etc.

Let W_L be the sum of all linear words in $W(x_1, x_2, x_3)^*$ of length 5, where the letter x_2 does not occur just before the letter x_1 and the letter x_3 does not occur just before the letters x_1 and x_2 , i.e., x_2x_1 , x_3x_1 and x_3x_2 do not occur in the words as a consecutive pair. So,

$$W_L = x_1x_1x_1x_1x_1 + x_1x_1x_1x_1x_2 + \dots + x_2x_2x_2x_2x_2 + x_1x_1x_1x_1x_3 + \dots + x_3x_3x_3x_3x_3.$$

Now we compute W_L by using the *Principle of Inclusion and Exclusion (PIE)* rule.

The sum of all possible words of length 5 is $(x_1 + x_2 + x_3)^5$. This can be written as

$$(x_1 + x_2 + x_3)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3).$$

The sum of all possible words of length 5, where there is an occurrence of x_2x_1 at least as the first two letters, is $x_2x_1(x_1 + x_2 + x_3)^3$. Similarly, the sum of all possible words of length 5, where there is an occurrence of x_3x_2 and x_3x_1 at least as the first two letters, is given by $x_3x_2(x_1 + x_2 + x_3)^3$ and $x_3x_1(x_1 + x_2 + x_3)^3$, respectively. The sum of all possible words of length 5, where there is an occurrence of $x_3x_2x_1$ at least as the first three letters, is $x_3x_2x_1(x_1 + x_2 + x_3)^2$. Proceeding this way, and using the PIE rule, we get

$$\begin{aligned} W_L = & (x_1 + x_2 + x_3)^5 - x_2x_1(x_1 + x_2 + x_3)^3 - x_3x_1(x_1 + x_2 + x_3)^3 - x_3x_2(x_1 + x_2 + x_3)^3 - \\ & (x_1 + x_2 + x_3)x_2x_1(x_1 + x_2 + x_3)^2 - (x_1 + x_2 + x_3)x_3x_1(x_1 + x_2 + x_3)^2 - \\ & (x_1 + x_2 + x_3)x_3x_2(x_1 + x_2 + x_3)^2 + \dots + x_3x_2x_1(x_1 + x_2 + x_3)^2 + \\ & (x_1 + x_2 + x_3)x_3x_2x_1(x_1 + x_2 + x_3) + (x_1 + x_2 + x_3)^2x_3x_2x_1. \end{aligned}$$

Now the terms appearing in the above PIE expression of W_L are $(x_1 + x_2 + x_3)^5$, $x_2x_1(x_1 + x_2 + x_3)^3$, $x_3x_1(x_1 + x_2 + x_3)^3$, $x_3x_2(x_1 + x_2 + x_3)^3$, \dots , $x_3x_2x_1(x_1 + x_2 + x_3)^2$, $(x_1 + x_2 + x_3)x_3x_2x_1(x_1 + x_2 + x_3)$, $(x_1 + x_2 + x_3)^2x_3x_2x_1$, and the signs of the corresponding terms are $+, -, -, -, \dots, +, +, +$, respectively. We define the weight of a word w in $W(x_1, x_2, x_3)^*$ as the homomorphic image $f(w)$, where $f : W(x_1, x_2, x_3)^* \rightarrow \mathbb{Z}[x_1, x_2, x_3]$ is a ring homomorphism, with $f(x_1) = x_1$, $f(x_2) = x_2$ and $f(x_3) = x_3$. Figure 1 illustrates the sign- and

weight-preserving bijection between the terms appearing in the PIE expression and the linear subdigraphs of $D(E(e_1, \dots, e_m))$. What is really meant by this is that the weight of the linear subdigraph on the right side of Figure 1 is the image of the corresponding word on the left, under the ring homomorphism $f : W(x_1, x_2, x_3)^* \rightarrow \mathbb{Z}[x_1, x_2, x_3]$. \square

Using the same combinatorial model as in the proof above, one can rederive some interesting identities of Sury and McLaughlin [7, 11]. Note that the original proofs of those identities involve, to some extent, cumbersome algebraic manipulations, whereas our approach is purely combinatorial.

Corollary 2.1 (Theorem 1 [7]). *Let x_1, \dots, x_k be independent variables. Let e_1, \dots, e_k denote the elementary symmetric polynomials in the x_i of degrees $1, \dots, k$ respectively. Then in the polynomial ring $K[x_1, \dots, x_k]$, where K is a field of characteristic 0, one has the following identity for each $n \geq 1$:*

$$\sum_{r_1+r_2+\dots+r_k=n, r_i \geq 0} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k} = \sum_{2i_2+3i_3+\dots+ki_k \leq n} c(i_2, \dots, i_k, n) e_1^{n-2i_2-3i_3-\dots-ki_k} \times (-e_2)^{i_2} \times e_3^{i_3} \times \dots \times ((-1)^{k-1} e_k)^{i_k},$$

where

$$c(i_2, \dots, i_k, n) = \frac{(n - i_2 - 2i_3 - \dots - (k - 1)i_k)!}{i_2! \dots i_k! (n - 2i_2 - 3i_3 - \dots - ki_k)!}.$$

