ecajournal.haifa.ac.il
Enumerative Combinatorics and Applications
ECA **4:3** (2024) Article #S2R22
`https://doi.org/10.54550/ECA2024V4S3R22`

Enumerating Colored Permutations by the Parity of Descent Positions

Qiongqiong Pan†, Chao Xu‡, and Jiang Zeng‡

† *College of Mathematics and Physics, Wenzhou University*
Wenzhou 325035, PR China
qpan@math.univ-lyon1.fr

‡ *Universite Claude Bernard Lyon 1, ICJ UMR5208, CNRS, Centrale Lyon, INSA Lyon, Université Jean Monnet*
69622, Villeurbanne Cedex, France
xu@math.univ-lyon1.fr, zeng@math.univ-lyon1.fr

Received: September 16, 2023, **Accepted**: March 14, 2024, **Published**: April 5, 2024

Abstract: Motivated by recent works on the enumeration of Coxeter groups by the parity of descent positions, we prove a formula for the generating function of the vector statistic $(\mathrm{odes}_G, \mathrm{edes}_G, \mathrm{col}_G, \ell_G)$ over the group of colored permutations $G(r,n)$. Here odes_G, edes_G, col_G and ℓ_G denote the number of odd descent positions, even descent positions, colors, and length of colored permutation, respectively. This generalises and unifies several known results over Coxeter groups of type A and B. In particular, a special case of our formula permits to evaluate the signed alternating descent polynomials over $G(r,n)$ by the usual Eulerian polynomials, which extends Dey and Sivasubramanian's recent results in the special cases when $r = 1, 2$.

Keywords: Euler-Mahonian polynomial; Colored permutation; Descent; Signed alternating descent; Inversion
2020 Mathematics Subject Classification: 05A15; 05A05; 05A19

1. Introduction

For any positive integer n, let $\mathfrak{S}_n$ be the symmetric group of permutations of $[n] := \{1, 2, \ldots, n\}$. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, an index i $(1 \le i \le n-1)$ is a *descent* (respectively, *ascent*) of σ if $\sigma_i > \sigma_{i+1}$ (respectively, $\sigma_i < \sigma_{i+1}$). The number of descents (respectively, ascents) of σ are denoted by $\mathrm{des}(\sigma)$ (respectively, $\mathrm{asc}(\sigma)$). A descent i of σ is an *odd descent* (respectively, *even descent*) if i is *odd* (respectively, *even*). Let $\mathrm{odes}(\sigma)$ (respectively, $\mathrm{edes}(\sigma)$) denote the number of odd (respectively, even) descents of σ. Similarly, we define the number of even (respectively, odd) ascents by easc (respectively, oasc). The *inversion number* (inv) of σ is the number of pairs $(i,j) \in [n] \times [n]$ such that $\sigma_i > \sigma_j$ and $i < j$.

It is well known that the enumerative polynomial of permutations of $[n]$ by descents is the Eulerian polynomial $A_n(x)$, which can also be defined by the exponential generating function [15, 19, Chap.1]

$$\sum_{n\ge 0} A_n(x)\frac{t^n}{n!} = \frac{x-1}{x-\exp((x-1)t)}. \tag{1.1}$$

In 1973 Carlitz and Scoville [7] enumerated permutations according to the parity of both descents and ascents. Recently, Pan and Zeng considered the problem of enumerating permutations by the vector statistic (easc, oasc, edes, odes, inv) and established the exponential generating function [14, Theorem 1.1]. The following is one of the four equivalent forms of their formula [14, Eq. (1.6)].

Theorem A (Pan and Zeng)**.** *Let* $M = \sqrt{(1-x)(1-y)}$. *We have*

$$\begin{aligned}\sum_{n\ge 1} \frac{t^n}{[n]_q!} \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{odes}(\sigma)} y^{\mathrm{edes}(\sigma)} q^{\mathrm{inv}(\sigma)} \\ = \frac{(1+x)\cosh(Mt;q) + M\sinh(Mt;q) - x(\cosh^2(Mt;q) - \sinh^2(Mt;q)) - 1}{1-(x+y)\cosh(Mt;q) + xy(\cosh^2(Mt;q) - \sinh^2(Mt;q))},\end{aligned} \tag{1.2}$$

where

$$\cosh(t;q)=\sum_{n\geq 0}\frac{t^{2n}}{[2n]_q!},\quad \sinh(t;q)=\sum_{n\geq 1}\frac{t^{2n-1}}{[2n-1]_q!} \tag{1.3}$$

with $[0]_q!=1$ *and* $[n]_q!=\prod_{i=1}^{n}(1+q+\cdots+q^{i-1})$ *for* $n\geq 1$.

