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A Catalanization Map on the Symmetric Group

Mahir Bilen Can[†], Luke Nelson[‡], and Kevin Treat[††]

[†] *Tulane University*
Email: mahirbilencan@gmail.com

[‡] *Air Force Research Laboratory*

[††] *United States Air Force Academy*
Email: kevin.treat@afacademy.af.edu

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Abstract: In this article we investigate a self-map of the symmetric group. We show that the set of plus-indecomposable permutations is stable under our map. Furthermore, the fixed points are precisely the 231-avoiding permutations. Finally, by using our map we obtain a nested sequence of polytopes that starts with the permutahedron and ends with the associahedron.

Keywords: Associahedron; Bruhat order; Catalanization map; Pattern avoidance; Permutahedron; Scope sequences; Tamari lattice; Weak order
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1 Introduction

The starting point of our paper is a map, $\partial : S_n \to S_n$, that we introduce by using the inversion sets of permutations; it depends on the total order on the transpositions in S_n. For convenience, we work with lexicographic ordering. We define our map by viewing the inversions of a permutation as transpositions in S_n, and then multiplying them in the lexicographic ordering. This map leads us to a distinguished set of permutations whose count is given by the Catalan numbers, whereby we get to the Tamari lattice $\mathcal{T}_n$.

Our main result is as follows.

Theorem 1.1. *Let $r \in \mathbb{N}$, and let $S_n^{(r)}$ denote the image of S_n under the r-fold application of ∂. Denote by S_n^{231} the set of all 231-avoiding permutations in S_n. For each $n \in \mathbb{Z}_+$,*

1. *For each $w \in S_n$, $\partial(w) = w$ if and only if $w \in S_n^{231}$.*
2. *$S_n = S_n^{(0)} \supset S_n^{(1)} \supset \cdots \supset S_n^{(t_n)} = S_n^{231}$, for some $t_n \in \mathbb{N}$.*

We denote by $\leq_L$ (respectively, $\leq_R$) the left weak order (respectively, the right weak order) on S_n. Likewise, we denote by $\leq_{BC}$ Bruhat-Chevalley order on S_n.

A result of Disanto, Ferrari, Pinzani, and Rinaldi in [5] states that the restriction of the right weak order on 312-avoiding permutations agrees with the Tamari lattice poset, and the restriction of the Bruhat-Chevalley order on the same set of permutations gives the Dyck path lattice. The 231-avoiding permutations are the inverses of 312-avoiding permutations. Thus, we have the following version of the main theorem of the article [5].

1. $(S_n^{231}, \leq_L)$ is isomorphic to the Tamari lattice.
2. $(S_n^{231}, \leq_{BC})$ is isomorphic to the lattice of Dyck paths ordered by inclusion.

Therefore, the order theoretic thrust of our Theorem 1.1 is the systematic construction of a series of subposets of S_n that converge to the posets $(S_n^{231}, \leq_L)$ and $(S_n^{231}, \leq_{BC})$; each application of ∂ takes the previous iteration one step *closer*. This motivates our other results. In addition to our main result, we consider the two novel decompositions of permutations, and our subsequent results concerning them, to be interesting in their own right.

In [13], the second and the third authors considered an order on the equivalence classes of walks on the associahedron (Stasheff-Tamari polytope), called the Tamari Block Lattice, which turns out to be anti-isomorphic to the Higher Stasheff-Tamari Order in dimension 3. Let us note in passing that our map ∂ helps to establish similar equivalence relations on the walks on the 1-skeleton of the permutohedron.

Reading defined the Cambrian lattices as lattice quotients of the weak order on S_n (more generally any Coxeter group) modulo certain congruences [16]. It is well-known that the 231-avoiding permutations form a Cambrian lattice, which is isomorphic to the Tamari lattice. Our focus in the present article is on the existence of intermediate posets obtained between the weak order and Tamari lattice. We anticipate our ∂ map will lead to more generalized approaches for studying orders on arbitrary Coxeter groups and Cambrian lattices. In this regard, there are several questions that remain open. For example, *what kind of posets (or lattices) do we get by using a map that is analogous to our catalanization map for other Coxeter groups?*

In Section 2, we introduce notation, and we present some additional motivation for our work. In Section 3, we prove Theorem 1.1. Open problems are discussed in Section 4.

2 Preliminaries

In this section we review basic definitions, establish notation, and give background information on the orders $(S_n^{(r)}, \leq_L)$ and $\mathcal{T}_n$.

2.1 Notation

We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$, and $[i, n]$ for $\{i, i+1, \ldots, n\}$. We denote by S_n the set of all permutations on n distinct elements taken from $[n]$. It is inferred a subscript i on a permutation $w \in S_n$ means the i-th entry of w (so $w_1 w_2 \ldots w_n$ is taken as the one-line notation of w). A permutation $w \in S_n$ is 231-*avoiding*, i.e., $w \in S_n^{231}$, if there does not exist $i, j, k \in [n]$ with $i < j < k$ and $w_j > w_i > w_k$. Let us mention that in the literature 231-avoiding permutations are sometimes called *stack sortable permutations*.

For P a finite partially ordered set, or *poset* (see [21] for more on posets), we use $s < t$ (respectively, $s > t$) to mean s is less than (respectively, greater than) t, and we use $s \lessdot t$ to mean s is *covered* by t. If P has an element less than (respectively, greater than) or equal to every element of P, that element is denoted $\hat{0}_P$ (respectively, $\hat{1}_P$). The *length* of a chain C in P is its number of elements minus one, denoted $\ell(C)$. A chain is *maximal* if it is not contained in a longer chain of P. If every maximal chain of P has the same length n, then we say P is *graded* of *rank* n. If every pair of elements of P have a least upper bound (called a *join*) and a greatest lower bound (called a *meet*) then P is a *lattice*.

2.2 Tamari lattices

In [8], Huang and Tamari describe $\mathcal{T}_n$ as a poset of certain n-tuples of nonnegative integers, where the ordering is given by the coordinate-wise comparisons. The Tamari lattice, $\mathcal{T}_n$ can also be represented as triangulations of an $(n+2)$-gon with covering relations involving edge flips, as Rambau and Reiner do in [19]. A very natural encoding of $\mathcal{T}_n$ views the vertices of $\mathcal{T}_n$ as binary trees, and the covering relations correspond to right rotations (see [1] for example). In [1], Bernardi and Bonichon represent the Tamari lattice in terms of Dyck paths. [7] and [12] adapted this to use Young diagrams.

Knuth identifies forests on n nodes with the sequences, $(s_1, \ldots, s_n)$, obtained from the nodes' descendant counts; the sequences on 4 nodes are listed in [9, Table 7.2.1.6(2)]. He characterizes and places an order on the set of sequences of length n, which is another interpretation of $\mathcal{T}_n$ [9, Exercise 7.2.1.6(27)]. An equivalent characterization is given in Knuth's video lecture [10]. This encoding appears in [3, Definition 9.1] as well. We make a slight twist to Knuth's scope sequences (by including the parent node with the descendant counts) as it was done in [13].

