

On the Formal Power Series of Involutory Functions

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ABSTRACT: It is shown that the coefficients of any involutory function f represented as a power series can be expressed in terms of multivariable Lah polynomials. This result is based on the fact that any such f (\neq identity) can be regarded as a (compositional) conjugate of negative identity. Moreover, a constructive proof of this statement is given.

Keywords: Bell polynomials; Faà di Bruno's formula; Formal power series; Higher derivatives; Involutory function; Lah numbers; Lah polynomials; Stirling numbers; Stirling polynomials

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1. Preliminaries

In the following, a *function* shall be understood as a formal power series with coefficients from a fixed commutative field \mathcal{K} of characteristic zero. For our purposes, we mostly restrict ourselves to (compositionally) *invertible* functions, i. e., $f \in \mathcal{K}[[x]]$ with $f \circ \bar{f} = \bar{f} \circ f = \iota$, where \bar{f} denotes the (unique) inverse of f and ι denotes the identity: $\iota(x) = x$. According to a simple criterion [10, Proposition 5.4.1], $f = f(x) = \sum_{n \geq 0} f_n \frac{x^n}{n!}$ is invertible if and only if $f_0 = f(0) = 0$ and $f_1 = f'(0) \neq 0$. The invertible functions form a non-commutative group \mathcal{G} with respect to the composition \circ . *Involutory functions* (or *involutions*) are the elements f in this group with $f \circ f = \iota$. The two simplest examples of involutory functions are the identity ι (*trivial involution*) and the negative identity $-\iota$.

A sequence of constants $g_1, g_2, g_3, \dots \in \mathcal{K}$, $g_1 \neq 0$, uniquely determines a function $g = g(x) = g_1x + g_2x^2/2! + g_3x^3/3! + \dots \in \mathcal{G}$, from which we can in turn recover the g_1, g_2, g_3, \dots by differentiation as Taylor coefficients: $D^n(g)(0) = g_n$ ($n = 1, 2, 3, \dots$). The operator D here denotes the unique extension of the ordinary algebraic derivation on $\mathcal{K}[x]$ (with $D(\iota) = 1$) to the algebra $\mathcal{K}[[x]]$ of formal power series (cf. [1]). D^n denotes, as usual, the n -th iterate of D .

Of special interest is the task of finding the general Taylor coefficient of a composite function $f \circ g$ ($g \in \mathcal{G}$, f not necessarily invertible). Its well-known solution is given by the famous formula of Faà di Bruno (see [2, p. 137], [7, eqs. (1.3), (1.4)]), which in modern notation is

$$D^n(f \circ g)(0) = \sum_{k=0}^n f_k B_{n,k}(g_1, \dots, g_{n-k+1}). \tag{1}$$

As is common, the symbol $B_{n,k}$ stands for the *partial Bell* (or *exponential*) *polynomials* belonging to $\mathcal{K}[X_1, \dots, X_{n-k+1}]$ (see, e.g., [2, p. 133], [5, p. 65]). Replacing the function g with its inverse \bar{g} leads to the following nontrivial counterpart of (1):

$$D^n(f \circ \bar{g})(0) = \sum_{k=0}^n f_k A_{n,k}(g_1, \dots, g_{n-k+1}). \tag{2}$$

Here the family $A_{n,k}$ consists of (Laurent) polynomials $\in \mathcal{K}[X_1^{-1}, X_2, \dots, X_{n-k+1}]$ which are ‘ortho-inverse’ companions of the $B_{n,k}$ in the sense that for $1 \leq k \leq n$

$$\sum_{j=k}^n A_{n,j} B_{j,k} = \delta_{nk} \tag{3}$$

with $\delta_{nn} = 1$, $\delta_{nk} = 0$ for $n \neq k$ (Kronecker's symbol).