Proof. We consider the $n \times n$ matrix

$$E(e_1, \dots, e_k, 0, \dots, 0) = \begin{pmatrix} e_1 & e_2 & e_3 & \dots & e_k & 0 & \dots & 0 & 0 \\ 1 & e_1 & e_2 & \dots & e_{k-1} & e_k & \dots & 0 & 0 \\ 0 & 1 & e_1 & \dots & e_{k-2} & e_{k-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e_1 & e_2 & \dots & e_{n-k} & e_{n-k+1} \\ 0 & 0 & 0 & \dots & 1 & e_1 & \dots & e_{n-k-1} & e_{n-k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & e_1 & e_2 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & e_1 \end{pmatrix}.$$

By Theorem 2.1, we get

$$\det(E(e_1, \dots, e_k, 0, \dots, 0)) = \sum_{r_1+r_2+\dots+r_k=n, r_i \geq 0} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}. \tag{2}$$

We know that

$$\det(E(e_1, \dots, e_k, 0, \dots, 0)) = \sum_L (-1)^{n-c(L)} w(L), \tag{3}$$

where the summation runs over all LSD L of the digraph $D(E(e_1, \dots, e_k, 0, \dots, 0))$. Clearly, each LSD in the digraph $D(E(e_1, \dots, e_k, 0, \dots, 0))$ contains cycles of length at most k . Suppose L is an arbitrary LSD and L contains i_t cycles of length t ($t = 2, 3, \dots, k$). Then L contains $(n - 2i_2 - 3i_3 - \dots - ki_k)$ loops. Note $(-1)^{n-c(L)} = (-1)^{i_2+2i_3+3i_4+\dots+(k-1)i_k}$. The weight of the LSD L is

$$e_1^{n-2i_2-3i_3-\dots-ki_k} \times (e_2)^{i_2} \times e_3^{i_3} \times \dots \times (e_k)^{i_k},$$

and

$$c(i_2, \dots, i_k, n) = \frac{(n - i_2 - 2i_3 - \dots - (k - 1)i_k)!}{i_2! \dots i_k! (n - 2i_2 - 3i_3 - \dots - ki_k)!}$$

is the number of L containing $n - 2i_2 - 3i_3 - \dots - ki_k$ loops and i_t ($t = 2, 3, \dots, k$) cycles of length t . Upon considering the weight and sign of each L in Equation (3), we get

$$\det(E(e_1, \dots, e_k, 0, \dots, 0)) = \sum_{2i_2+3i_3+\dots+ki_k \leq n} c(i_2, \dots, i_k, n) e_1^{n-2i_2-3i_3-\dots-ki_k} \times (-e_2)^{i_2} \times e_3^{i_3} \times \dots \times ((-1)^{k-1} e_k)^{i_k},$$

and hence the identity. \square

Corollary 2.2 (Theorem 3 [7]). *Let n be a positive integer and x, y, z be indeterminates. Then*

$$\sum_{2i+3j \leq n} (-1)^i \binom{i+j}{j} \binom{n-i-2j}{i+j} (x+y+z)^{n-2i-3j} (xy+yz+zx)^i (xyz)^j$$

$$= \frac{xy(x^{n+1} - y^{n+1}) - xz(x^{n+1} - z^{n+1}) + yz(y^{n+1} - z^{n+1})}{(x-y)(x-z)(y-z)}.$$

Proof. Let $\lambda = (n)$ denote the partition of n consisting of a single part. Then the right-hand side of this identity is the Schur polynomial

$$s_\lambda(x, y, z) = \frac{a_{\lambda+\delta}(x, y, z)}{a_\delta(x, y, z)}, \text{ where}$$

$$a_{\lambda+\delta}(x, y, z) = \det \begin{pmatrix} x^{n+2} & y^{n+2} & z^{n+2} \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix} \text{ and } a_\delta(x, y, z) = \det \begin{pmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{pmatrix}.$$

So for the partition λ , we have that $s_\lambda(x, y, z)$ is the complete homogeneous polynomial of degree n in the variables x, y, z . Thus, by Theorem 2.1,

$$h_n(x, y, z) = \det \begin{pmatrix} e_1 & e_2 & e_3 & \cdots & 0 & 0 \\ 1 & e_1 & e_2 & \cdots & 0 & 0 \\ 0 & 1 & e_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_1 & e_2 \\ 0 & 0 & 0 & \cdots & 1 & e_1 \end{pmatrix} = \sum_L (-1)^{n-c(L)} w(L),$$

where e_1, e_2, e_3 are the elementary symmetric polynomials in the variables x, y, z and the order of the matrix is n . Now applying the same argument used in the proof of Corollary 2.1, we get the desired identity. \square

Corollary 2.3. *Let x, y be indeterminates. Then the following polynomial identity holds:*

$$\sum_{0 \leq i \leq n/2} (-1)^i \binom{n-i}{i} (x+y)^{n-2i} (xy)^i = x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n.$$