Definition 2.1 ([13], Definition 2.1)**.** *The set of* scope sequences *of length n, denoted $\mathcal{SS}_n$, is the set of n-tuples, $(h_1, \ldots, h_n)$, such that for each $i \in [n]$,*

1. $1 \leq h_i \leq n - i + 1$*, and*
2. $h_{i+r} \leq h_i - r$*, for* $0 < r < h_i$.

Definition 2.2 ([13], Proposition 2.2)**.** *The Tamari lattice, $\mathcal{T}_n$, may be represented on the set $\mathcal{SS}_n$, denoted $\mathcal{T}_{\mathcal{SS}_n}$, where $(h_1, \ldots, h_n) \leq (h'_1, \ldots, h'_n)$ if and only if $h_i \leq h'_i$, for each $i \in [n]$.*

The bottom and top elements of $\mathcal{T}_{\mathcal{SS}_n}$ are $\hat{0} = (1, \dots, 1)$ and $\hat{1} = (n, n-1, \dots, 1)$.

Remark 2.1. *Pallo discovered the Tamari lattices independently, and encoded $\mathcal{T}_n$ with sequences in the reverse order as given in Definition 2.1; see [14, Theorems 1,2]. His encoding is also a Catalan set found in Stanley's* Catalan Numbers *[22, Chapter 2 #85].*

2.3 The left weak and the Bruhat-Chevalley orders.

Permutations have various representations. For $w \in S_n$, when we write $w = w_1 \dots w_n$, we mean that w is the permutation that maps the number i ($i \in \{1, \dots, n\}$) to w_i. The *inversion set* of a permutation $w = w_1 \dots w_n$ is defined as

$$\mathrm{inv}(w) := \{(i, j) \mid 1 \leq i < j \leq n,\ w_i > w_j\}.$$

A *simple transposition* in S_n is a permutation of $\{1, \dots, n\}$ that interchanges only two consecutive numbers and keeps everything else fixed. More generally, a *transposition* is a permutation that interchanges two, but not necessarily consecutive, numbers and fixes everything else. It is well-known that the map from $w \in S_n$ to $\mathrm{inv}(w)$ is one-to-one. The cardinality of the inversion set $\mathrm{inv}(w)$ is equal to the minimum number of simple transpositions that is required to write w as their product. For this reason, $|\mathrm{inv}(w)|$ will be called the *length* of w, and we will denote it by $\ell(w)$.

Let i and j be two integers such that $1 \leq i < j \leq n$. Let s_i denote the simple transposition $s_i = (i\, i{+}1)$, and let t_{ij} denote the transposition $(i\, j)$. (In this section, to avoid confusion between the notations of inversion pairs and transpositions, we do not use commas in the latter. More precisely, if we write $(i\, j)$ for some $i, j \in \{1, \dots, n\}$ with $i < j$, then we mean the permutation t_{ij}.) Then the *weak order on S_n*, denoted by $\leq_L$, is the transitive closure of the covering relations:

$$v \lessdot_L s_i v \iff \ell(v) = \ell(s_i v) - 1 \qquad (v \in S_n).$$

The *Bruhat-Chevalley order on S_n*, denoted by $\leq$, is the transitive closure of the covering relations:

$$v \lessdot_{BC} v t_{ij} \iff \ell(v) = \ell(v t_{ij}) - 1 \qquad (v \in S_n).$$

The following facts about these orders are well-known [2]:

1. The definition of the left weak order depends on the multiplication by simple transpositions on the left. However, we can equally define the Bruhat-Chevalley order by multiplying with transpositions on the left.
2. Every covering relation of $\leq_L$ is a covering relation of $\leq_{BC}$ but not conversely.
3. The inclusions of the inversion sets characterize the left weak order: For $v, w \in S_n$, we have

$$v \leq_L w \iff \mathrm{inv}(v) \subseteq \mathrm{inv}(w).$$

3 The Catalanization map

Now it is time to introduce our *catalanization map*, $\partial : S_n \to S_n$. We view every pair of integers (i, j) such that $1 \leq i < j \leq n$ as a transposition $(i\, j)$ in S_n. More precisely, $(i\, j)$ is the permutation that interchanges i and j and does not change any other numbers.

Definition 3.1. *Let $w = w_1 \dots w_n$ be a permutation from S_n, and let $\{t_1', \dots, t_r'\}$ be its inversion set, ordered lexicographically. Then the* catalanization *of w, denoted $\partial(w)$, is defined as*

$$\partial(w) := t_1 \cdots t_r,$$

where t_i ($1 \leq i \leq r$) is the transposition that is defined by t_i'. For the identity permutation, $\partial(id) = id$.

Example 3.1. *Let $w = 52341$. Then*

$$inv(w) = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 5), (3, 5), (4, 5)\},$$

hence, the catalanization of w is given by

$$\partial(w) = (1\,2)(1\,3)(1\,4)(1\,5)(2\,5)(3\,5)(4\,5) = 54123.$$

Definition 3.2. *Let $w \in S_n$. The* inversion cycle decomposition *of w, denoted $w_{\mathcal{ICD}}$, is the sequence $C_1, \dots, C_n$ defined as follows. C_i, $i \in [n]$, is a permutation cycle, called an* inversion cycle, *written in decreasing order made up of all the elements in*

$$\{j \mid (i,j) \in inv(w)\} \sqcup \{i\}.$$

Note the first part of the union may be empty. In this case, w is a cycle of size one. Hence, it is one of the cycles, $(1), (2), \dots, (n)$. Any of these cycles is equivalent to the identity permutation cycle. We keep these cycles throughout the paper in an effort to normalize notation.

It follows that $\partial(w)$ is the product of inversion cycles,

$$\partial(w) = C_1 C_2 \cdots C_n. \tag{1}$$

For if $\{(i, j_1), (i, j_2), \dots, (i, j_s)\}$ is the set of all inversion pairs with first coordinate i (ordered lexicographically), then

$$C_i = (j_s\, j_{s-1}\, \dots\, j_1\, i) = (i\, j_1)(i\, j_2) \cdots (i\, j_s).$$

The inversion cycles appear in increasing order by the least (and last) entry of each cycle. Due to the fact that C_j, for $j > i$, acts as identity on i,

$$(\partial(w))_i = (C_1 C_2 \cdots C_n)_i = (C_1 C_2 \cdots C_i)_i, \tag{2}$$

$$(\partial(w))([1]) = C_1([1]),\ (\partial(w))([2]) = (C_1C_2)([2]),\ (\partial(w))([3]) = (C_1C_2C_3)([3]),\ \dots \tag{3}$$

Example 3.2. *Continuing with Example 3.1, the inversion cycle decomposition of* 52341 *is*

$$52341_{\mathcal{ICD}} = (5\,4\,3\,2\,1), (5\,2), (5\,3), (5\,4), (5).$$

There is an important distinction to make between the decomposition and the catalanization map.

