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Positivity of the Second Shifted Difference of Partitions and Overpartitions: a Combinatorial Approach

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Abstract: This note is devoted to the study of inequalities related to the second shifted difference of the number of integer partitions $p(n)$ and of overpartitions $\overline{p}(n)$ by an elementary combinatorial approach. Recently Gomez, Males, and Rolen proved the positivity of $\Delta_j^2(p(n)) = p(n)-2p(n-j)+p(n-2j)$ by employing the Hardy-Ramanujan-Rademacher formula for $p(n)$ and Lehmer's error bound. Our goal is to prove $\Delta_j^2(p(n)) \geq 0$ (resp. $\Delta_j^2(\overline{p}(n)) > 0$) by an explicit description of a non-empty subset, say $X_p^2(n,j)$ of the set of integer partitions $P(n)$ (resp. $X_{\overline{p}}^2(n,j)$ and the set of overpartitions $\overline{P}(n)$) with $|X_p^2(n,j)| = \Delta_j^2(p(n))$ (resp. $|X_{\overline{p}}^2(n,j)| = \Delta_j^2(\overline{p}(n))$).

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1. Introduction

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ such that $\sum_{i=1}^{\ell} \lambda_i = n$, denoted by $\lambda \vdash n$. The set of partitions of n is denoted by $P(n)$ and $|P(n)| = p(n)$. For $\lambda \vdash n$, we define $\ell(\lambda)$ to be the total number of parts of λ and $\mathrm{mult}_\lambda(\lambda_i)$ to be the multiplicity of the part λ_i in λ. For $\lambda \vdash n$ with $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu \vdash m$ with $\mu = (\mu_1, \ldots, \mu_{\ell'})$, define the union $\lambda \cup \mu \vdash m + n$ to be the partition with parts $\{\lambda_i, \mu_j\}$ arranged in nonincreasing order.

Inequalities for the partition function have been studied in many directions and proofs of such inequalities were by employing analytic tools such as the Hardy-Ramanujan-Rademacher formula for $p(n)$, see [6, 11–13], and Lehmer's error bound [7, 8]. Let Δ be the backward difference operator defined on a sequence $a(n)$ by $\Delta(a(n)) := a(n) - a(n-1)$ and, for $r \geq 1$, $\Delta^r(a(n)) := \Delta\Big(\Delta^{r-1}(a(n))\Big)$. In 1977, Good [4] conjectured that $\Delta^r(p(n))$ alternates in sign up to a certain value $n = n(r)$, and then it stays positive. Using the Hardy-Ramanujan-Rademacher series for $p(n)$, Gupta [5] proved that for any given $r \in \mathbb{Z}_{\geq 1}$, $\Delta^r(p(n)) > 0$ for sufficiently large n. In 1988, Odlyzko [10] proved the conjecture of Good and obtained the following asymptotic formula for $n(r)$:

$$n(r) \sim \frac{6}{\pi^2} r^2 \log^2 r \quad \text{as} \quad r \to \infty.$$

For a more detailed study on $\Delta(p(n))$, we refer to [1]. Recently, Gomez, Males, and Rolen studied the second-order j-shifted difference of $p(n)$, defined by

$$\Delta_j^2(p(n)) = p(n) - 2p(n-j) + p(n-2j)$$

and proved the following theorem.

Theorem 1.1 (Theorem 1.2, [3]). *Let $n \geq 2$ and $j \leq \frac{1}{4}\sqrt{n - \frac{1}{24}}$. Then we have that*

$$\Delta_j^2(p(n)) \geq 0.$$

In other words, $p(n)$ satisfies the extended convexity result $p(n) + p(n-2j) \geq 2p(n-j)$.

An overpartition of n is a nonincreasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. We denote the number of overpartitions of n by $\overline{p}(n)$ and the set of overpartitions of n by $\overline{P}(n)$. For example, the 4 overpartitions of 2 are $2, \overline{2}, 1+1, \overline{1}+1$. The study on overpartitions dates back to MacMahon [9] but under different nomenclature, an extensive study on the overpartitions began with the work of Corteel and Lovejoy [2]. A Hardy-Ramanujan-Rademacher type series expansion for $\overline{p}(n)$ was due to Zuckerman [16]. Recently, Wang, Xie, and Zhang [15] proved that $\Delta^r(\overline{p}(n)) > 0$ for $n \geq n(r)$, where $n(r)$ is a positive integer depending on r.