As shown in [14], Formula (1.2) is a q-analogue of Carlitz-Scoville's formula [7, Theorem 3.1] and encompasses both Stanley's formula for the bi-statistic (des, inv) [18] and Chebikin's formula for alternating descent polynomials [8]. In a follow-up, among other things, Dey, Shankar, and Sivasubramanian [10, Theorems 1.5 and 1.8] established analog formulas of Theorem A for types B and D Coxeter groups. The following is one of their type B formulas [10, Theorems 1.5], which is also a q-analogue of a formula due to Pan-Zeng [14, Theorem 1.4].

Theorem B (Dey, Shankar, and Sivasubramanian)**.** *Let* $M=\sqrt{(1-x)(1-y)}$. *We have*

$$\begin{aligned}\sum_{n\geq 0}\frac{t^n}{(-q;q)_n[n]_q!}&\sum_{\sigma\in\mathcal{B}_n}x^{\mathrm{odes}_B(\sigma)}y^{\mathrm{edes}_B(\sigma)}q^{\mathrm{inv}_B(\sigma)}\\ &=\frac{(1-y)\big((1-x\cosh(Mu;q))\cosh_B(Mt;q)+x\sinh(Mt;q)\sinh_B(Mt;q)\big)}{1-(x+y)\cosh(Mt;q)+xy\exp(Mt;q)\exp(-Mt;q)}\\ &\qquad+\frac{M\big((1-y\cosh(Mt;q))\sinh_B(Mt;q)+y\sinh(Mt;q)\cosh_B(Mt;q)\big)}{1-(x+y)\cosh(Mt;q)+xy\exp(Mt;q)\exp(-Mt;q)},\end{aligned}$$

where

$$\cosh_B(t;q)=\sum_{n\geq 0}\frac{t^{2n}}{(-q;q)_{2n}[2n]_q!},\quad \sinh_B(t;q)=\sum_{n\geq 1}\frac{t^{2n-1}}{(-q;q)_{2n-1}[2n-1]_q!} \tag{1.4}$$

and odes_B *(respectively,* edes_B *and* inv_B*) denotes the number of odd descent positions (respectively, even descent positions and inversions) over type B permutations in* $\mathcal{B}_n$*, see Remark 2.3.*

In this paper, as a natural continuation of the work done in [10,14], we provide a formula for the generating function of the vector statistic $(\mathrm{odes}_G, \mathrm{edes}_G, \mathrm{col}_G, \ell_G)$ over the group of colored permutations $G(r,n)$, which permits to put Theorems A and B under the same umbrella. Here odes_G (respectively, edes_G, col_G and ℓ_G) denotes the number of odd descent positions (respectively, even descent positions, colors, and length) of permutation. We shall achieve our goal by extending Dey, Shankar, and Sivasubramanian's arguments in type B Coxeter groups [10].

The study of signed Eulerian polynomials was initiated by Loday, Désarménien, Foata, Wachs, and Reiner in the 1990's and has attracted great attention of researchers [9,13,16,17,21], with two recent references being [11,12]. The alternating descent statistic on permutations was introduced by Chebikin [8] as a variant of the descent statistic. Dey and Sivasubramanian [11] further studied the signed enumeration of alternating descents for classical Weyl groups. Applying our generating function formula for colored permutations (see Theorem 2.1), we shall evaluate the signed alternating descent polynomials over $G(r,n)$ by the usual Eulerian polynomials. The resulting formula (see Theorem 2.2) extends Dey and Sivasubramanian's recent results in the special cases when $r=1,2$.

The rest of this paper is organized as follows. We introduce definitions and main results, i.e., Theorem 2.1 and Theorem 2.2 in Section 2 and prove them in Section 3 and Section 4, respectively.