Example 3.3. *The map that takes $w \in S_n$ to $w_{\mathcal{ICD}}$ is one-to-one (as the inversions may be recovered from the decomposition), however the catalanization map is not.*

$$41532_{\mathcal{ICD}} = (5\,4\,2\,1), (2), (5\,4\,3), (5\,4), (5), \quad \partial(41532) = 51423,$$
$$51423_{\mathcal{ICD}} = (5\,4\,3\,2\,1), (2), (5\,4\,3), (4), (5), \quad \partial(51423) = 51423.$$

The next proposition lists equivalent statements for 231-avoiding permutations, which happen to have special significance to the catalanization map. Note the well-known fact that the numbers $|S_n^{231}|$ are given by the Catalan numbers.

Proposition 3.1. *Let $w \in S_n$. The following are equivalent.*

1. *$w \in S_n^{231}$.*
2. *For each inversion cycle of w, the entries of w whose indices appear in the cycle form an interval.*
3. *Each inversion cycle of w consists of (decreasing) consecutive integers.*
4. *For any two inversion cycles of w that have a common entry, the one to the left in $w_{\mathcal{ICD}}$ contains all the entries of the other.*

Proof. (1) ⇒ (2). Suppose some inversion cycle C with least entry i does not satisfy (2). Then there are j, k with $w_i > w_j > w_k$ such that C has the entry k but not j, which implies $j < i < k$. Thus $w \notin S_n^{231}$.

(2) ⇒ (3). Suppose some inversion cycle C with least entry i does not satisfy (3). Then there are j, k with $i < j < k$ such that C has the entry k but not j, which implies $w_j > w_i > w_k$. The inversion cycle with least entry j has the entry k but not i, so it does not satisfy (2).

(3) ⇒ (4). Assume (3) and suppose $C = (l\ l-1 \cdots k)$ and $C' = (j\ j-1 \cdots i)$ are two inversion cycles that have a common entry, with C left of C'. By construction, $k < i$, so by assumption, $i \le l$. Thus $w_k > w_i$ and $w_i > w_j$. So $w_k > w_j$ which implies j is an entry in C. Thus C contains all the entries of C'.

(4) ⇒ (1). Supposing $w \notin S_n^{231}$. Then there are $i < j < k$ with $w_j > w_i > w_k$. The inversion cycle C with least entry i has the entry k, but not j. The inversion cycle C' with least entry j has the entry k. Thus C and C' have a common entry, but C does not contain all the entries of C'. □

Remark 3.1. *We will add three more equivalent conditions for* 231*-avoiding permutations. See Proposition 3.3, Corollary 3.4, and Proposition 3.6.*

Definition 3.3. *For $r \in \mathbb{N}$, we call $\partial^r(w)$ the* r-th catalanization *of w; we call w a* catalanized permutation *if $\partial(w) = w$.*

Example 3.4. *In this example we list the complete image of ∂ on S_4. The catalanized permutations are highlighted in bold fonts.*

w	$w_{\mathcal{ICD}}$	$\partial(w)$	w	$w_{\mathcal{ICD}}$	$\partial(w)$
1234	(1),(2),(3),(4)	**1234**	**3124**	(3 2 1),(2),(3),(4)	**3124**
1243	(1),(2),(4 3),(4)	**1243**	3142	(4 2 1),(2),(4 3),(4)	**4123**
1324	(1),(3 2),(3),(4)	**1324**	**3214**	(3 2 1),(3 2),(3),(4)	**3214**
1342	(1),(4 2),(4 3),(4)	**1423**	3241	(4 2 1),(4 2),(4 3),(4)	**4213**
1423	(1),(4 3 2),(3),(4)	**1423**	3412	(4 3 1),(4 3 2),(3),(4)	**4321**
1432	(1),(4 3 2),(4 3),(4)	**1432**	3421	(4 3 1),(4 3 2),(4 3),(4)	**4312**
2134	(2 1),(2),(3),(4)	**2134**	**4123**	(4 3 2 1),(2),(3),(4)	**4123**
2143	(2 1),(2),(4 3),(4)	**2143**	**4132**	(4 3 2 1),(2),(4 3),(4)	**4132**
2314	(3 1),(3 2),(3),(4)	**3124**	**4213**	(4 3 2 1),(3 2),(3),(4)	**4213**
2341	(4 1),(4 2),(4 3),(4)	**4123**	4231	(4 3 2 1),(4 2),(4 3),(4)	**4312**
2413	(3 1),(4 3 2),(3),(4)	3421	**4312**	(4 3 2 1),(4 3 2),(3),(4)	**4312**
2431	(4 1),(4 3 2),(4 3),(4)	**4132**	**4321**	(4 3 2 1),(4 3 2),(4 3),(4)	**4321**

A permutation $w = w_1 w_2 \dots w_n \in S_n$ is a *plus-indecomposable permutation* (or, *connected*, or, *irreducible permutation*) if there does not exist $m < n$ such that $w([m]) = [m]$ [4, 20]. We denote by $\mathscr{C}_n$ the subset of all plus-indecomposable permutations of S_n.

We may partition the set of permutations $w \in S_n$ by the number $t \in [n]$ such that $w_1 w_2 \dots w_t$ is plus-indecomposable.

Thus, the following recurrence is readily obtained:

$$n! = |S_n| = |\mathscr{C}_1|(n-1)! + |\mathscr{C}_2|(n-2)! + \cdots + |\mathscr{C}_{n-1}|1! + |\mathscr{C}_n|0!,$$

which leads to the following sequence,

$$|\mathscr{C}_1|, |\mathscr{C}_2|, \dots = 1, 1, 3, 13, 71, 461, 3447, 29093, \dots$$

The importance of these permutations for algebraic combinatorics stems from the fact that they label bases for the algebra of free quasi-symmetric functions [6, Section 3.3]. In our setup, plus-indecomposable permutations are important because the map ∂ preserves the plus-indecomposability.

Proposition 3.2. *Let $w \in S_n$, and let $t \in [n]$. Then $w([t]) = [t]$ if and only if $(\partial w)([t]) = [t]$. In particular, w is plus-indecomposable if and only if $\partial(w)$ is plus-indecomposable.*

Proof. Let $w_{\mathcal{ICD}} = C_1, \dots, C_n$.

($\Rightarrow$) Suppose $w([t]) = [t]$. This implies each entry of each C_i, $i \in [t]$, is less than or equal to t. Thus

$$(\partial(w))([t]) = (C_1 C_2 \cdots C_t)([t]) = [t].$$

($\Leftarrow$) Suppose $w([t]) \neq [t]$, so there is at least one pair i, j with $i \le t < j$ and $w_i > w_j$. We choose the pair with i smallest, and to adjudicate tiebreakers, we choose j to be the largest. This means that the largest entry of C_i is j, and j does not appear in any cycle to the left of C_i in $w_{\mathcal{ICD}}$. Thus $\partial(w)$ sends i to j, from which $(\partial(w))([t]) \neq [t]$. □

Our next aim is to show that ∂ fixes every 231-avoiding permutation and no others.