The main motivation of this paper is to prove Theorem 1.1 using a combinatorial approach rather than the analytic one; i.e., by studying an asymptotic estimate of $\frac{p(n-j)}{p(n)}$ as in [3, Theorem 1.1]. Moreover, we will show $\Delta_j^2(p(n)) \geq 0$ for all $n \geq 2j$, a weaker assumption in comparison to $n \geq \max\{2, 16j^2 + \frac{1}{24}\}$ assumed in Theorem 1.1. Moreover, we show $\Delta_j^2(\overline{p}(n)) > 0$ with a similar combinatorial approach as that for $p(n)$. Gomez, Males, and Rolen [3] proved the positivity of $\Delta_j^2(p(n))$ using the asymptotic estimate of the quotient $p(n-j)/p(n)$ whereas our main objective is to show that $(\Delta_j^2(p(n)))_{n\geq 2j}$ (resp. $(\Delta_j^2(\overline{p}(n)))_{n\geq 2j}$) can be enumerated by a non-empty proper subset of $P(n)$ (resp. of $\overline{P}(n)$) so as to prove positivity of the respective sequences.

We organize the paper in the following way. Below we list all the theorems, Theorem 1.2-1.5, with two corollaries Corollary 1.1 and 1.2. The proofs of Theorem 1.2-1.5 are given in Section 2.

Definition 1.1. *For all positive integers n and j, define*

$$X_a^1(n,j) = A(n) \setminus A(n-j) \quad \textit{and} \quad |X_a^1(n,j)| = \Delta_j^1(a(n)),$$

$$X_a^2(n,j) = X_a^1(n) \setminus X_a^1(n-j) \quad \textit{and} \quad |X_a^2(n,j)| = \Delta_j^2(a(n)),$$

where $|A(n)| := a(n)$.

In our context, $A(n)$ is $P(n)$, resp. $\overline{P}(n)$; consequently, we will consider $X_a^i(n,j) = X_p^i(n,j)$, resp. $X_a^i(n,j) = X_{\overline{p}}^i(n,j)$.

Theorem 1.2. *For all positive integers n and j with $n \geq j$,*

$$X_p^1(n,j) = \Big\{\lambda \in P(n) : 0 \leq \lambda_1 - \lambda_2 \leq j-1\Big\}. \tag{1}$$

Remark 1.1. *Plugging in $j=1$ into Theorem 1.2, $X_p^1(n,j)$ is described as the set of non-unitary partitions of n as well as the set of partitions of $n-1$ in which the least part occurs exactly once [14, A002865]. For any $j \geq 1$, the set $X_p^1(n,j)$ is also known to be the set of non-j-ary partitions of n, see [3, p. 69]. A detailed analytic discussion on Theorem 1.2 has been documented in [3, Theorem 1.1].*

Theorem 1.3. *For all positive integers n and j with $n \geq 2j$,*

$$X_p^2(n,j) = \Big\{\lambda \in X_p^1(n,j) : 0 \leq \text{mult}_\lambda(1) \leq j-1\Big\}. \tag{2}$$

Remark 1.2. *Plugging in $j=1$ into Theorem 1.3, $X_p^2(n,j)$ is described as the set of partitions of $n-2$ with all parts >1 and with the largest part occurring more than once [14, A053445].*

Corollary 1.1. *For all positive integers n and j with $n \geq 2j$,*

$$\Delta_j^2(p(n)) \geq 0. \tag{3}$$

Proof. For $j=1$ and $n \in \{3,5,7\}$, $X_p^2(n,1) = \emptyset$ and so $\Delta_1^2(p(n)) = 0$ and for $n=2$, $\Delta_1^2(p(n)) = 1$. Next, if $n = 2k$ with $k \geq 2$, then $\lambda = (k,k) \in X_p^2(2k,1)$, and if $n = 2k+1$ with $k \geq 4$,

$$\lambda = \left(\left\lceil \frac{2k+1}{3} \right\rceil, \left\lceil \frac{2k+1}{3} \right\rceil, (2k+1) - 2\left\lceil \frac{2k+1}{3} \right\rceil\right) \in X_p^2(2k+1,1),$$

as $(2k+1) - 2\left\lceil \frac{2k+1}{3} \right\rceil > 1$ for all $k \geq 4$. So, $\Delta_1^2(p(n)) \geq 0$ for all $n \geq 2$.

