ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **4:3** (2024) Article #S2R18
https://doi.org/10.54550/ECA2024V4S3R18

Generating Functions for the cd-indices of Simplices and Cubes

Richard Ehrenborg

Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027, USA
Email: richard.ehrenborg@uky.edu

Received: September 13, 2023, **Accepted**: December 30, 2023, **Published**: January 19, 2024

ABSTRACT: Inspired by the grammatical calculus of William Chen, we derive an expression for the exponential generating function for the **cd**-indices of the n-dimensional simplices and the n-dimensional cubes. This expression for simplices differs from a similar expression obtained by Stanley.

Keywords: **cd**-indices of simplices and cubes; Exponential generating functions; Pyramid and prism operators
2020 Mathematics Subject Classification: 05A15, 06A07, 52B05

1. Introduction

William Chen introduced the grammatical calculus, where derivative operators played an important role in order to obtain identities [4]. As an example, Dumont introduced the derivative G on (commutative) polynomials in the variables x and y such that $G(x) = G(y) = xy$ and used this derivative to generate the Eulerian polynomials $A_n(x)$; see [7] and [5, Section 1]. Recall that the Eulerian polynomials enumerate permutations, taking into account the number of descents of the permutation.

The **ab**-index of a graded poset is a non-commutative polynomial encoding the flag h-vector of the poset. In the special case when the poset is the Boolean algebra, this information is exactly the descent set statistic of permutations. This fact is straightforward to observe using the classical R-labeling of the Boolean algebra. Hence we may view the **ab**-index of the Boolean algebra as a non-commutative extension of the Eulerian polynomial. Since the Boolean lattice is the face lattice of the simplex Δ_n, we denote this polynomial by $\Psi(\Delta_n)$ in the non-commutative variables $\mathbf{a}$ and $\mathbf{b}$.

What is more striking is that Dumont's derivation appears independently in the non-commutative world in order to understand the geometric operation of taking the pyramid of a polytope [9]. There the derivation is $G(\mathbf{a}) = \mathbf{ba}$ and $G(\mathbf{b}) = \mathbf{ab}$, which is a non-commutative analogue of the Dumont derivative. By considering the operator $\mathrm{Pyr}(w) = w \cdot (\mathbf{a} + \mathbf{b}) + G(w)$, called the pyramid operator, the paper [9] proved that the **ab**-index of the pyramid of a polytope P is the pyramid of the **ab**-index of the original polytope P. In particular, iterating the pyramid operator yields the **ab**-index of the simplex. There is a similar operator Prism whose iteration yields **ab**-indices of the cubes.

The purpose of this note is to lift techniques from the field of grammatical calculus of Chen, Dumont, Fu, and others to the non-commutative setting and obtain new results for the **ab**- and **cd**-indices of simplices and cubes. The main step is to obtain how powers of the pyramid and prism operators behave on a product; see Proposition 3.1. From these identities it is straightforward to obtain the desired generating functions for simplices and cubes.

The reader might object that we use generating functions without a variable. However, since our coefficients are homogenous polynomials, an extra variable of the generating function does not add any extra information. We end this note with open questions.

2. Preliminaries

Given a graded poset P of rank $n+1$. The **ab**-index is a non-commutative polynomial, homogenous of degree n, in two variables $\mathbf{a}$ and $\mathbf{b}$. It encodes the flag f-vector of the poset, equivalently, the flag h-vector. When the poset is Eulerian, the **ab**-index can be written in terms of $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$; see [1]. In particular, for an n-dimensional polytope P, its face lattice is Eulerian and we write $\Psi(P)$ for the associated **cd**-index. For more details; see [10].

Define the two derivations G and D on $\mathbb{R}\langle \mathbf{a}, \mathbf{b} \rangle$ by $G(\mathbf{a}) = \mathbf{ba}$, $G(\mathbf{b}) = \mathbf{ab}$ and $D(\mathbf{a}) = D(\mathbf{b}) = \mathbf{ab} + \mathbf{ba}$. Observe that these two derivations restrict to the subalgebra $\mathbb{R}\langle \mathbf{c}, \mathbf{d} \rangle$ by $G(\mathbf{c}) = \mathbf{d}$, $G(\mathbf{d}) = \mathbf{cd}$, $D(\mathbf{c}) = 2\mathbf{d}$ and $D(\mathbf{d}) = \mathbf{cd} + \mathbf{dc}$. It is also common to let $\mathbf{e}$ denote the difference $\mathbf{a} - \mathbf{b}$. Observe that $\mathbf{e}^2 = \mathbf{c}^2 - 2\mathbf{d}$ and $D(\mathbf{e}) = 0$. The pyramid and prism operators Pyr and Prism are defined by

$$\begin{aligned} \mathrm{Pyr}(w) &= w \cdot \mathbf{c} + G(w) = \frac{1}{2} \cdot (w \cdot \mathbf{c} + D(w) + \mathbf{c} \cdot w), \\ \mathrm{Prism}(w) &= w \cdot \mathbf{c} + D(w). \end{aligned}$$