Lemma 3.1. *Let $w = w_1 w_2 \dots w_n \in S_n$ and let $\partial(w) = w'_1 w'_2 \dots w'_n$. Suppose there is $1 \le t < n$ such that $w([t]) = [t]$. Let*

$$x = w_1 w_2 \dots w_t \in S_t, \quad y = w_{t+1} - t \;\; w_{t+2} - t \; \dots \; w_n - t \in S_{n-t}.$$

Then

$$\partial(x) = w'_1 w'_2 \dots w'_t, \quad \partial(y) = w'_{t+1} - t \;\; w'_{t+2} - t \; \dots \; w'_n - t.$$

Proof. Assume for some $1 \le t < n$, $w([t]) = [t]$. Then also $w([t+1, n]) = [t+1, n]$. Let $w_{\mathcal{ICD}} = C_1, \dots, C_n$. The entries of the cycles $C_1, \dots, C_t$ are precisely $[t]$, and the entries of the cycles $C_{t+1}, \dots, C_n$ are precisely $[t+1, n]$. Moreover, x and y maintain the same relative order as their respective entries in w. Thus by equation (2),

$$(\partial(x))_i = (C_1 \cdots C_t)_i = w'_i, \quad i \in [t],$$
$$(\partial(y))_i = (C_1 \cdots C_n)_i - t = (C_{t+1} \cdots C_n)_i - t = w'_i - t, \quad i \in [t+1, n].$$
□

Example 3.5. *We demonstrate Lemma 3.1 on* $w = 2315764$ *with* $w([3]) = [3]$.

$$\partial(2315764) = (3\,1)(3\,2)(3)\ \ (7\,4)(7\,6\,5)(7\,6)(7) = 3127465,$$
$$\partial(231) = (3\,1)(3\,2)(3) = 312, \quad \partial(2431) = (4\,1)(4\,3\,2)(4\,3)(4) = 4132.$$

Lemma 3.2. *Let* $w = w_1w_2\dots w_n \in S_n$ *and let* $\partial(w) = w_1'w_2'\dots w_n'$. *Suppose* $w_1 = n$. *Then* $w_1' = n$ *and* $\partial(w_2w_3\dots w_n) = w_2'w_3'\dots w_n'$ *(viewed in* S_{n-1}*).*

Proof. Suppose $w_1 = n$, and let $w_{\mathcal{ICD}} = C_1,\dots,C_n$. It follows $C_1 = (n\,n-1\,\cdots\,1)$, which implies $(\partial(w))_1 = w_1' = n$. After writing
$$C_2C_3\cdots C_n = C_1^{-1}\ \partial(w) = (n\,1\,2\,\cdots\,n-1)\ \partial(w),$$
we observe that
$$(C_2C_3\cdots C_n)_1 = 1,$$
$$(C_2C_3\cdots C_n)_i = (\partial(w))_i + 1 = w_i' + 1, \quad i \in [2,n],$$
and finally,
$$(\partial(w_2w_3\cdots w_n))_{i-1} = (C_2C_3\cdots C_n)_i - 1 = w_i',\ i \in [2,n].$$
□

Example 3.6. *We demonstrate Lemma 3.2 on* $w = 623541$.

$$\partial(623541) = (6\,5\,4\,3\,2\,1)\ \ (6\,2)(6\,3)(6\,5\,4)(6\,5)(6) = 651243,$$
$$\partial(23541) = (5\,1)(5\,2)(5\,4\,3)(5\,4)(5) = 51243.$$

Proposition 3.3. *For* $w \in S_n$, $\partial(w) = w$ *if and only if* $w \in S_n^{231}$.

Proof. We prove this statement by induction on n. The base case $n = 1$ is clear so suppose $n > 1$. Let t be minimum with $w([t]) = [t]$.

Suppose $t < n$ (hence, w is not plus-indecomposable). It's easy to check that $w \in S_n^{231}$ if and only if
$$x = w_1w_2\dots w_t \in S_t^{231},\ \text{and}\ y = w_{t+1} - t\ \ w_{t+2} - t\ \dots\ w_n - t \in S_{n-t}^{231}.$$
Given Lemma 3.1, $\partial(w) = w$ if and only if $\partial(x) = x$ and $\partial(y) = y$, so this case follows by induction.

Suppose $t = n$ (hence, w is plus-indecomposable). Let $w_{\mathcal{ICD}} = C_1,\dots,C_n$. Notice that the cycle size of C_1 is the value of w_1 (simply by counting inversions), and its maximum entry is $(\partial(w))_1$; this latter statement follows from the two facts (1) the maximum entry of an inversion cycle C_i is $(C_i)_i$, and (2) $(\partial(w))_1 = (C_1)_1$ (by equation (2)). This means $(\partial(w))_1 \geq w_1 > 1$ (if $w_1 = 1$, this would imply $t = 1$, contradicting that $t = n > 1$). There are two subcases to consider.

In the first subcase, suppose $(\partial(w))_1 > w_1$. This means $\partial(w) \neq w$, and furthermore it means C_1 does not satisfy Proposition 3.1(3) (thus $w \notin S_n^{231}$).

In the second subcase, suppose $(\partial(w))_1 = w_1$. This means C_1 consists of (decreasing) consecutive integers, which further implies (from the perspective of the associated inversions) that $w([w_1]) = [w_1]$. So by assumption, $n = t \leq w_1$. Then it is clear that $w_1 = n$ and $C_1 = (n\,n-1\,\cdots\,1)$. The proof follows by induction from the following two facts: (1) $\partial(w) = w$ if and only if $\partial(w_2w_3\dots w_n) = w_2w_3\dots w_n$ (Lemma 3.2), and (2) $w \in S_n^{231}$ if and only if $w_2w_3\dots w_n \in S_{n-1}^{231}$. Fact (2) is easy to see (with $w_1 = n$). □

Corollary 3.1. *If* $w \in S_n$ *is not* 231*-avoiding, then* $\partial(w)$ *is lexicographically greater than* w *(in one-line notation).*

Proof. For $n \leq 3$, 231 is the only permutation which is not 231-avoiding, and $\partial(231) = 312$. The proof follows easily utilizing Proposition 3.3 and Lemmas 3.1, 3.2. □

In each application of ∂, unless a permutation is fixed, the resulting permutation is greater in the lexicographic ordering. Therefore, we conclude that, by applying ∂ successively, we reach a fixed point.

Corollary 3.2. *Let* $w \in S_n$. *The sequence* $\partial^0(w), \partial^1(w), \partial^2(w), \dots$ *converges to a* 231*-avoiding permutation.*

The proof of the main theorem as stated in the introduction readily follows.