Remark 1.1. *The fundamental properties (explicit representations, recurrences, inverse relations, reciprocity laws) of these double-indexed polynomial families are treated in detail in [7–9]. The collective term multivariate Stirling polynomials (MSP) of the first and second kind was proposed for $A_{n,k}$ and $B_{n,k}$ because the associated coefficient sums $A_{n,k}(1, \dots, 1)$ and $B_{n,k}(1, \dots, 1)$ turn out to be just the signed Stirling numbers of the first kind and the Stirling numbers of the second kind, respectively. This happens when $g(x) = e^x - 1$ is chosen.*

Specializing $f = \iota^k/k!$, we obtain from (1) and (2) after a short calculation

$$\begin{aligned} D^n\left(\frac{g^k}{k!}\right)(0) &= D^n\left(\frac{\iota^k}{k!} \circ g\right)(0) = B_{n,k}(g_1, \dots, g_{n-k+1}), \\ D^n\left(\frac{\bar{g}^k}{k!}\right)(0) &= D^n\left(\frac{\iota^k}{k!} \circ \bar{g}\right)(0) = A_{n,k}(g_1, \dots, g_{n-k+1}), \end{aligned}$$

which clearly illustrates the meaning of the MSP. — Finally, note the following remarkable identity [7, Theorem 5.3], which holds for $1 \leq k \leq n$:

$$A_{n,k} = B_{n,k}(A_{1,1}, \dots, A_{n-k+1,1}). \tag{4}$$

The reader will find a set of tables for the MSP in [7, p. 2471] and [9, p. 51–54]. The partial Bell polynomials $B_{n,k}$, $1 \leq k \leq n \leq 12$, are tabulated in [2, p. 307–308].

2. The coefficients of involutory functions

With the aid of the Faà di Bruno formula (1) it is possible to characterize the general Taylor coefficient f_n of an involutory function f by a recurrence.

Theorem 2.1. *Let $f \in \mathcal{G}$ be any function with $f \neq \iota$ and $f_n = D^n(f)(0)$. Then, f is involutory if and only if there is a sequence of constants $a_1, a_2, a_3, \dots \in \mathcal{K}$ such that for every $n \geq 1$*

$$f_n = \begin{cases} -1, & \text{if } n = 1; \\ a_{n/2}, & \text{if } n \text{ even}; \\ \frac{1}{2} \sum_{k=2}^{n-1} f_k B_{n,k}(-1, f_2, \dots, f_{n-k+1}), & \text{if } n \geq 3 \text{ odd.} \end{cases} \tag{5}$$

Proof. We calculate the coefficients of the two sides of $f \circ f = \iota$ separately. This yields $D^n(\iota)(0) = \delta_{n1}$ and further by applying (1) to $D^n(f \circ f)(0)$ the following statement, which is equivalent to f being involutory:

$$\sum_{k=1}^n f_k B_{n,k}(f_1, \dots, f_{n-k+1}) = \delta_{n1} \quad \text{for all } n \geq 1. \tag{6}$$

For $n = 1$ equation (6) becomes $1 = \delta_{11} = f_1 B_{1,1}(f_1) = f_1^2$. Thus, to prove $f_1 = -1$, we must show that $f_1 = 1$ leads to a contradiction. For this purpose, we first consider (6) for $n \geq 2$ and by splitting off the first and the last summand on the left side of the equation we obtain

$$f_1 B_{n,1}(f_1, \dots, f_n) + \sum_{k=2}^{n-1} f_k B_{n,k}(f_1, \dots, f_{n-k+1}) = -f_n B_{n,n}(f_1). \tag{7}$$

We now show by induction that indirectly assuming $f_1 = 1$ implies $f_n = 0$ for all $n \geq 2$, which contradicts $f \neq \iota$. In the case $n = 2$ we have $f_2 = f_1 B_{2,1}(f_1, f_2) = -f_2 B_{2,2}(f_1) = -f_2$, and therefore $f_2 = 0$. Now assume $f_j = 0$ for $3 \leq j \leq n$ (induction hypothesis). From this follows by (7)

$$f_{n+1} = f_1 B_{n+1,1}(f_1, \dots, f_{n+1}) + 0 = -f_{n+1} B_{n+1,n+1}(f_1) = -f_{n+1},$$

hence $f_{n+1} = 0$. Thus $f_1 = -1$ is proved.