Finally, for $j \geq 2$ and $n = 2m \geq 2j$, observe that $\lambda = (m,m) \in X_p^2(n,j)$ and for $n = 2m+1 > 2j$, $\lambda = (m+1,m) \in X_p^2(n,j)$. Therefore, $\Delta_1^2(p(n)) > 0$ for all $n \geq 2j$ with $j \geq 2$. $\square$

Remark 1.3. *A combinatorial proof of Corollary 1.1 is also provided in [3, p. 77]. But our proof of $\Delta_j^2(p(n)) \geq 0$ is based on studying the elements of residual set $X_p^2(n,j)$.*

Theorem 1.4. *For all positive integers n and j with $n \geq j$,*

$$\begin{aligned} X_{\overline{p}}^1(n,j) = &\Big\{\lambda \in \overline{P}(n) : 0 \leq \lambda_1 - \lambda_2 \leq j-1 \textit{ and } \lambda_1, \lambda_2 \textit{ may be overlined}\Big\} \\ &\cup \Big\{\lambda \in \overline{P}(n) : \lambda_1 - \lambda_2 = j \textit{ and } \lambda_2 \textit{ is overlined}\Big\}. \end{aligned} \tag{4}$$

Theorem 1.5. *For all positive integers n and j with $n \geq 2j$,*

$$X^2_{\overline{p}}(n,j) = \Big\{\lambda \in X^1_{\overline{p}}(n,j) : 0 \leq \text{mult}_\lambda(1) \leq j-1 \text{ and } 0 \leq \text{mult}_\lambda(\overline{1}) \leq 1\Big\}. \tag{5}$$

Corollary 1.2. *For all positive integers n and j with $n \geq 2j$,*

$$\Delta^2_j(\overline{p}(n)) > 0. \tag{6}$$

Proof. For $j = 1$ and $n = 2$, $\Delta^2_j(\overline{p}(n)) = 1$. For $j \geq 1$, $n = 2k \geq 2j$ with $k \in \mathbb{Z}_{\geq 2}$, $\lambda = (\overline{k}, k) \in X^2_{\overline{p}}(n,j)$ and when $n = 2k+1 > 2j$ with $k \in \mathbb{Z}_{\geq 1}$, $\lambda = (\overline{k+1}, \overline{k}) \in X^2_{\overline{p}}(n,j)$. This concludes the proof. □

2. Proofs of Theorem 1.2-1.5

Proof of Theorem 1.2: For all positive integers n, j with $n \geq j$, we define an injective map $i_1 : P(n-j) \longrightarrow P(n)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto i_1(\lambda) = (\lambda_1 + j, \lambda_2, \ldots, \lambda_r). \tag{7}$$

It is immediate that $i_1(\lambda) \in P(n)$, and the image set can be described as

$$\text{Im}(i_1) = \Big\{\pi \in P(n) : \pi_1 - \pi_2 \geq j\Big\}.$$

Note that i_1 is an injective map: for any two partitions, say, for $\lambda, \mu \in P(n-j)$, there are two possible cases, either $\ell(\lambda) = \ell(\mu)$ or $\ell(\lambda) \neq \ell(\mu)$. When $\ell(\lambda) \neq \ell(\mu)$, $\ell(i_1(\lambda)) \neq \ell(i_1(\mu))$ and therefore i_1 is injective. If $\ell(\lambda) = \ell(\mu)$, then $i_1(\lambda) = i_1(\mu)$ immediately implies that $\lambda_m = \mu_m$ for all $1 \leq m \leq \ell(\lambda)$. Hence,

$$P(n) \setminus i_1(P(n-j)) = \Big\{\pi \in P(n) : 0 \leq \pi_1 - \pi_2 \leq j\Big\} = X^1_p(n,j).$$