The name of these operators comes from that they correspond to the same operators on polytopes. That is, for a polytope P, let $\mathrm{Pyr}(P)$ be the cone of P and let $\mathrm{Prism}(P)$ be the Cartesian product with a line segment. Then the following two relations hold:

$$\Psi(\mathrm{Pyr}(P)) = \mathrm{Pyr}(\Psi(P)), \qquad \Psi(\mathrm{Prism}(P)) = \mathrm{Prism}(\Psi(P));$$

see [9, Theorems 4.4 and 5.2]. Especially, for the n-dimensional simplex Δ_n and the n-dimensional cube $\square_n$ we have

$$\Psi(\Delta_n) = \mathrm{Pyr}^n(1), \qquad \Psi(\square_n) = \mathrm{Prism}^n(1),$$

since the **cd**-index of the 0-dimensional polytope, that is, a point, is 1.

Define the **cd**-polynomials A_n by $A_0 = 1$,

$$\begin{aligned} A_{2k} &= \frac{1}{2} \cdot \left((\mathbf{c}^2 - 2\mathbf{d})^k + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{k-1} \cdot \mathbf{c} \right), \\ A_{2k+1} &= -\frac{1}{2} \cdot \left((\mathbf{c}^2 - 2\mathbf{d})^k \cdot \mathbf{c} + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^k \right). \end{aligned}$$

The negatives of these polynomials were defined in [2, Section 3]. Similarly define the polynomial sequence B_n by

$$B_{2k} = (\mathbf{c}^2 - 2\mathbf{d})^k, \qquad B_{2k+1} = -\mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^k.$$

These polynomials appear in Stanley's proof of the existence of the **cd**-index; see [10, Proof of Theorem 1.1].

Lemma 2.1. *The polynomial sequences A_n and B_n satisfy the recursions:*

$$A_{n+1} = -\mathbf{c} \cdot A_n + \mathrm{Pyr}(A_n) - A_n \cdot \mathbf{c}, \tag{2.1}$$
$$B_{n+1} = -\mathbf{c} \cdot B_n + \mathrm{Prism}(B_n) - B_n \cdot \mathbf{c}. \tag{2.2}$$

The recursion (2.1) is basically the third equation on page 428 in [2], but for completeness we include a proof.

Proof of Lemma 2.1. The two relations in the lemma can be rewritten as

$$A_{n+1} = \frac{1}{2} \cdot (-\mathbf{c} \cdot A_n + D(A_n) - A_n \cdot \mathbf{c}), \tag{2.3}$$
$$B_{n+1} = -\mathbf{c} \cdot B_n + D(B_n). \tag{2.4}$$

We prove these identities by considering the even versus odd cases. First note that $D(\mathbf{e}) = 0$, hence

$$D(u \cdot (\mathbf{c}^2 - 2\mathbf{d})^k \cdot v) = D(u) \cdot (\mathbf{c}^2 - 2\mathbf{d})^k \cdot v + u \cdot (\mathbf{c}^2 - 2\mathbf{d})^k \cdot D(v).$$

To make expressions shorter let $X = (\mathbf{c}^2 - 2\mathbf{d})^k$ and $Y = (\mathbf{c}^2 - 2\mathbf{d})^{k-1}$ such that $X = (\mathbf{c}^2 - 2\mathbf{d}) \cdot Y = Y \cdot (\mathbf{c}^2 - 2\mathbf{d})$. When $n = 2k$ is even, the left-hand side of (2.3) and the left-hand side of (2.4) are:

$$\begin{aligned} \frac{-\mathbf{c} \cdot A_n + D(A_n) - A_n \cdot \mathbf{c}}{2} &= \frac{-\mathbf{c} \cdot (X + \mathbf{c}Y\mathbf{c}) + D(X + \mathbf{c}Y\mathbf{c}) - (X + \mathbf{c}Y\mathbf{c}) \cdot \mathbf{c}}{4} \\ &= \frac{-\mathbf{c}X + (2\mathbf{d} - \mathbf{c}^2) \cdot Y\mathbf{c} - X\mathbf{c} + \mathbf{c}Y \cdot (2\mathbf{d} - \mathbf{c}^2)}{4} \\ &= \frac{-\mathbf{c}X - X\mathbf{c}}{2} = A_{n+1}, \\ -\mathbf{c} \cdot B_n + D(B_n) &= -\mathbf{c} \cdot X + D(X) = -\mathbf{c} \cdot X = B_{n+1}. \end{aligned}$$