Proof. Clearly, the right-hand side of the above identity is the complete homogeneous polynomial $h_n(x, y)$ of degree n . By Theorem 2.1,

$$\det \begin{pmatrix} x+y & xy & 0 & \cdots & 0 & 0 \\ 1 & x+y & xy & \cdots & 0 & 0 \\ 0 & 1 & x+y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x+y & xy \\ 0 & 0 & 0 & \cdots & 1 & x+y \end{pmatrix} = h_n(x, y), \tag{4}$$

where the matrix is of order n . Also, we have

$$\det \begin{pmatrix} x+y & xy & 0 & \cdots & 0 & 0 \\ 1 & x+y & xy & \cdots & 0 & 0 \\ 0 & 1 & x+y & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x+y & xy \\ 0 & 0 & 0 & \cdots & 1 & x+y \end{pmatrix} = \sum_L (-1)^{n-c(L)} w(L),$$

where the summation runs over all linear subdigraphs L . Proceeding now as in the proof of Corollary 2.1 above, one obtains the identity. \square

3. Determinantal Interpretation of General Linear Recurrences

In this section, we prove that the solution of a general linear recurrence with constant coefficients can be expressed as the determinant of a suitable matrix.

Definition 3.1. Let c_1, c_2, \dots, c_r be complex numbers. Let u_n be the sequence defined by the r -th order linear recurrence

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + \dots + c_r u_{n-r}, \quad n \geq 1,$$

with initial conditions $u_0 = 1$ and $u_j = 0$ for $j < 0$.

There is a more or less a well-known combinatorial interpretation of general linear recurrences (see [4]). In fact, u_n is the sum of weights of all tilings of a board of length n with tiles of length at most r , where for each i ($1 \leq i \leq r$), the weight of each tile of length i is c_i and the weight of a tiling is the product of the weights of all its tiles. Now we consider the following $n \times n$ matrix

$$C = \begin{pmatrix} c_1 & -c_2 & c_3 & \dots & (-1)^{r+1}c_r & 0 & \dots & 0 & 0 \\ 1 & c_1 & -c_2 & \dots & (-1)^r c_{r-1} & (-1)^{r+1}c_r & \dots & 0 & 0 \\ 0 & 1 & c_1 & \dots & (-1)^{r-1}c_{r-2} & (-1)^r c_{r-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & c_1 & -c_2 & \dots & (-1)^{n-r+1}c_{n-r} & (-1)^{n-r+2}c_{n-r+1} \\ 0 & 0 & 0 & \dots & 1 & c_1 & \dots & (-1)^{n-r}c_{n-r-1} & (-1)^{n-r+1}c_{n-r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & c_1 & -c_2 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & c_1 \end{pmatrix}.$$

Theorem 3.1. Let $u_n (n \geq 1)$ be a general linear recurrence defined as (3.1). Then $u_n = \det(C)$.

Proof. To prove this result, we use the combinatorial interpretation of u_n (stated above) and the fact $\det(C) = \sum_L (-1)^{n-c(L)} w(L)$, where the summation runs over all LSD L in $D(C)$ (as in Section 2). In fact, we demonstrate a sign- and weight-preserving bijection between tilings of the n -board and the linear subdigraphs of $D(C)$. Let τ be a tiling of the n -board and suppose τ contains tiles of length at most r . Assume for each fixed $i \in [r]$ that τ contains t_i tiles $\tau_{i1}, \dots, \tau_{it_i}$ of length i . For fixed $j \in [t_i]$, suppose the tile τ_{ij} occupies the positions $k_{ij}, (k_{ij} + 1), \dots, (k_{ij} + i - 1)$, where $k_{ij} \in [n - i + 1]$. (For example, the left side of Figure 2 contains two tiles of lengths 2 and 3, which occupy the positions 1, 2, and 3, 4, 5, respectively.) For this tiling, we choose the LSD L_τ containing t_i cycles C_{i1}, \dots, C_{it_i} each of length i . Moreover, the cycle C_{ij} , corresponding to the tile τ_{ij} , contains the vertices $k_{ij}, (k_{ij} + 1), \dots, (k_{ij} + i - 1)$ and $k_{ij} \rightarrow (k_{ij} + i - 1) \rightarrow (k_{ij} + i - 2) \rightarrow \dots \rightarrow (k_{ij} + 1) \rightarrow k_{ij}$ (here, $u \rightarrow v$ denotes an edge directed from u to v). Clearly, this is a bijection. See Figure 2 for an illustration.

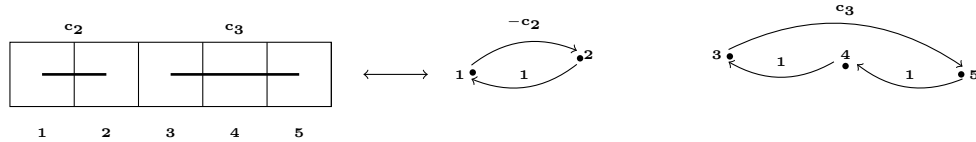


Figure 2: The bold line in each tile represents the length of the corresponding tile.