□

Proof of Theorem 1.3: For all positive integers n, j with $n \geq 2j$, we first define an injective map $i_2 : X^1_p(n-j,j) \longrightarrow X^1_p(n,j)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto i_2(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_r) \cup (\underbrace{1, 1, \ldots, 1}_{j \text{ times}}). \tag{8}$$

Now $i_2(\lambda) \in X^1_p(n,j)$ and consequently,

$$\text{Im}(i_2) = \Big\{\pi \in X^1_p(n,j) : \text{mult}_\pi(1) \geq j\Big\}.$$

Clearly, i_2 is an injective map, since we adjoin the partition of j with all parts being 1 to any partition $\lambda \in X^1_p(n-j,j)$. Therefore,

$$X^1_p(n,j) \setminus i_2(X^1_p(n-j,j)) = \Big\{\pi \in X^1_p(n,j) : 0 \leq \text{mult}_\pi(1) \leq j-1\Big\} = X^2_p(n,j).$$

□

Proof of Theorem 1.4: For all positive integers n, j with $n \geq j$, we define an injective map $\overline{i}_1 : \overline{P}(n-j) \longrightarrow \overline{P}(n)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto \overline{i}_1(\lambda) = (\lambda_1 + j, \lambda_2, \ldots, \lambda_r) \in \overline{P}(n). \tag{9}$$

Here we consider two separate cases depending on whether $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$.

For $\lambda_1 = \lambda_2$, we observe that only the first occurrence of λ_1 can be overlined and the image of $\overline{i}_1$ is given by

$$\text{Im}(\overline{i}_1) = \Big\{\pi \in \overline{P}(n) : \pi_1 - \pi_2 = j \text{ and } \pi_2 \text{ is not overlined}\Big\}.$$

For the other case $\lambda_1 \neq \lambda_2$,

$$\text{Im}(\overline{i}_1) = \Big\{\pi \in \overline{P}(n) : \pi_1 - \pi_2 \geq j \text{ and } \pi_1, \pi_2 \text{ may be overlined}\Big\}.$$

Clearly, $\overline{i}_1$ is an injective map in each of the cases. Therefore

$$\begin{aligned}\overline{P}(n) \setminus \overline{i}_1(\overline{P}(n-j)) =& \Big\{\pi \in \overline{P}(n) : 0 \leq \pi_1 - \pi_2 \leq j-1 \text{ and } \pi_1, \pi_2 \text{ may be overlined}\Big\} \\ & \cup \Big\{\pi \in \overline{P}(n) : \pi_1 - \pi_2 = j \text{ and } \pi_2 \text{ is overlined}\Big\} \\ =& \overline{X}^1_{\overline{p}}(n,j).\end{aligned}$$

□

Proof of Theorem 1.5: For all positive integers n, j with $n \geq 2j$, we define an injective map $\bar{i}_2 : \overline{X}_p^1(n-j, j) \longrightarrow \overline{X}_p^1(n, j)$ by

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \mapsto \bar{i}_2(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_r) \cup (\underbrace{1, 1, \ldots, 1}_{j \text{ times}}) \in \overline{X}_p^1(n, j). \tag{10}$$

Consequently,

$$\mathrm{Im}(\bar{i}_2) = \Big\{\pi \in X_p^1(n, j) : \mathrm{mult}_\pi(1) \geq j\Big\}.$$

Note that i_2 is an injective map as we adjoin the overpartition of j with all parts being 1 to any overpartition $\lambda \in \overline{X}_p^1(n-j, j)$. Therefore,

$$\begin{aligned}\overline{X}_p^1(n, j) \setminus \bar{i}_2(\overline{X}_p^1(n-j, j)) &= \Big\{\pi \in X_p^1(n, j) : 0 \leq \mathrm{mult}_\pi(1) \leq j-1 \text{ and } 0 \leq \mathrm{mult}_\pi(\bar{1}) \leq 1\Big\} \\ &= \overline{X}_p^2(n, j),\end{aligned}$$

since if $\bar{1}$ is a part of an overpartition, say $\pi \in \overline{P}(n)$, then according to the definition $0 \leq \mathrm{mult}_\pi(\bar{1}) \leq 1$.

□

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