If we now substitute this value into (7), we obtain after a few transformations the following equivalent statement:

$$\sum_{k=2}^{n-1} f_k B_{n,k}(-1, f_2, \dots, f_{n-k+1}) = (1 + (-1)^{n+1})f_n \quad \text{for all } n \geq 2. \tag{8}$$

For odd $n \geq 3$ the right-hand side becomes $2f_n$ and the third line of the assertion (5) is immediately clear. In the case of an even $n \geq 2$, the right-hand side vanishes, with the consequence that the same coefficients, namely f_2, \dots, f_{n-1} , occur both in the equation (8) for n and in that for its odd predecessor $n - 1$, while f_n itself remains undetermined. The coefficients f_2, f_4, f_6, \dots can thus be taken as a sequence of arbitrary constants $a_k := f_{2k}$, $k \geq 1$, on which the solution system of the equation (6) depends. This shows the rest of the assertion. \square

Example 2.1. We illustrate here the statement of Theorem 2.1 by listing the first odd indexed coefficients of any nontrivial involution f in the form of polynomial expressions of the arbitrary constants $f_{2k} = a_k$ ($1 \leq k \leq 4$):

$$\begin{aligned} f_1 &= -1, \\ f_3 &= -\frac{3}{2}a_1^2, \\ f_5 &= 15a_1^4 - \frac{15}{2}a_1a_2, \\ f_7 &= -\frac{4095}{4}a_1^6 + \frac{945}{2}a_1^3a_2 - \frac{35}{2}a_2^2 - 14a_1a_3, \\ f_9 &= \frac{411075}{2}a_1^8 - \frac{208845}{2}a_1^5a_2 + 7875a_1^2a_2^2 + 2205a_1^3a_3 - 105a_2a_3 - \frac{45}{2}a_1a_4. \end{aligned}$$

It remains open (if not questionable) whether a closed-form representation for the general coefficient f_{2k-1} can be achieved from the calculation procedure (5).

3. Multivariable Lah polynomials

The recursive representation of the coefficients established in the previous section is based upon a direct evaluation of the defining functional equation $f \circ f = \iota$ using the Faà di Bruno formula. As shown in [8, 9], it is nevertheless possible to obtain a closed representation for the f_n by using a decomposition of f in the form $f(x) = g(-\bar{g}(x))$ with some $g \in \mathcal{G}$; in argument-free notation:

$$f = g \circ (-\iota) \circ \bar{g}. \tag{9}$$

Throughout Subsection 5.5 of [8], which among other things deals with the representation of involutory functions, use is made of (9) without proof. Instead, there is a reference to McCarthy [6], which, however, misses the point, since this paper is about continuous bijective involutions $\mathbb{R} \rightarrow \mathbb{R}$ and based on some set-theoretic arguments. On closer inspection, it becomes clear that the statements derived in this context cannot be simply carried over to functions in the form of formal power series. Thus, the possibility of a conjugate representation (9) requires a proof of its own, which also closes the gap in [8, Subsection 5.5].

In preparing this, we first show that equation (9) allows us to give the general form of the coefficients of an involution.

Theorem 3.1. Let $f, g \in \mathcal{G}$ be arbitrarily given with $f \neq \iota$, $f_n = D^n(f)(0)$ and $g_n = D^n(g)(0)$. Then, $f = g \circ (-\iota) \circ \bar{g}$ holds if and only if $f_n = L_{n,1}(g_1, g_2, \dots, g_{2\lfloor n/2 \rfloor})$ for all $n \geq 1$.