When $n = 2k+1$ is odd, the two left-hand sides are:

$$\begin{aligned}
\frac{-\mathbf{c} \cdot A_n + D(A_n) - A_n \cdot \mathbf{c}}{2} &= \frac{\mathbf{c} \cdot (X\mathbf{c} + \mathbf{c}X) - D(X\mathbf{c} + \mathbf{c}X) + (X\mathbf{c} + \mathbf{c}X) \cdot \mathbf{c}}{4} \\
&= \frac{(\mathbf{c}^2 - 2\mathbf{d}) \cdot X + 2 \cdot \mathbf{c}X\mathbf{c} + X \cdot (\mathbf{c}^2 - 2\mathbf{d})}{4} \\
&= \frac{(\mathbf{c}^2 - 2\mathbf{d})^{k+1} + \mathbf{c}X\mathbf{c}}{2} = A_{n+1}, \\
-\mathbf{c} \cdot B_n + D(B_n) &= \mathbf{c}^2 X - D(\mathbf{c}X) = (\mathbf{c}^2 - 2\mathbf{d}) \cdot X = B_{n+1}.
\end{aligned}$$

Finally observe the $n = 0$ case for the first recursion is $(-\mathbf{c} \cdot A_0 + D(A_0) - A_0 \cdot \mathbf{c})/2 = -(2\mathbf{c})/2 = -\mathbf{c} = A_1$, completing the lemma. □

3. Results

We now explore the powers of the pyramid and prism operators. They satisfy a rule related to the Leibniz' product rule for derivatives.

Proposition 3.1. *For two* **ab***-polynomials u and v the following two identities hold:*

$$\mathrm{Pyr}^n(u \cdot v) = \sum_{i+j+k=n} \binom{n}{i,j,k} \cdot \mathrm{Pyr}^i(u) \cdot A_j \cdot \mathrm{Pyr}^k(v), \tag{3.1}$$

$$\mathrm{Prism}^n(u \cdot v) = \sum_{i+j+k=n} \binom{n}{i,j,k} \cdot \mathrm{Prism}^i(u) \cdot B_j \cdot \mathrm{Prism}^k(v). \tag{3.2}$$

Proof. We begin by proving (3.1) for the pyramid operation. For $n = 0$ there is nothing to prove since $A_0 = 1$. For $n = 1$ we have

$$\begin{aligned}
\mathrm{Pyr}(u \cdot v) &= u \cdot v \cdot \mathbf{c} + G(u \cdot v) \\
&= u \cdot \mathbf{c} \cdot v + G(u) \cdot v - u \cdot \mathbf{c} \cdot v + u \cdot v \cdot \mathbf{c} + u \cdot G(v) \\
&= \mathrm{Pyr}(u) \cdot v - u \cdot \mathbf{c} \cdot v + u \cdot \mathrm{Pyr}(v).
\end{aligned} \tag{3.3}$$

Since $A_1 = -\mathbf{c}$ this completes the case $n = 1$. By applying (3.3) twice we have

$$\mathrm{Pyr}(u \cdot v \cdot w) = \mathrm{Pyr}(u) \cdot v \cdot w - u \cdot \mathbf{c} \cdot v \cdot w + u \cdot \mathrm{Pyr}(v) \cdot w - u \cdot v \cdot \mathbf{c} \cdot w + u \cdot v \cdot \mathrm{Pyr}(w). \tag{3.4}$$

We prove the identity by induction on n. The base case $n = 1$ is already done. For the induction step it is enough to observe by equation (3.4) that

$$\begin{aligned}
&\mathrm{Pyr}(\mathrm{Pyr}^i(u) \cdot A_j \cdot \mathrm{Pyr}^k(v)) \\
&= \mathrm{Pyr}^{i+1}(u) \cdot A_j \cdot \mathrm{Pyr}^k(v) \\
&- \mathrm{Pyr}^i(u) \cdot \mathbf{c} \cdot A_j \cdot \mathrm{Pyr}^k(v) + \mathrm{Pyr}^i(u) \cdot \mathrm{Pyr}(A_j) \cdot \mathrm{Pyr}^k(v) - \mathrm{Pyr}^i(u) \cdot A_j \cdot \mathbf{c} \cdot \mathrm{Pyr}^k(v) \\
&+ \mathrm{Pyr}^i(u) \cdot A_j \cdot \mathrm{Pyr}^{k+1}(v).
\end{aligned}$$