2. Definitions and main results

For positive integers m and n with $m\leq n$, we denote by $[m,n]$ the set $\{m,m+1,\ldots,n\}$. The cardinality of a set A will be denoted by $|A|$. For $r,n\in\mathbb{P}$, we define the *wreath product* $\mathbb{Z}_r\wr\mathfrak{S}_n$ of $\mathbb{Z}_r$ by $\mathfrak{S}_n$, i.e., the group of colored permutations $G(r,n)$, by

$$G(r,n):=\{(c_1,\ldots,c_n;\sigma)\mid c_i\in[0,r-1],\sigma=\sigma_1\cdots\sigma_n\in\mathfrak{S}_n\}. \tag{2.1}$$

The product in $G(r,n)$ is defined by

$$(c;\sigma)\cdot(c';\tau):=(c_1+c'_{\tau_1^{-1}},\ldots,c_n+c'_{\tau_n^{-1}};\,\sigma\circ\tau),$$

where the addition $+$ is in $\mathbb{Z}_r$ and composition $\circ$ in $\mathfrak{S}_n$. The entry c_i is called the color of the σ_i, for $1\leq i\leq n$. The elements of $\mathbb{Z}_r\wr\mathfrak{S}_n$ can be viewed as r-colored permutations, see Steingrímsson [20] and Bagno et al. [2]. We will represent an element $\gamma\in G(r,n)$ in *window notation* as

$$\gamma=[\gamma(1),\ldots,\gamma(n)]=[\sigma_1^{c_1},\ldots,\sigma_n^{c_n}],$$

and call σ_i the *absolute value* of $\gamma(i)$, denoted by $|\gamma(i)|$. For $\gamma \in G(r,n)$, we define the color set of γ by

$$\mathrm{Col}_G(\gamma) := \{i \in [n] : c_i \neq 0\}$$

and its size will be denoted by $\mathrm{col}_G(\gamma)$. If $c_i = 0$, it will be omitted in the window notation. For example, $\gamma = [3^1, 2, 1^3, 4^2, 6^2, 5^1] \in G(5,6)$. The group $G(r,n)$ is generated by $S_G := \{s_0, s_1, \ldots, s_{n-1}\}$, where for $i \in [n-1]$

$$s_i := [1, \ldots, i-1, i+1, i, i+2, \ldots, n] \ \text{ and } \ s_0 := [1^1, 2, \ldots, n],$$

with relations given by the Dynkin-like diagram (see Figure 1).

Figure 1: The Dynkin-like diagram of $G(r,n)$.

The *length* of $\gamma \in G(r,n)$ is the minimal number of generators in S_G whose product is γ,

$$\ell_G(\gamma) := \min\{r \in \mathbb{N} : \gamma = s_{i_1} \cdots s_{i_r}, \text{ for some } s_{i_j} \in S_G\}. \tag{2.2}$$

Thus, the *descent set* of $\gamma \in G(r,n)$ is

$$\mathrm{Des}_G(\gamma) := \{s \in S_G : \ell_G(\gamma s) < \ell_G(\gamma)\}$$

and its size is denoted by $\mathrm{des}_G(\gamma)$. To give a combinatorial description of ℓ_G and $\mathrm{Des}_G(\gamma)$ we use the following linear order

$$n^{r-1} < \cdots < n^1 < \cdots < 1^{r-1} < \cdots < 1^1 < 0 < 1 < \cdots < n \tag{2.3}$$

on the set $\{0, 1, \ldots, n, 1^1, \ldots, n^1, \ldots, 1^{r-1}, \ldots, n^{r-1}\}$ of colored integers (see [4]). If $\gamma = [\gamma(1), \ldots, \gamma(n)] \in G(r,n)$, the length of γ is then characterized by (see [3,16,20])

$$\ell_G(\gamma) = \mathrm{inv}(\gamma) + \sum_{c_i \neq 0} \left(|\gamma(i)| + c_i - 1\right), \tag{2.4}$$

where the inversion number is defined by

$$\mathrm{inv}(\gamma) = |\{(i,j) : 1 \leq i < j \leq n, \text{ and } \gamma(i) > \gamma(j)\}|.$$