Here the polynomials $L_{n,1}$ form a subfamily of the double-indexed *multivariable Lah polynomials* introduced in [8] and defined by

$$L_{n,k} = \sum_{j=k}^n (-1)^j A_{n,j} B_{j,k}. \tag{10}$$

Remark 3.1. Similar to the Stirling polynomials mentioned above, the naming is justified by the hardly surprising fact that the associated number sequence $L_{n,k}(1, \dots, 1)$, more precisely the sum of coefficients of $L_{n,k}$, is just that of the well-known signed Lah numbers $(-1)^n \frac{n!}{k!} \binom{n-1}{k-1}$; see [8, p. 19].

Example 3.1. The following list contains the first 6 members of the subfamily $L_{n,1}$:

$$\begin{aligned} L_{1,1} &= -1, & L_{2,1} &= \frac{2X_2}{X_1^2}, & L_{3,1} &= -\frac{6X_2^2}{X_1^4}, & L_{4,1} &= \frac{30X_2^3}{X_1^6} - \frac{8X_3X_2}{X_1^5} + \frac{2X_4}{X_1^4}, \\ L_{5,1} &= -\frac{210X_2^4}{X_1^8} + \frac{120X_3X_2^2}{X_1^7} - \frac{30X_4X_2}{X_1^6}, \\ L_{6,1} &= \frac{1890X_2^5}{X_1^{10}} - \frac{1680X_3X_2^3}{X_1^9} + \frac{420X_4X_2^2}{X_1^8} + \frac{140X_3^2X_2}{X_1^8} - \frac{12X_5X_2}{X_1^7} + \frac{2X_6}{X_1^6} - \frac{40X_3X_4}{X_1^7}. \end{aligned}$$

We now prove Theorem 3.1.

Proof. The equation $f = g \circ (-\iota) \circ \bar{g}$ being satisfied is equivalent to the statement that $f_n = D^n(g \circ (-\iota) \circ \bar{g})(0)$ holds for all $n \geq 1$. Hence, only $D^n(g \circ ((-\iota) \circ \bar{g}))(0)$ needs to be evaluated. We first determine the coefficients of $(-\iota) \circ \bar{g}$. By means of (2) we obtain

$$D^r((-\iota) \circ \bar{g})(0) = \sum_{k=1}^r D^k(-\iota)(0) A_{r,k}(g_1, \dots, g_{r-k+1}) = \sum_{k=1}^r -\delta_{k,1} A_{r,k}(g_1, \dots, g_{r-k+1}) = -A_{r,1}(g_1, \dots, g_r). \tag{11}$$

Next, we apply the Faà di Bruno formula (1) as well as equation (4), which gives

$$f_n = D^n(g \circ ((-\iota) \circ \bar{g}))(0) = \sum_{k=1}^n g_k B_{n,k}(-A_{1,1}(g_1), -A_{2,1}(g_1, g_2), \dots, -A_{n-k+1,1}(g_1, \dots, g_{n-k+1})) =: (*).$$

Considering that the $B_{n,k}$ are homogeneous polynomials of degree k and further that $g_k = B_{k,1}(g_1, \dots, g_k)$ holds, the last expression becomes

$$(*) = \sum_{k=1}^n (-1)^k A_{n,k}(g_1, \dots, g_{n-k+1}) B_{k,1}(g_1, \dots, g_k) = L_{n,1}(g_1, g_2, \dots, g_{\lfloor n/2 \rfloor}). \quad \square$$

In order to prove that any involution admits a decomposition of the kind (9), we need the following two fundamental properties of Lah polynomials.

Proposition 3.1. *For all $1 \leq k \leq n$,*

$$\sum_{j=k}^n L_{n,j} L_{j,k} = \delta_{nk}.$$

Proof. A direct consequence from (10) and (3). See [8, Proposition 5.6]. □

The next statement says that the Lah polynomials $L_{n,k}$ can be represented using the partial Bell polynomials with the indeterminates X_j replaced by the subfamily terms $L_{j,1}$ for each $j \geq 1$.