Proof of Theorem 1.1. The first claim of our theorem (1) is Proposition 3.3. The second claim (2) is a consequence of Corollary 3.2. □

Example 3.7. *We consider $w = 2413$. Then we have*

$$\begin{aligned} inv(2413) = \{(1,3),(2,3),(2,4)\} &\implies \partial(w) = (3\,1)(4\,3\,2) = 3421 \\ inv(3421) = \{(1,3),(1,4),(2,3),(2,4),(3,4)\} &\implies \partial^2(w) = (4\,3\,1)(4\,3\,2)(4\,3) = 4312 \\ inv(4312) = \{(1,2),(1,3),(1,4),(2,3),(2,4)\} &\implies \partial^3(w) = (4\,3\,2\,1)(4\,3\,2) = 4312. \end{aligned}$$

Example 3.8. *We consider the powers of ∂ applied to $w = 5714263$. The sequence $\partial^0(w), \partial^1(w), \partial^2(w), \ldots$ converges to the 231-avoiding permutation 7654312.*

r	$\partial^r(w)$	$(\partial^r(w))_{\mathcal{ICD}}$	$\partial^{r+1}(w)$
0	5714263	(7 5 4 3 1),(7 6 5 4 3 2),(3),(7 5 4),(5),(7 6),(7)	7526134
1	7526134	(7 6 5 4 3 2 1),(7 6 5 3 2),(5 3),(7 6 5 4),(5),(6),(7)	7625314
2	7625314	(7 6 5 4 3 2 1),(7 6 5 4 3 2),(6 3),(7 6 5 4),(6 5),(6),(7)	7645321
3	7645321	(7 6 5 4 3 2 1),(7 6 5 4 3 2),(7 6 5 3),(7 6 5 4),(7 6 5),(7 6),(7)	7654312
4	7654312	(7 6 5 4 3 2 1),(7 6 5 4 3 2),(7 6 5 4 3),(7 6 5 4),(7 6 5),(6),(7)	7654312

In the remainder of Section 3, we develop some implications of our main result and pursue some interesting aspects of ∂.

The last non-identity cycle in the inversion cycle decomposition exhibits a *nice* property.

Lemma 3.3. *For all $w \in S_n$, the last non-identity cycle of $w_{\mathcal{ICD}}$ consists of (decreasing) consecutive integers.*

Proof. Let $w_{\mathcal{ICD}} = C_1, \ldots, C_n$. Suppose some cycle C_i does not consist of decreasing consecutive integers. Then for some $i < j < k$, C_i has the entry k but not j, and thus $w_j > w_i > w_k$. Then C_j (which contains k) is a non-identity cycle to its right. □

Corollary 3.3. *Let $w \in S_n$. If $w_{\mathcal{ICD}}$ has at most one non-identity cycle, then $\partial(w) = w$.*

The *code* of a permutation w ([21, Chapter 1.3]) is the sequence

$$\mathbf{c}(w) = (c_1, \ldots, c_n) \text{ where } c_i := |\{(i,j) \in \mathrm{inv}(w)\}|, \text{ for } i \in [n]. \tag{4}$$

Definition 3.4. *Let $w \in S_n$. The* shifted code *of w, denoted by $\mathbf{sc}(w)$, is the sequence $(c_1 + 1, \ldots, c_n + 1)$, where the code of w is $(c_1, \ldots, c_n)$.*

Definition 3.5. *Let $w \in S_n$. The* consecutive cycle decomposition *of w, denoted $w_{\mathcal{CCD}}$, is the sequence $D_1, \ldots, D_n$ of permutation cycles, called* consecutive cycles*, with cycle sizes,*

$$(|D_1|, \ldots, |D_n|) = \mathbf{sc}(w) = (s_1, \ldots, s_n),$$

where each cycle D_i consists of decreasing consecutive integers with least entry i,

$$D_i = (s_i + i - 1 \quad s_i + i - 2 \ \cdots \ i).$$

Remark 3.2. *As with $w_{\mathcal{ICD}}$, the cycles of $w_{\mathcal{CCD}}$ appear in increasing order, according to the least (and last) entry of each cycle. Moreover, the size of each cycle and the least entry of each cycle in $w_{\mathcal{CCD}}$ agree with those of its respective cycle in $w_{\mathcal{ICD}}$.*

Example 3.9. *We continue Example 3.8 with $w = 5714263$. Its shifted code is $\mathbf{sc}(w) = (5,6,1,3,1,2,1)$. $w_{\mathcal{ICD}}$ easily converts to $w_{\mathcal{CCD}}$.*

$$\begin{aligned} w_{\mathcal{ICD}} &= (7\,5\,4\,3\,1),(7\,6\,5\,4\,3\,2),(3),(7\,5\,4),(5),(7\,6),(7), \\ \Rightarrow w_{\mathcal{CCD}} &= (5\,4\,3\,2\,1),(7\,6\,5\,4\,3\,2),(3),(6\,5\,4),(5),(7\,6),(7). \end{aligned}$$

In our next proposition, we show that the product of the cycles of $w_{\mathcal{CCD}}$ is actually again w.

Proposition 3.4. *The map,*

$$\begin{aligned} \alpha_n : S_n &\to S_n \\ w &\mapsto D_1 D_2 \ldots D_n, \end{aligned}$$

with $w_{\mathcal{CCD}} = D_1, D_2, \ldots, D_n$, is the identity map.

Proof. Let $w \in S_n$. We induct on the number of non-identity cycles, say t, of $w_{\mathcal{CCD}}$. If $t \in \{0,1\}$, the proof follows by Lemma 3.3 and Corollary 3.3, since then $w_{\mathcal{CCD}} = w_{\mathcal{ICD}}$. Suppose $t > 1$. Let $D_m = (p\ p-1\ \cdots\ m)$, some $t \le m < p \le n$, be the last non-identity cycle of $w_{\mathcal{CCD}}$. Again by Lemma 3.3, D_m is also the last non-identity cycle of $w_{\mathcal{ICD}}$, so

$$w_{m+1} < w_{m+2} < \cdots < w_p < w_m < w_{p+1} < w_{p+2} < \cdots < w_n. \tag{5}$$

It follows $(D_m)^{-1} = (m\ m+1\ \cdots\ p)$, and $w(D_m)^{-1}$ is obtained from w by replacing

$$w_m w_{m+1} \dots w_p \quad \text{with} \quad w_{m+1} \dots w_p w_m,$$

leaving everything else fixed. Furthermore, by equation (5),

$$(w(D_m)^{-1})_m, (w(D_m)^{-1})_{m+1}, \dots, (w(D_m)^{-1})_n$$

is strictly increasing. This implies

$$(w(D_m)^{-1})_{\mathcal{CCD}} = D_1, D_2, \dots, D_{m-1}, (m), (m+1), \dots, (n).$$

(Notice for each $i \in [m-1]$, the cycle size of the ith consecutive cycle in $(w(D_m)^{-1})_{\mathcal{CCD}}$ agrees with the size of D_i.) The proof then follows by induction. □

Remark 3.3. *The consecutive cycle decomposition $w_{\mathcal{CCD}} = D_1, D_2, \dots, D_n$ of every $w \in S_n$ readily provides for a reduced decomposition of $w = D_1 D_2 \cdots D_n$ simply by rewriting each non-identity cycle as the product of simple transpositions. Notice the length (number of inversions) is*

$$\ell(w) = \sum_{i=1}^{n} (|D_i| - 1),$$

where $|D_i|$ denotes the size of D_i. Write the consecutive cycle $D = (d_t\, d_{t-1}\ \cdots\ d_1)$, of size $t > 1$, as the product of the $t-1$ simple transpositions,

$$(d_t\, d_{t-1})(d_{t-1}\, d_{t-2})\ \cdots\ (d_2\, d_1).$$

Example 3.10. *Continuing with Example 3.9, we exhibit a reduced decomposition of*

$$w = 5714263 = (5\,4\,3\,2\,1)(7\,6\,5\,4\,3\,2)(3)(6\,5\,4)(5)(7\,6)(7)$$

with $\ell(w) = 12$ in the manner of Remark 3.3:

$$w = (5\,4)(4\,3)(3\,2)(2\,1)\ (7\,6)(6\,5)(5\,4)(4\,3)(3\,2)\ (6\,5)(5\,4)\ (7\,6).$$

Corollary 3.4. *For each $w \in S_n$, $\partial(w) = w$ if and only if $w_{\mathcal{ICD}} = w_{\mathcal{CCD}}$.*

Proof. (⇒) Suppose $\partial(w) = w$. Then $w \in S_n^{231}$ (Proposition 3.3). The proof follows by Proposition 3.1(3) and Remark 3.2.