To complete the proof, we must show that this bijection preserves both the sign and weight. Suppose that L_τ contains k cycles of even length and m cycles of odd length. Let the total number of vertices in the k cycles be $2s$. Then the remaining $n - 2s$ (n is total number of vertices in L_τ) vertices are among the m odd cycles. Now,

$$(-1)^{n-(k+m)} w(L_\tau) = (-1)^k (-1)^{n-m} w(L_\tau).$$

If m is odd, then $n - 2s$ is odd, so n is odd. If m is even, then n is also even. So $(-1)^{n-m}$ is always 1. Note that only a cycle of even length contributes a negative sign towards the weight of the LSD L_τ . Hence

$(-1)^{n-(k+m)}w(L_\tau)$ is positive. From the construction of the bijection, the weight of τ is the same as the weight of the LSD L_τ . For example, Figure 3 describes the required bijection for finding u_4 in the recurrence $u_n = c_1u_{n-1} + c_2u_{n-2}$. \square

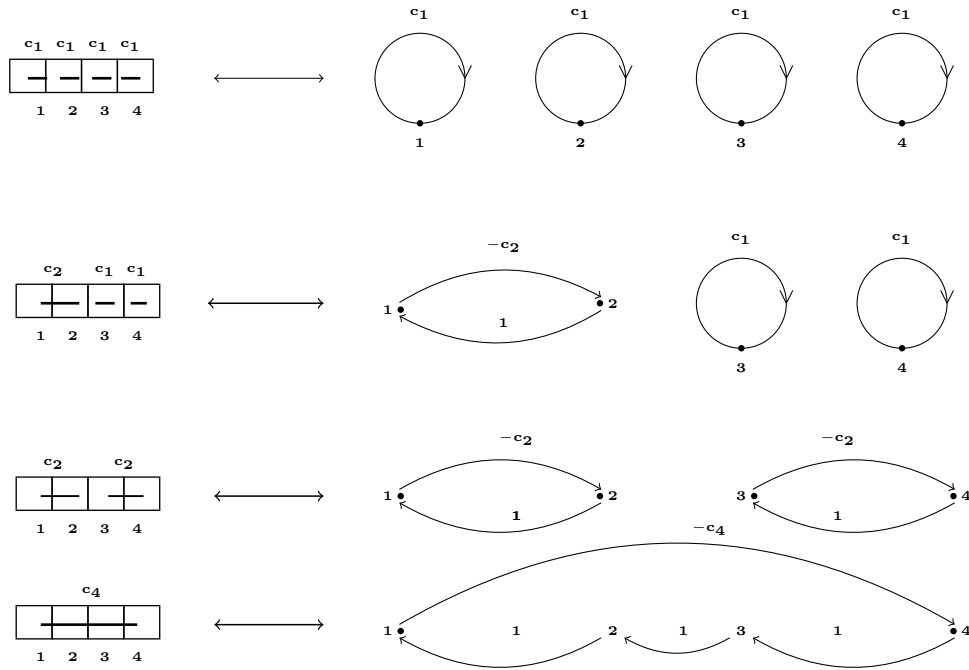


Figure 3: Each bold line in every tile represents the length of the corresponding tile. For example, the last tiling of the board contains exactly one tile of length 4 occupying the positions 1, 2, 3, 4, whereas the second tiling contains one tile of length 2 occupying the position 1, 2 and two tiles of length 1 occupying the positions 3 and 4. The numbers appearing above the tiles are the weights of the corresponding tiles and the numbers appearing along the edges are the weights of the corresponding edges.

Definition 3.2. The r -acci sequence F_n is defined by the recurrence

$$F_n = F_{n-1} + F_{n-2} + \dots + F_{n-r}, \quad n \geq 1,$$

with initial conditions $F_0 = 1$ and $F_i = 0$ for $i < 0$.

One combinatorial interpretation for F_n is the number of ways to tile a board of length n using tiles of length at most r . In [7], the authors proved a formula for the r -acci numbers. Here we give a determinantal expression of the same. Consider the following $n \times n$ matrix

$$G = \begin{pmatrix} 1 & -1 & 1 & \dots & (-1)^{r+1} & 0 & \dots & 0 & 0 \\ 1 & 1 & -1 & \dots & (-1)^r & (-1)^{r+1} & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & (-1)^{r-1} & (-1)^r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 & \dots & (-1)^{n-r+1} & (-1)^{n-r+2} \\ 0 & 0 & 0 & \dots & 1 & 1 & \dots & (-1)^{n-r} & (-1)^{n-r+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 1 \end{pmatrix}. \quad (5)$$

Corollary 3.1. Let F_n be the n -th term of the r -acci sequence defined above. Then $F_n = \det(G)$.