Proposition 3.2. *For all $1 \leq k \leq n$,*

$$L_{n,k} = B_{n,k}(L_{1,1}, \dots, L_{n-k+1,1}).$$

Remark 3.2. *This result is derived in [8] directly from the context of the proof of Theorem 5.3 (see Corollary 5.1), so it is not immediately obvious whether or not it makes use of the decomposition (9), the possibility of which remains to be proved here. In any way, the proof that follows is independent of (9).*

Proof. Assume g to be any function $\in \mathcal{G}$. Putting $\tilde{g} := g \circ (-\iota)$ and $h := \tilde{g} \circ \bar{g}$ we get by Theorem 3.1

$$h_n = D^n(h)(0) = L_{n,1}(g_1, g_2, \dots). \quad (12)$$

From (10) and (4) follows

$$L_{n,k} = \sum_{j=k}^n B_{n,j}(A_{1,1}, A_{2,1}, \dots) B_{j,k}(-X_1, X_2, -X_3, \dots)$$

and after replacing X_r with g_r ($r = 1, 2, \dots$)

$$L_{n,k}(g_1, g_2, \dots) = \sum_{j=k}^n B_{n,j}(\bar{g}_1, \bar{g}_2, \dots) B_{j,k}(\tilde{g}_1, \tilde{g}_2, \dots), \quad (13)$$

where $\bar{g}_r = D^r(\bar{g})(0)$ and $\tilde{g}_r = D^r(\tilde{g})(0)$ for $r = 1, 2, \dots$. If we now take a closer look at the right-hand side of (13), we see that it is equal to $B_{n,k}(h_1, h_2, \dots)$, according to a formula established by Jabotinsky [3, 4] (cf. also [2, Section 3.7] and the *Second Composition Rule* in [8, Theorem 4.2 (i)]). This yields

$$L_{n,k}(g_1, g_2, \dots) = B_{n,k}(L_{1,1}(g_1), L_{2,1}(g_1, g_2), \dots)$$

and the asserted identity if we recall that polynomials over an infinite integral domain (here the field \mathcal{K}) that give rise to the same polynomial function are identical. □

Combining Proposition 3.1 and 3.2 yields the following lemma, which we need to prove our main results in the next section.

Lemma 3.1. *For all $n \geq 2$ we have*

$$\sum_{k=2}^{n-1} L_{k,1} B_{n,k}(L_{1,1}, \dots, L_{n-k+1,1}) = (1 + (-1)^{n+1}) L_{n,1}.$$

Proof. From Proposition 3.1 and 3.2 we conclude

$$\sum_{k=1}^n L_{k,1} B_{n,k}(L_{1,1}, \dots, L_{n-k+1,1}) = \delta_{n1}.$$

Taking into account $L_{1,1} B_{n,1}(L_{1,1}, \dots, L_{n,1}) = -L_{n,1}$, this results in

$$\sum_{k=2}^n L_{k,1} B_{n,k}(L_{1,1}, \dots, L_{n-k+1,1}) = L_{n,1} + \delta_{n1}$$

and further

$$\sum_{k=2}^{n-1} L_{k,1} B_{n,k}(L_{1,1}, \dots, L_{n-k+1,1}) = L_{n,1} + \delta_{n1} - L_{n,1} B_{n,n}(L_{1,1}).$$

Here, the right-hand side is equal to $L_{n,1} - L_{n,1} \cdot (-1)^n$ for $n \geq 2$, which proves the assertion. □

4. The main results

We are now in a position to prove that any nontrivial involution can be represented in terms of the equation (9).