(⇐) Suppose $w_{\mathcal{ICD}} = w_{\mathcal{CCD}}$. Then $w \in S_n^{231}$ (Proposition 3.1(3)). The proof follows by Proposition 3.3. □

Corollary 3.5. *The map,*

$$\begin{aligned} \beta_n : S_n &\to \{(s_1, \dots, s_n) \mid s_i \in [n-i+1],\ \textit{for } i \in [n]\} \\ w &\mapsto \mathbf{sc}(w), \end{aligned}$$

is a bijection. Moreover, S_n, viewed as $\alpha_n(S_n)$, where α is as in Proposition 3.4, is the set of products of permutation cycles,

$$\{(a_1\ a_1 - 1 \cdots 1)(a_2\ a_2 - 1 \cdots 2) \cdots (a_n\ a_n - 1 \cdots n) \mid a_i \in [i, n],\ \textit{for } i \in [n]\}. \tag{6}$$

Proof. According to Definition 3.5, the size of any cycle D_i of $w_{\mathcal{CCD}}$ is restricted to $|D_i| \in [n-i+1]$ just by counting the number of possible inversions. So there are a maximum of $n!$ possible sequences $(|D_1|, |D_2|, \dots, |D_n|)$. However, every one of these must be a valid sequence due to Proposition 3.4. The image of α_n on S_n is then given by equation (6). □

The first statement of Corollary 3.5 is equivalent to [21, Proposition 1.3.12], for which Proposition 3.4 is an alternative approach. See Example 3.13 for the complete list of shifted codes on S_4.

Corollary 3.6. *Let $w \in S_n$, $w_{\mathcal{ICD}} = C_1, \ldots, C_n$, and $w_{\mathcal{CCD}} = D_1, \ldots, D_n$. Then for each $t \in [n]$,*

$$\partial(D_t D_{t+1} \cdots D_n) = C_t C_{t+1} \cdots C_n.$$

Proof. The case $t = 1$ is Proposition 3.4. Consider $t = 2$. Obtain the permutation $w' = D_1^{-1} w$ from w by increasing entries less than w_1 by one, and replacing w_1 with 1, thereby maintaining the same relative order in $w'_2 w'_3 \ldots w'_n$ as $w_2 w_3 \ldots w_n$. The resulting permutation w', with $w'_{\mathcal{CCD}} = (1), D_2, D_3, \ldots, D_n$, must then satisfy $w'_{\mathcal{ICD}} = (1), C_2, C_3, \ldots, C_n$. For arbitrary t,

$$D_t D_{t+1} \cdots D_n = x_1 x_2 \ldots x_n,$$

such that $x_1 x_2 \ldots x_{t-1} = 12 \ldots t-1$, and such that $x_t x_{t+1} \ldots x_n$ is made from the entries of $\{t, t+1, \ldots, n\}$ with the same relative order as $w_t w_{t+1} \ldots w_n$ (see Example 3.12). □

In general, the map ∂ does not preserve $\leq_L$, hence, it does not preserve $\leq_{BC}$.

Example 3.11. *For example* $41523 \lessdot_L 41532$ *but* $\partial(41532) = 51423 \lessdot_L 51432 = \partial(41523)$.

Despite these, not so pleasant properties, ∂ preserves parity and is monotone with respect to the length function.

Corollary 3.7. *For every $w \in S_n$, the difference of the lengths, $\ell(\partial(w)) - \ell(w)$, is divisible by 2.*

Proof. This is an immediate consequence from the fact that for every $w \in S_n$, the cycle sizes of $w_{\mathcal{CCD}}$ agree with those of $w_{\mathcal{ICD}}$. Recall a permutation is odd or even (but not both), based on whether it can be written as the product of an odd or even, respectively, number of transpositions. An odd (respectively, even) sized cycle may be written with an even (respectively, odd) number of transpositions. □

If multiplication of a permutation p by a transposition t results with the product $t\,p$ lexicographically greater than p (in one line notation), then $\ell(t\,p) > \ell(p)$.

Proposition 3.5. *For every $w \in S_n$, $\ell(\partial(w)) \geq \ell(w)$.*

Proof. This is trivial for $n = 1$, so suppose $n > 1$. In the notation of Corollary 3.6, let $w \in S_n$ with

$$w = D_1 D_2 \cdots D_n, \quad \partial(w) = C_1 C_2 \cdots C_n,$$

in terms of consecutive cycles and inversion cycles of w, respectively. We claim

$$\ell(C_i C_{i+1} \cdots C_n) - \ell(C_{i+1} C_{i+2} \cdots C_n) \geq |C_i| - 1, \text{ for } i \in [n-1], \tag{7}$$

where $|C_j|$ denotes the size of C_j (and of D_j). That would imply

$$\ell(\partial(w)) - \ell(C_n) \geq \sum_{i=1}^{n-1} (|C_i| - 1) = \sum_{i=1}^{n-1} (|D_i| - 1) = \ell(w) - (|D_n| - 1),$$

or $\ell(\partial(w)) \geq \ell(w)$ (since $C_n = D_n = (n) = id$). As for the claim, let $i \in [n-1]$ and let

$$C_i = (c_m\, c_{m-1}\, \cdots\, c_1 = i), \quad v = 12 \ldots i\, v_{i+1} v_{i+2} \ldots v_n = C_{i+1} C_{i+2} \cdots C_n,$$

with $m \geq 1$ and $c_m > c_{m-1} > \cdots > c_1$, and with v the one-line notation of the cycle product. Equation (7) is satisfied in case $m = 1$ with $C_i = (i) = id$, so suppose $m > 1$. In the fashion of Remark 3.3, rewrite C_i as the product of the $m - 1$ transpositions,

$$(c_m\, c_{m-1})(c_{m-1}\, c_{m-2}) \cdots (c_2\, c_1).$$

The validity of the claim stems from the fact mentioned before the proposition. The product $(c_2\, c_1)\, v$ is lexicographically greater than v since multiplication by $(c_2\, c_1)$ places c_2 into i-th position and does not affect entries to the left. Similarly, each successive transposition multiplication increases the i-th entry and does not affect entries to the left. Thus $\ell(C_i\, v) - \ell(v) \geq m - 1 = |C_i| - 1$ as advertised. □

Example 3.12. *We demonstrate Corollary 3.6 by breaking out the consecutive cycles and inversion cycles on* $w = 35241$.