The second through fourth terms in this expression combine to

$$\mathrm{Pyr}^i(u) \cdot (-\mathbf{c} \cdot A_j + \mathrm{Pyr}(A_j) - A_j \cdot \mathbf{c}) \cdot \mathrm{Pyr}^k(v) = \mathrm{Pyr}^i(u) \cdot A_{j+1} \cdot \mathrm{Pyr}^k(v),$$

using equation (2.1). Observe that applying the pyramid operator to the term indexed by (i,j,k) in the expansion for n yields the terms indexed by $(i+1,j,k)$, $(i,j+1,k)$ and $(i,j,k+1)$ in the expansion for $n+1$, proving the identity (3.1). The proof for the identity (3.2) is parallel to the proof of the first identity with the difference that we use the derivative D, $B_0 = 1$, $B_1 = -\mathbf{c}$ and the recursion in (2.2). □

Let e^{Pyr} and e^{Prism} denote the two operators

$$e^{\mathrm{Pyr}} = \sum_{n \geq 0} \frac{\mathrm{Pyr}^n}{n!}, \qquad e^{\mathrm{Prism}} = \sum_{n \geq 0} \frac{\mathrm{Prism}^n}{n!}.$$

Proposition 3.2. *For two* **ab***-polynomials u and v the following two identities hold:*

$$e^{\mathrm{Pyr}}(u \cdot v) = e^{\mathrm{Pyr}}(u) \cdot \sum_{n \geq 0} \frac{A_n}{n!} \cdot e^{\mathrm{Pyr}}(v), \tag{3.5}$$

$$e^{\mathrm{Prism}}(u \cdot v) = e^{\mathrm{Prism}}(u) \cdot \sum_{n \geq 0} \frac{B_n}{n!} \cdot e^{\mathrm{Prism}}(v). \tag{3.6}$$

Proof. Divide the two identities in Proposition 3.1 by $n!$ and sum over all n. □

Theorem 3.1. *The exponential generating function for the* **cd***-indices of the simplices is given by*

$$\sum_{n \geq 0} \frac{\Psi(\Delta_n)}{n!} = \frac{2}{1 + \cosh(\mathbf{e}) + \mathbf{c}\mathbf{e}^{-1}(\cosh(\mathbf{e}) - 1)\mathbf{e}^{-1}\mathbf{c} - \sinh(\mathbf{e})\mathbf{e}^{-1}\mathbf{c} - \mathbf{c}\mathbf{e}^{-1}\sinh(\mathbf{e})}.$$

Proof. First observe that the sought-after generating function is $e^{\mathrm{Pyr}}(1)$. Next set $u = 1$ (or $v = 1$) in equation (3.5) and solve for $e^{\mathrm{Pyr}}(1)$. We obtain

$$\begin{aligned} \frac{1}{e^{\mathrm{Pyr}}(1)} = \sum_{n \geq 0} \frac{A_n}{n!} &= 1 + \frac{1}{2} \cdot \sum_{k \geq 1} \frac{(\mathbf{c}^2 - 2\mathbf{d})^k + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^{k-1} \cdot \mathbf{c}}{(2k)!} \\ &\quad - \frac{1}{2} \cdot \sum_{k \geq 0} \frac{(\mathbf{c}^2 - 2\mathbf{d})^k \cdot \mathbf{c} + \mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^k}{(2k+1)!}. \end{aligned}$$

We now use that $\mathbf{c}^2 - 2\mathbf{d} = (\mathbf{a} - \mathbf{b})^2 = \mathbf{e}^2$ and can thus express the above generating function in terms of the hyperbolic functions:

$$\frac{1}{e^{\mathrm{Pyr}}(1)} = \frac{2 + \cosh(\mathbf{e}) - 1 + \mathbf{c}\mathbf{e}^{-1}(\cosh(\mathbf{e}) - 1)\mathbf{e}^{-1}\mathbf{c} - \sinh(\mathbf{e})\mathbf{e}^{-1}\mathbf{c} - \mathbf{c}\mathbf{e}^{-1}\sinh(\mathbf{e})}{2}.$$ □

Note that $(\cosh(\mathbf{e}) - 1)\mathbf{e}^{-2} = \mathbf{e}^{-1}(\cosh(\mathbf{e}) - 1)\mathbf{e}^{-1} = \mathbf{e}^{-2}(\cosh(\mathbf{e}) - 1)$ and $\sinh(\mathbf{e})\mathbf{e}^{-1} = \mathbf{e}^{-1}\sinh(\mathbf{e})$. But as a matter of taste we have picked the most palindromic expressions. Note also how our expression in the theorem differs from [10, Corollary 1.4].