Theorem 4.1. *Let f be any function from \mathcal{G} with $f \neq \iota$. Then, f is involutory if and only if there exists $g \in \mathcal{G}$ such that $f = g \circ (-\iota) \circ \bar{g}$.*

Proof. The sufficiency ('if') is immediate, since $g \circ (-\iota) \circ \bar{g}$ is clearly involutory. For necessity ('only if') assume f to be any involutory function $\neq \iota$. Then, by Theorem 2.1 there are constants $a_1, a_2, \dots \in \mathcal{K}$ such that the coefficients $f_n = D^n(f)(0)$ satisfy the recurrence (5). We now construct the function

$$g(x) = \sum_{n \geq 1} g_n \frac{x^n}{n!}$$

we are looking for in the following way: The odd indexed coefficients g_1, g_3, g_5, \dots are chosen as arbitrary constants $\in \mathcal{K}$, where $g_1 \neq 0$; for even $n \geq 2$, we define g_n recursively by setting

$$g_n = \frac{1}{2} \sum_{k=2}^n f_k B_{n,k}(g_1, \dots, g_{n-k+1}). \tag{14}$$

We must prove that g satisfies (9). According to Theorem 3.1, it suffices to show that $L_{n,1}(g_1, g_2, \dots) = f_n$ holds for all $n \geq 1$. The following induction is split into two parts, depending on the parity of n .

The cases $n = 1$ and $n = 2$ are settled readily by observing Theorem 2.1 and Example 3.1 as follows:

$$\begin{aligned} L_{1,1}(g_1, g_2, \dots) &= -1 = f_1, \\ L_{2,1}(g_1, g_2, \dots) &= \frac{2g_2}{g_1^2} \stackrel{(14)}{=} \frac{1}{g_1^2} \cdot f_2 B_{2,2}(g_1) = f_2, \end{aligned}$$

where $f_2 = a_1$ (arbitrary constant).

Induction hypothesis: Assume $L_{r,1}(g_1, g_2, \dots) = f_r$ for $3 \leq r \leq n - 1$.

First, we consider the case of an odd $n \geq 3$. From Lemma 3.1 we get

$$L_{n,1} = \frac{1}{2} \sum_{k=2}^{n-1} L_{k,1} B_{n,k}(L_{1,1}, L_{2,1}, \dots)$$

and by replacing each X_j with g_j

$$\begin{aligned} L_{n,1}(g_1, g_2, \dots) &= \frac{1}{2} \sum_{k=2}^{n-1} L_{k,1}(g_1, g_2, \dots) B_{n,k}(L_{1,1}(g_1, g_2, \dots), L_{2,1}(g_1, g_2, \dots), \dots) \\ &= \frac{1}{2} \sum_{k=2}^{n-1} f_k B_{n,k}(f_1, f_2, \dots, f_{n-k+1}) && \text{(induction hypothesis)} \\ &= f_n. && \text{(Theorem 2.1, (5))} \end{aligned}$$

Let us now assume that $n \geq 4$ is even. Recalling $B_{j,1} = X_j$ equation (10) can be written as

$$L_{n,1} = \sum_{j=1}^n (-1)^j X_j A_{n,j}.$$

We apply to this the *inversion of sequences* based on (3) and explained in [7, Corollary 5.2] (cf. also [8, Proposition 5.4]) thus obtaining

$$\sum_{k=1}^n L_{k,1} B_{n,k} = (-1)^n X_n$$

and after substituting g_j for X_j :

$$\sum_{k=1}^n L_{k,1}(g_1, g_2, \dots) B_{n,k}(g_1, g_2, \dots) = (-1)^n g_n.$$

From this follows by the induction hypothesis

$$\sum_{k=1}^{n-1} f_k B_{n,k}(g_1, g_2, \dots) = (-1)^n g_n - g_1^n L_{n,1}(g_1, g_2, \dots).$$

Now consider the first summand on the left-hand side, which is $f_1 B_{n,1}(g_1, g_2, \dots) = -g_n$. We split it off and then solve the equation for $L_{n,1}(g_1, g_2, \dots)$:

$$L_{n,1}(g_1, g_2, \dots) = -\frac{1}{g_1^n} \sum_{k=2}^{n-1} f_k B_{n,k}(g_1, g_2, \dots) + \frac{g_n}{g_1^n} (1 + (-1)^n).$$

By (14) we have

$$2g_n = \sum_{k=2}^{n-1} f_k B_{n,k}(g_1, g_2, \dots) + f_n B_{n,n}(g_1)$$

and thus from the above

$$L_{n,1}(g_1, g_2, \dots) = -\frac{1}{g_1^n} (2g_n - f_n g_1^n) + \frac{2g_n}{g_1^n} = f_n,$$

which also proves the assertion for the case of even n . □

From the Theorems 3.1 and 4.1 follows immediately our second main result, which describes the form of the coefficients of involutory functions.