	Consecutive Cycles	ℓ		*Inversion Cycles*		ℓ
35241 =	(321)(5432)(43)(54)(5)	*7*	∂(35241) =	(531)(5432)(53)(54)(5)	= 53421	*9*
15342 =	(1)(5432)(43)(54)(5)	*5*	∂(15342) =	(1)(5432)(53)(54)(5)	= 15423	*5*
12453 =	(1)(2)(43)(54)(5)	*2*	∂(12453) =	(1)(2)(53)(54)(5)	= 12534	*2*
12354 =	(1)(2)(3)(54)(5)	*1*	∂(12354) =	(1)(2)(3)(54)(5)	= 12354	*1*
12345 =	(1)(2)(3)(4)(5)	*0*	∂(12345) =	(1)(2)(3)(4)(5)	= 12345	*0*

We exhibit how multiplication by each successive inversion cycle increases length by at least the cycle size minus 1 (see equation (7)). The length (number of inversions) of the cycle $(5\,4)$ *is 1 and multiplication by the cycle* $(5\,3)$ *increases length by 1 (the cycle size minus 1) resulting with* $\ell(12534) = 2$. *Multiplication by* $(5\,4\,3\,2)$ *increases length by 3 (again the cycle size minus 1) resulting with* $\ell(15423) = 5$. *Finally, multiplication by* $(5\,3\,1)$ *increases length by 4 (more than the cycle size minus 1) resulting with* $\ell(53421) = 9$.

The main observation of this section, which leads to a proof of Theorem 1.1, is yet another equivalent condition to those in Proposition 3.1 for a 231-avoiding permutation.

Proposition 3.6. *Let* $w \in S_n$. *The shifted code of* w *is a scope sequence if and only if* $w \in S_n^{231}$.

Proof. The map that takes a permutation to its shifted code is one-to-one, and every scope sequence in $\mathcal{SS}_n$ is the shifted code for a (unique) permutation in S_n (see Definition 2.1(1) and Corollary 3.5). Moreover, $\mathcal{SS}_n$ and S_n^{231} are equal sized (Catalan) sets. Thus to complete the proof, it suffices to prove one direction. We show that if $\mathbf{sc}(w)$ is a scope sequence, then $w \in S_n^{231}$.

Let $\mathbf{sc}(w) = (s_1, \dots, s_n)$ and suppose $w \notin S_n^{231}$. Then there exists a triple $i < j < k$, with $w_j > w_i > w_k$. Choose one with i largest and then j smallest. This implies $w_x < w_i$ for $i < x < j$ with $j-i-1$ entries between w_i and w_j. By counting inversions (add one for shifted codes), and the fact that w_k is to the right of w_j, $s_i \geq j-i+1, \quad s_j \geq s_i-(j-i-1)$. After manipulation, this leads to $s_{i+(j-i)} > s_i-(j-i)$, where $0 < j-i < s_i$, which shows $\mathbf{sc}(w)$ is not a scope sequence; see Definition 2.1(2). □

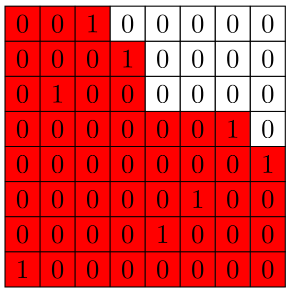
Example 3.13. *In this example we give the complete listing of shifted codes on* S_4. *Only in the case* $\partial(w) = w$ *(w is 231-avoiding) is the shifted code of w a scope sequence (Proposition 3.6). Those catalanized permutations are highlighted.*

w	$\mathbf{sc}(w)$	$\partial(w)$	w	$\mathbf{sc}(w)$	$\partial(w)$
1234	*(1,1,1,1)*	*(1)(2)(3)(4)=**1234***	***3124***	*(3,1,1,1)*	*(3 2 1)(2)(3)(4)=**3124***
1243	*(1,1,2,1)*	*(1)(2)(4 3)(4)=**1243***	*3142*	*(3,1,2,1)*	*(4 2 1)(2)(4 3)(4)=**4123***
1324	*(1,2,1,1)*	*(1)(3 2)(3)(4)=**1324***	***3214***	*(3,2,1,1)*	*(3 2 1)(3 2)(3)(4)=**3214***
1342	*(1,2,2,1)*	*(1)(4 2)(4 3)(4)=**1423***	*3241*	*(3,2,2,1)*	*(4 2 1)(4 2)(4 3)(4)=**4213***
1423	*(1,3,1,1)*	*(1)(4 3 2)(3)(4)=**1423***	*3412*	*(3,3,1,1)*	*(4 3 1)(4 3 2)(3)(4)=**4321***
1432	*(1,3,2,1)*	*(1)(4 3 2)(4 3)(4)=**1432***	*3421*	*(3,3,2,1)*	*(4 3 1)(4 3 2)(4 3)(4)=**4312***
2134	*(2,1,1,1)*	*(2 1)(2)(3)(4)=**2134***	***4123***	*(4,1,1,1)*	*(4 3 2 1)(2)(3)(4)=**4123***
2143	*(2,1,2,1)*	*(2 1)(2)(4 3)(4)=**2143***	***4132***	*(4,1,2,1)*	*(4 3 2 1)(2)(4 3)(4)=**4132***
2314	*(2,2,1,1)*	*(3 1)(3 2)(3)(4)=**3124***	***4213***	*(4,2,1,1)*	*(4 3 2 1)(3 2)(3)(4)=**4213***
2341	*(2,2,2,1)*	*(4 1)(4 2)(4 3)(4)=**4123***	*4231*	*(4,2,2,1)*	*(4 3 2 1)(4 2)(4 3)(4)=**4312***
2413	*(2,3,1,1)*	*(3 1)(4 3 2)(3)(4)=3421*	***4312***	*(4,3,1,1)*	*(4 3 2 1)(4 3 2)(3)(4)=**4312***
2431	*(2,3,2,1)*	*(4 1)(4 3 2)(4 3)(4)=**4132***	***4321***	*(4,3,2,1)*	*(4 3 2 1)(4 3 2)(4 3)(4)=**4321***

An explicit bijection due to Krattenthaler [11] between the set of Dyck paths (or the associated partitions) and the catalanized permutations is shown in the next example.