Proof. The proof follows from the combinatorial interpretation of r -acci numbers and the proof of Theorem 3.1. \square

Corollary 3.2 (Corollary 5 [7]). *Let F_n be the n -th term of the r -acci sequence. Then*

$$F_n = \sum_{2i_2+\dots+ri_r \leq n} \frac{(n - i_2 - 2i_3 - \dots - (r - 1)i_r)!}{i_2! \cdots i_r! ((n - 2i_2 - 3i_3 - \dots - ri_r)!)}.$$

Proof. By Corollary 3.1, $F_n = \det(G)$ and each LSD of $D(G)$ contributes 1 to $\det(G)$. Again,

$$\frac{(n - i_2 - 2i_3 - \dots - (r - 1)i_r)!}{i_2! \cdots i_r! ((n - 2i_2 - 3i_3 - \dots - ri_r)!)}$$

is the number of LSD containing $(n - 2i_2 - 3i_3 - \dots - ri_r)$ loops and i_t ($t = 2, 3, \dots, r$) cycles of length t . Hence,

$$\det(G) = \sum_{2i_2+\dots+ri_r \leq n} \frac{(n - i_2 - 2i_3 - \dots - (r - 1)i_r)!}{i_2! \cdots i_r! ((n - 2i_2 - 3i_3 - \dots - ri_r)!)}.$$

□

Let f_n be the n -th Fibonacci number defined by

$$f_n = f_{n-1} + f_{n-2}, \quad n \geq 2, \tag{6}$$

with $f_0 = f_1 = 1$. Then f_n may be interpreted combinatorially as the number of linear tilings of a board of length n using tiles of length at most 2. Consider now the $n \times n$ matrix

$$F = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}. \tag{7}$$

Then we have the following corollary.

Corollary 3.3. *Let F be the matrix defined as (7). Then $f_n = \det(F)$.*

Proof. We use the same argument as in the proof of Theorem 3.1 above and the combinatorial interpretation for the Fibonacci numbers. □

Corollary 3.4 (Binet's Formula). *Let f_n be the n -th Fibonacci number. Then Binet's formula says that*

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}.$$

Proof. To show this, consider the matrix (4) from the proof of Corollary 2.3 above. If $x \neq y$, then the determinant of (4) can be written as $\frac{x^{n+1}-y^{n+1}}{x-y}$. Now put $x+y = 1$ and $xy = -1$ in (4). Then one gets the matrix F . Clearly, x and y are the roots of the polynomial $t^2 - t - 1 = 0$, whence $x = \frac{1+\sqrt{5}}{2}$ and $y = \frac{1-\sqrt{5}}{2}$. By Corollary 3.3, we have $f_n = \det(F)$. By Corollary 2.3, $\det(F) = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}$, which implies the result. □

4. Determinantal Expression of the Lucas Numbers and a New Identity

In this section, we derive a determinantal formula for members of the Lucas sequence. In the course of doing so, we first prove a new determinantal identity and, as a consequence, one obtains the desired Lucas determinantal formula. Recall that the Lucas numbers are defined recursively by

$$l_n = l_{n-1} + l_{n-2}, \quad n \geq 2,$$

with $l_0 = 2$ and $l_1 = 1$. Then l_n can be interpreted combinatorially as the number of tilings of a circular n -board with tiles of length at most 2 (see [4]).

For the variables a and b , we consider the $n \times n$ ($n \geq 3$) matrix

$$S = \begin{pmatrix} a+b & (-1)^{n+1}a & 0 & \cdots & 0 & b \\ (-1)^{n+1}b & a+b & a & \cdots & 0 & 0 \\ 0 & b & a+b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a+b & a \\ a & 0 & 0 & \cdots & b & a+b \end{pmatrix}. \tag{8}$$

Then the following result holds.

Theorem 4.1. *Let S be an $n \times n$ matrix defined as (8). Then $\det(S) = 2(a^n + b^n)$.*

Proof. We prove this theorem in the case $n = 4$ (the general case can be done in a similar fashion). For $n = 4$, we have

$$S = \begin{pmatrix} a+b & -a & 0 & b \\ -b & a+b & a & 0 \\ 0 & b & a+b & a \\ a & 0 & b & a+b \end{pmatrix}.$$

It must be shown that $\det(S) = 2(a^4 + b^4)$. Regard $A = \{a, b\}$ as the set of letters. Let C be the set of all cyclic words of length 4 formed by a and b . For cyclic words, we always take the starting point to be 1 and the orientation to be clockwise. For example, Figure 4 contains three cyclic words of length 4 with starting point 1 and clockwise orientation.

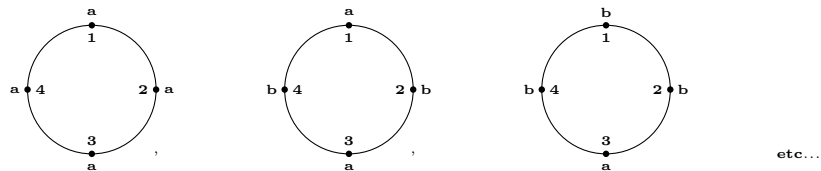


Figure 4: Illustrated are three cyclic words formed by a, b .