Theorem 3.2. *The exponential generating function for the* **cd***-indices of the cubes is given by*

$$\sum_{n \geq 0} \frac{\Psi(\square_n)}{n!} = \frac{1}{\cosh(\mathbf{e}) - \mathbf{c}\mathbf{e}^{-1}\sinh(\mathbf{e})}.$$

Proof. Our goal is to determine $e^{\mathrm{Prism}}(1)$. Again set $u = 1$ (or $v = 1$) but now in equation (3.6) and solve for $e^{\mathrm{Prism}}(1)$. We obtain

$$\frac{1}{e^{\mathrm{Prism}}(1)} = \sum_{n \geq 0} \frac{B_n}{n!} = \sum_{k \geq 0} \frac{(\mathbf{c}^2 - 2\mathbf{d})^k}{(2k)!} - \sum_{k \geq 0} \frac{\mathbf{c} \cdot (\mathbf{c}^2 - 2\mathbf{d})^k}{(2k+1)!} = \cosh(\mathbf{e}) - \mathbf{c}\mathbf{e}^{-1}\sinh(\mathbf{e}).$$ □

4. Concluding remarks

Is there a poset theoretic approach to explain the two identities in Proposition 3.1? The polynomial $(-1)^j \cdot A_j$ is the **cd**-index of the Whitney stratified space consisting of a j-dimensional ball whose boundary is a point and a $(j-1)$-dimensional ball; see [8, Example 6.15]. Could this interpretation help in giving a more combinatorial proof of equation (3.1)? For equation (3.2) the polynomial $(-1)^j \cdot B_j$ has a similar interpretation; see [8, Example 6.14].

It is interesting to note that the generating function for the cubes is more compact than the corresponding generating function for simplices. Is this fact related to that there is a straightforward expression for the **cd**-index of the cube in terms of permutations? See [3, Proposition 8.1]. Furthermore, the **ab**-index of the n-dimensional cube is a non-commutative extension of the Eulerian polynomial of type B.

Are there other examples of the grammatical calculus that can be lifted to the non-commutative world? In the papers [5, 6] Chen and Fu also use the multiplicative inverses of the variables in their algebra to derive identities. Can the multiplicative inverses of **a** and **b** be used in a similar manner? For instance, we have $\mathrm{Pyr}(\mathbf{a}^{-1}) = \mathrm{Pyr}(\mathbf{b}^{-1}) = 1$.

Acknowledgments

The author thanks William Chen for inspiration for this note. The author thanks Theodore Ehrenborg for comments on an earlier draft. This work was partially supported by a grant from the Simons Foundation (#854548 to Richard Ehrenborg).

References

[1] M. M. Bayer and A. Klapper, *A new index for polytopes*, Discrete Comput. Geom. 6(1) (1991), 33–47.

[2] L. J. Billera and R. Ehrenborg, *Monotonicity of the cd-index for polytopes*, Math. Z. 233(3) (2000), 421–441.

[3] L. J. Billera, R. Ehrenborg, and M. Readdy, *The c-2d-index of oriented matroids*, J. Combin. Theory Ser. A, 80(1) (1997), 79–105.

[4] W. Y. C. Chen, *Context-free grammars, differential operators and formal power series*, Volume 117, 113–129, 1993. Conference on Formal Power Series and Algebraic Combinatorics, Bordeaux, 1991.

[5] W. Y. C. Chen and A. M. Fu, *Context-free grammars for permutations and increasing trees*, Adv. in Appl. Math. 82 (2017), 58–82.

[6] W. Y. C. Chen and A. M. Fu, *The Dumont Ansatz for the Eulerian Polynomials, Peak Polynomials and Derivative Polynomials*, Ann. Comb. 27(3) (2023), 707–735.

[7] D. Dumont, *Grammaires de William Chen et dérivations dans les arbres et arborescences*, Sém. Lothar. Combin. 37 (1996), Article B37a.

[8] R. Ehrenborg, M. Goresky, and M. Readdy, *Euler flag enumeration of Whitney stratified spaces*, Adv. Math. 268 (2015), 85–128.

[9] R. Ehrenborg and M. Readdy, *Coproducts and the cd-index*, J. Algebraic Combin. 8(3) (1998), 273–299.

[10] R. P. Stanley, *Flag f-vectors and the cd-index*, Math. Z. 216(3) (1994), 483–499.