Example 3.14. *Let* $w = w_1 w_2 \dots w_8 = 8\,3\,1\,2\,7\,6\,4\,5$. *Then w is a catalanized permutation, and its inversion cycle decomposition is given by* $w_{\mathcal{ICD}} = w_{\mathcal{CCD}} = (8\ 7\ 6\ 5\ 4\ 3\ 2\ 1), (4\ 3\ 2), (3), (4), (8\ 7\ 6\ 5), (8\ 7\ 6), (7), (8)$. *The matrix of w is given below to the left. Then we shade the entries of the matrix that are to the left or below a nonzero entry as shown in the next figure. Clearly, this region of the matrix contains all entries on or below the main diagonal of the matrix. Thus, the boundary of the non-shaded area defines a Dyck path.*

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Example 3.15. *The purpose of this example is to demonstrate how the left weak and Bruhat-Chevalley orders on* S_4 *transform. In Figure 1, we consider the left weak order, and in Figure 2 we consider the Bruhat-Chevalley order.*

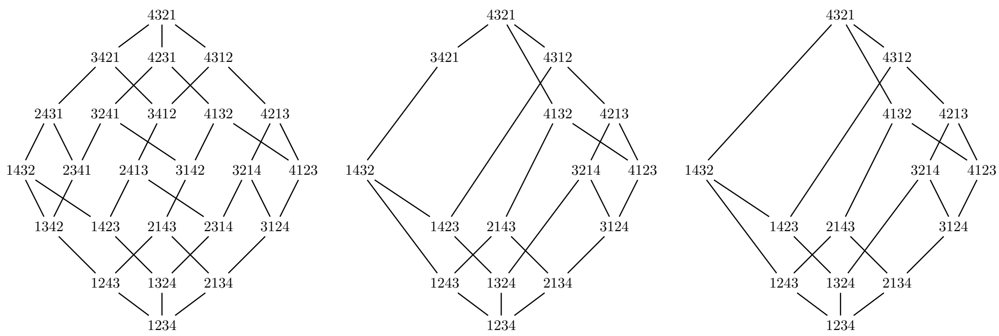


Figure 1: The catalanization of the left weak order $(S_4, \leq_L)$.

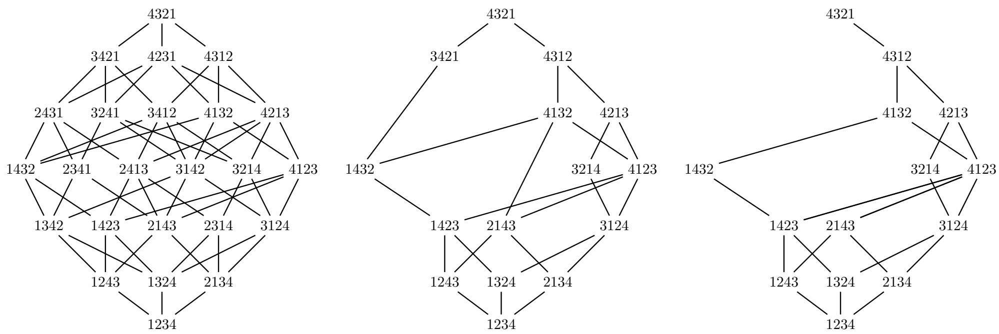


Figure 2: The catalanization of the Bruhat-Chevalley order $(S_4, \leq_{BC})$.

4 Open Questions

In this section, we pose several open problems. The first two questions are about enumeration.

1. What are the numbers t_n in Theorem 1.1(2)? As we mentioned earlier, t_n is minimum for $x \geq 0$ such that $\partial^x(S_n) = S_n^{231}$. Trivially, $t_1 = 0$, for $n \in \{2,3,4,5,6,7\}$, $t_n = n-2$ by computer analysis. But it gets tricky after that. We thought for a long time this should be the general pattern but a proof eluded us. We eventually identified the precisely three counter examples in S_8 that make $t_8 = 7$.

w	$\partial(w)$	$\partial^2(w)$	$\partial^3(w)$	$\partial^4(w)$	$\partial^5(w)$	$\partial^6(w)$	$\partial^7(w)$
24163785	35284167	63581427	76384125	87561423	87652413	87653421	87654312
24173586	35284167	63581427	76384125	87561423	87652413	87653421	87654312
24183567	35284167	63581427	76384125	87561423	87652413	87653421	87654312

The combinatorial numbers $|S_n^{(r)}| = \partial^r(S_n)$, for arbitrary r, n, also spark interest. Below is a table for $n \in [8]$. The first number in each row is $n!$ and the last is the Catalan number $|S_n^{231}|$. The numbers, $24, 15, 14$ for $n = 4$ are the sizes of the posets of Figure 1. What is their formula?

n	$\lvert S_n^{(0)}\rvert$	$\lvert S_n^{(1)}\rvert$	$\lvert S_n^{(2)}\rvert$	$\lvert S_n^{(3)}\rvert$	$\lvert S_n^{(4)}\rvert$	$\lvert S_n^{(5)}\rvert$	$\lvert S_n^{(6)}\rvert$	$\lvert S_n^{(7)}\rvert$
1	1							
2	2							
3	6	5						
4	24	15	14					
5	120	56	45	42				
6	720	261	169	139	132			
7	5040	1437	734	520	442	429		
8	40320	9208	3712	2160	1600	1454	1431	1430

2. For a permutation $w \in S_n$, denote by rank(w), the minimum $x \geq 0$ such that $\partial^x(w) \in S_n^{231}$. For arbitrary r and n, what is the formula for

$$|\ \{w \in S_n \mid \text{rank}(w) = r\}\ |\ ?$$

Below is a table of values for $n \in [8]$. The first column is the sequence of Catalan numbers.

$n \setminus r$	0	1	2	3	4	5	6	7
1	1							
2	2							
3	5	1						
4	14	9	1					
5	42	59	15	4				
6	132	354	155	62	17			
7	429	2059	1407	760	325	60		
8	1430	11930	12265	8423	4618	1408	243	3

3. There is a geometric perspective that draws our interest. The Hasse diagram of the weak order gives the 1-skeleton of the permutahedron. Likewise, the Hasse diagram of the Tamari lattice gives the 1-skeleton of the associahedron. Starting from $(S_n, \leq_L)$, each application of ∂ gives us a new Hasse diagram (graph). For $n = 4$, the intermediate graphs are shown in Figure 1. The convex hulls of the vertex sets of these graphs give subpolytopes of the permutahedron. In particular, the last polytope in this sequence is the associahedron. Are the intermediate polytopes equivariant fiber polytopes [18]? Also, are they brick polytopes [15]?

4. Both of the posets $(S_n, \leq_L)$ and $(S_n^{231}, \leq_L)$ are lattices. Also, both of the posets $(S_n, \leq_{BC})$ and $(S_n^{231}, \leq_{BC})$ are ranked. What are the values of n and k for which $(\partial^k S_n, \leq_L)$ is a lattice? What are the values of n and k for which $(\partial^k S_n, \leq_{BC})$ is a ranked poset?

5. Since ∂ preserves the plus-indecomposable permutations, it defines an automorphism of the algebra of free quasi-symmetric functions. What is the algebraic significance of this automorphism for the various algebras that are defined in [6]?

6. As we hinted at it in the introduction, it is possible (and natural) to develop similar results by varying 1) the total order on the set of inversions, 2) underlying Coxeter group. It would be interesting to know what kind of posets/lattices one gets by applying the "new" catalanization maps. For example, if we start with a different underlying Coxeter group (not type A), and apply a generalized catalanization map, is the resulting poset a Cambrian lattice [16]? Likewise, what kind of poset do we get if we apply the catalanization map to c-sortable elements of a Coxeter group [17, Theorem 1.2]?

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