Theorem 4.2. *Let f be any function from \mathcal{G} , $f \neq \iota$. Then, f is involutory if and only if there exist constants $g_1 (\neq 0), g_2, g_3, \dots \in \mathcal{K}$ such that the general Taylor coefficient f_n of f meets $f_n = L_{n,1}(g_1, g_2, \dots, g_{2\lfloor n/2 \rfloor})$.*

5. Discussion

Let us look more closely at the statement of Theorem 4.1. If there exists $g \in \mathcal{G}$ satisfying $f = g \circ (-\iota) \circ \bar{g}$, this says that f and the negative identity $-\iota$ are conjugate elements in the group (\mathcal{G}, \circ) . Conversely, the nontrivial part of the statement says that every involution $f \neq \iota$ of \mathcal{G} is indeed a conjugate of $-\iota$. In summary, *all elements of \mathcal{G} of order 2 form just one conjugacy class*.

One can by no means expect the function g to be uniquely determined by an equation $f = g \circ (-\iota) \circ \bar{g}$. As a quite simple example, consider $g(x) = e^x - 1$ and the corresponding sequence $g_1 = g_2 = \dots = 1$, for which $f_n = L_{n,1}(1, \dots, 1) = (-1)^n n!$ (cf. Remark 3.1) and the resulting involutory function becomes

$$f(x) = -x + x^2 - x^3 + x^4 - \dots = -\frac{x}{1+x}. \tag{15}$$

There are also other sequences leading to the same involution, for instance the coefficients $g_j = c^j$, ($j = 1, 2, 3, \dots$) with $c \in \mathcal{K}$, $c \neq 0$, belonging to the function $g(x) = e^{cx} - 1$ and also satisfying $L_{n,1}(c, c^2, c^3, \dots) =$

$(-1)^n n!$. So, the obvious question is how to describe the set of all functions $g \in \mathcal{G}$ satisfying $f = g \circ (-\iota) \circ \bar{g}$ for a given involution f . The answer is easily obtained by assuming any two functions $g, h \in \mathcal{G}$ for which

$$g \circ (-\iota) \circ \bar{g} = h \circ (-\iota) \circ \bar{h}. \tag{16}$$

If we set $\psi := \bar{g} \circ h$, (16) can be equivalently transformed into $\psi \circ (-\iota) = (-\iota) \circ \psi$, which shows ψ is an odd function. Conversely, the general solution h of the equation $f = h \circ (-\iota) \circ \bar{h}$ is obtained in the form $h = g \circ \psi$, where g is any particular solution and ψ is some odd function from \mathcal{G} . Note that the subgroup of odd functions is the centralizer of $-\iota$ in \mathcal{G} .

Applying this to the involutory function f in (15), it results that the equation $f(x) = g(-\bar{g}(x))$ holds if and only if g has the form $g(x) = e^{\psi(x)} - 1$ with ψ being an odd function. Suitable examples are $\psi(x) = cx$, which gives the above sequence c, c^2, c^3, \dots , or $\psi(x) = \sin x$. Of course, the seemingly unwieldy sequence $1, 1, 0, -3, -8, -3, 56, 217, 64, -2951, \dots$ produced by $e^{\sin x} - 1$ nevertheless leads to the same coefficients $L_{n,1}(1, 1, 0, -3, -8, -3, \dots) = (-1)^n n!$, as the reader may verify from the list in Example 3.1 for some values.

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