Let us introduce the following notation. We will denote by $\mathbb{Z}C(n)$ the set of all formal linear combinations

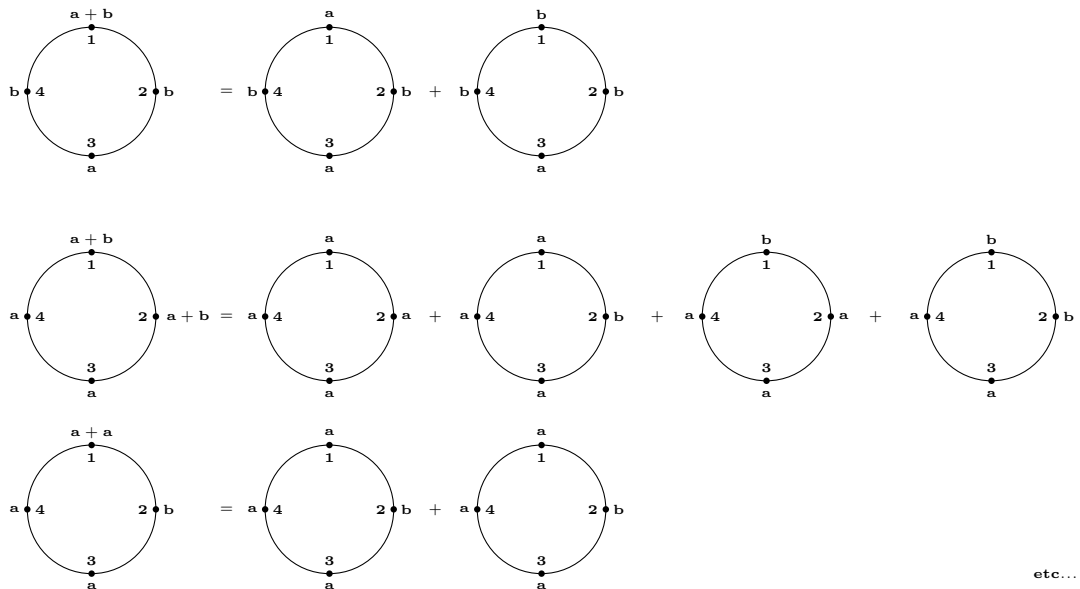


Figure 5:

of cyclic words of length n (with the starting point and orientation already prescribed) with integer coefficients. For example, $c_1, c_2, c_1 + c_2, 2c_1 - c_2, 0$ (the empty word) are some typical elements of $\mathbb{Z}C(n)$, where c_1, c_2 are cyclic words. Now let us consider Figure 5. Note that none of the items on the left side of Figure 5 is a cyclic word but may be regarded as an element of $\mathbb{Z}C(n)$ by the corresponding item on the right side.

We wish to evaluate the sum of all possible cyclic words of length 4 formed by a and b such that ab does not appear as a subword, i.e., a and b do not occur as a consecutive pair (note that ba may appear as a subword). Figure 6 shows some cyclic words containing ab as a subword. Let $C_0(\in \mathbb{Z}C(4))$ be the formal sum of all cyclic

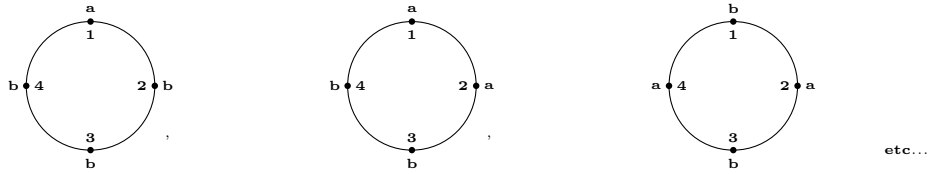


Figure 6: These are some cyclic words, where ab appears as a subword. For example, in the first cyclic word, the letters a and b occupy the consecutive positions 1, 2, whereas the last cyclic word contains two pairs of consecutive positions 2, 3 and 4, 1 occupied by ab .

words with constant coefficient 1, i.e., $\sum_{\ell \in C} \ell$. Now the sum of all possible cyclic words of length 4 such that the

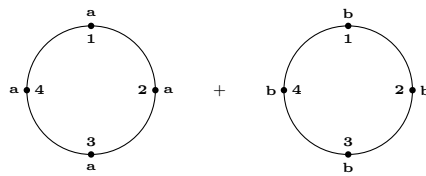


Figure 7:

letters a and b do not occupy two consecutive positions on the circle is clearly described in Figure 7. Again, we can calculate this sum by the PIE rule. One has that the sum of all cyclic words avoiding ab as a subword is given by $C_0 - C_1 + C_2$, where C_1 is the sum of all cyclic words of length 4 formed by letters a and b such that

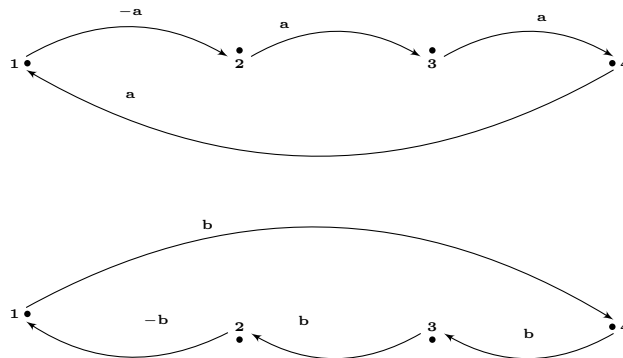


Figure 8: This figure contains two LSD of $D(S)$: L_1 (above) and L_2 (below). The number appearing along an edge is the corresponding weight.

there is at least one pair of consecutive positions (i.e., either 1, 2 or 2, 3 or 3, 4 or 4, 1 on a circle) occupied by ab (for example, see Figure 6) and C_2 is the sum of all cyclic words of length 4 formed by letters a and b such that there are at least two pairs of consecutive positions (i.e., 1, 2 and 3, 4 or 2, 3 and 4, 1) occupied by ab . See the last item of Figure 6 for an illustration of the latter. Now we describe a sign- and weight-preserving bijection

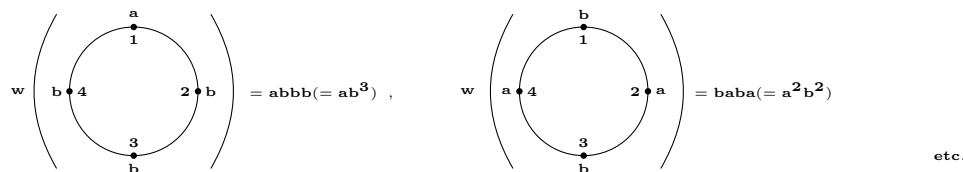


Figure 9: Two cyclic words and their weights.

between the terms in $C_0 - C_1 + C_2$ and the LSDs in the set $\mathcal{L} \setminus \{L_1, L_2\}$, where \mathcal{L} is the collection of all LSDs

in $D(S)$ and L_1, L_2 are the LSDs in Figure 8.

The *weight of a cyclic word* C , denoted by $w(C)$, is the product (here, we mean the usual polynomial product) of all letters present in that cyclic word and extend this to $\mathbb{Z}C(n)$ by linearity, i.e., for any two cyclic words ℓ and ℓ' , $w(x\ell + y\ell') := xw(\ell) + yw(\ell')$, $x, y \in \mathbb{Z}$. See Figures 9 and 10 for an illustration.

$$\begin{aligned}
 w\left(\begin{array}{c} \text{a+b} \\ \bullet \\ \text{1} \\ \bullet \\ \text{b} \bullet 4 \quad \quad \quad \bullet 2 \bullet \text{b} \\ \bullet \\ \text{3} \\ \bullet \\ \text{a} \end{array}\right) &= w\left(\begin{array}{c} \text{a} \\ \bullet \\ \text{1} \\ \bullet \\ \text{b} \bullet 4 \quad \quad \quad \bullet 2 \bullet \text{b} \\ \bullet \\ \text{3} \\ \bullet \\ \text{a} \end{array}\right) + w\left(\begin{array}{c} \text{b} \\ \bullet \\ \text{1} \\ \bullet \\ \text{b} \bullet 4 \quad \quad \quad \bullet 2 \bullet \text{b} \\ \bullet \\ \text{3} \\ \bullet \\ \text{a} \end{array}\right) \\
 &= w\left(\begin{array}{c} \text{a} \\ \bullet \\ \text{1} \\ \bullet \\ \text{b} \bullet 4 \quad \quad \quad \bullet 2 \bullet \text{b} \\ \bullet \\ \text{3} \\ \bullet \\ \text{a} \end{array}\right) + w\left(\begin{array}{c} \text{b} \\ \bullet \\ \text{1} \\ \bullet \\ \text{b} \bullet 4 \quad \quad \quad \bullet 2 \bullet \text{b} \\ \bullet \\ \text{3} \\ \bullet \\ \text{a} \end{array}\right) \\
 &= \text{abab} (= \text{a}^2\text{b}^2) + \text{bbab} (= \text{b}^3\text{a}) = \text{a}^2\text{b}^2 + \text{ab}^3
 \end{aligned}$$

Figure 10: This figure describes the weights of the sum of cyclic words.

Figure 11 shows the required bijection, which preserves both the sign and weight. Note that $(-1)^{4-1}w(L_1) = a^4$ and $(-1)^{4-1}w(L_2) = b^4$. Hence, $\det(S) = 2(a^4 + b^4)$. \square

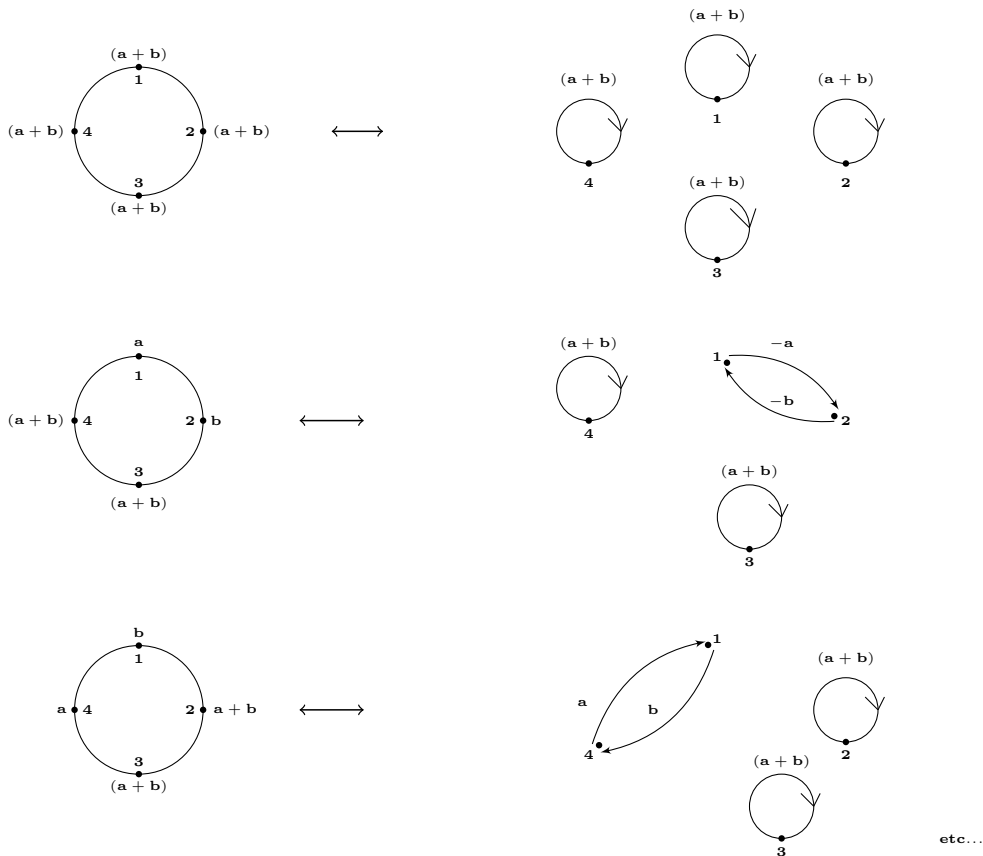


Figure 11: The number appearing along an edge in the above figure is the weight of the corresponding edge. The left-hand side of this figure describes some terms of the PIE expression.

Now we establish a determinantal expression for the Lucas numbers. First, we consider the $n \times n$ ($n \geq 3$)

matrix

$$A = \begin{pmatrix} 1 & (-1)^{n+1} \frac{1+\sqrt{5}}{2} & 0 & \cdots & 0 & \frac{1-\sqrt{5}}{2} \\ (-1)^{n+1} \frac{1-\sqrt{5}}{2} & 1 & \frac{1+\sqrt{5}}{2} & \cdots & 0 & 0 \\ 0 & \frac{1-\sqrt{5}}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & 0 & 0 & \cdots & \frac{1-\sqrt{5}}{2} & 1 \end{pmatrix}.$$

Corollary 4.1. *Let ℓ_n denote the n -th term of the Lucas sequence. Then $\ell_n = \frac{1}{2} \det(A)$ for $n \geq 3$.*

Proof. Putting $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$ in S , we get matrix A . So,

$$\det(A) = 2 \left[\left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

On the other hand, from the proof of Theorem 4.1 and the combinatorial interpretation of the Lucas numbers, one can write

$$\det(A) = \ell_n + \left(\frac{1+\sqrt{5}}{2} \right)^n + \left(\frac{1-\sqrt{5}}{2} \right)^n,$$

which implies the result. □

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References

- [1] A. Ayyer, *Determinants and perfect matchings*, J. Combin. Theory Ser. A 120 (2013), 304–314.
- [2] A.T. Benjamin, H. Derks and J.J. Quinn, *The combinatorialization of linear recurrences*, Electron. J. Combin. 18(2) (2011-2), P12.
- [3] A.T. Benjamin, G.M. Levin, K. Mahlburg and J.J. Quinn, *Random approaches to Fibonacci identities*, Amer. Math. Monthly 107 (2000), 511–516.
- [4] A.T. Benjamin and J.J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, Washington, DC, 2003.
- [5] S. Bera and S.K. Mukherjee, *Combinatorial proofs of some determinantal identities*, Linear and Multilinear Alg. 66 (2018), 1659–1667.
- [6] A.R. Brualdi and D. Cvetkovic, *A Combinatorial Approach to Matrix Theory and its Application*, Discrete Mathematics and Its Applications, CRC Press, Boca Raton, London, New York, 2009.
- [7] J. McLaughlin and B. Sury, *Powers of a matrix and combinatorial identities*, Integers 5(1) (2005), #A13.
- [8] S.K. Mukherjee and S. Bera, *Combinatorial proofs of the Newton–Girard and Chapman–Costas–Santos identities*, Discrete Math. 342 (2019), 1577–1580.
- [9] B.E. Sagan, *The Symmetric Group Representations, Combinatorial Algorithms, and Symmetric Functions*, Graduate Texts in Mathematics, Springer-Verlag, New York, 2001.
- [10] M. Shattuck, *Recounting some reciprocal Fibonacci number relations*, Util. Math. 79 (2009), 259–265.
- [11] B. Sury, *A curious polynomial identity*, Nieuw Arch. Wisk. 11 (1993), 93–96.
- [12] D. Zeilberger, *A combinatorial proof of the Newton’s identity*, Discrete Math. 49 (1984), 319.