The descent set of $\gamma \in G(r,n)$ has the following alternate definition

$$\mathrm{Des}_G(\gamma) = \{i \in [0, n-1] : \gamma(i) > \gamma(i+1)\},$$

where $\gamma(0) := 0$. The number of odd (respectively, even) descent positions of γ is denoted by $\mathrm{odes}_G(\gamma)$ (respectively, $\mathrm{edes}_G(\gamma)$). The ascent set of $\gamma \in G(r,n)$ is defined by

$$\mathrm{Asc}_G(\gamma) := \{i \in [0, n-1] : \gamma(i) < \gamma(i+1)\}.$$

Similarly, we define the statistics $\mathrm{asc}_G(\gamma)$, $\mathrm{easc}_G(\gamma)$ and $\mathrm{oasc}_G(\gamma)$. For any $\gamma \in G(r,n)$ the following identities hold

$$\begin{aligned} \mathrm{edes}_G(\gamma) + \mathrm{easc}_G(\gamma) &= \lfloor (n+1)/2 \rfloor, \\ \mathrm{odes}_G(\gamma) + \mathrm{oasc}_G(\gamma) &= \lfloor n/2 \rfloor. \end{aligned} \tag{2.5}$$

Note that $0 \in \mathrm{Des}_G(\gamma)$ if and only if $c_1 > 0$.

Example 2.1. If $\gamma = [3^1, 2, 1^3, 4^2, 6^2, 5^1] \in G(5,6)$, then $\mathrm{Des}_G(\gamma) = \{0, 2, 3, 4\}$, $\mathrm{des}_G(\gamma) = 4$, $\mathrm{inv}(\gamma) = 12$, $\sum_{c_i \neq 0} |\gamma(i)| = 19$, $\sum_{c_i \neq 0} (c_i - 1) = 4$, $\ell_G(\gamma) = 35$ and $\mathrm{col}_G(\gamma) = 5$.

For $n, r \in \mathbb{P}$, define the polynomials

$$\begin{aligned} G_{(r,n)}(a,q) &:= \sum_{\gamma \in G(r,n)} a^{\mathrm{col}_G(\gamma)} q^{\ell_G(\gamma)}, \\ G_{(r,n)}(x,y,a,q) &:= \sum_{\gamma \in G(r,n)} x^{\mathrm{odes}_G(\gamma)} y^{\mathrm{edes}_G(\gamma)} a^{\mathrm{col}_G(\gamma)} q^{\ell_G(\gamma)}, \end{aligned} \tag{2.6}$$

and the standard q-factorial notation

$$(a;q)_n := \begin{cases} (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1, \\ 1, & \text{if } n = 0. \end{cases}$$

By an argument similar to the proof of (4.13) in [4] we can prove the following result.

Proposition 2.1. *For $n, r \in \mathbb{P}$, we have*

$$G_{(r,n)}(a,q) = [n]_q!(-aq[r-1]_q;q)_n.$$

Remark 2.1. The case when $r = 1$ is a classical result about the inversion number over $\mathfrak{S}_n$ and the case when $r = 2$ is the counterpart in type B due to Brenti [6, Proposition 3.3].

For convenience, we use the convention

$$G_{(r,0)}(a,q) = 1, \qquad G_{(r,0)}(x,y,a,q) = 1.$$

The even and odd index generating functions of $G_{(r,n)}(x,y,a,q)$ are defined by

$$H_0^r := \sum_{n\geq 0} G_{(r,2n)}(x,y,a,q)\frac{t^{2n}}{G_{(r,2n)}(a,q)}, \tag{2.7a}$$

$$H_1^r := \sum_{n\geq 0} G_{(r,2n+1)}(x,y,a,q)\frac{t^{2n+1}}{G_{(r,2n+1)}(a,q)}. \tag{2.7b}$$

Define the q-analogue of exponential series over the wreath product

$$\exp_{G(r)}(t;a,q) := \sum_{n\geq 0} \frac{t^n}{G_{(r,n)}(a,q)}, \tag{2.8}$$

and the corresponding hyperbolic cosine and sine series

$$\cosh_{G(r)}(t;a,q) = \frac{\exp_{G(r)}(t;a,q) + \exp_{G(r)}(-t;a,q)}{2}, \tag{2.9a}$$

$$\sinh_{G(r)}(t;a,q) = \frac{\exp_{G(r)}(t;a,q) - \exp_{G(r)}(-t;a,q)}{2}. \tag{2.9b}$$

Remark 2.2. *When $r = 1, 2$, we recover the classic q-analogue of hyperbolic series in* (1.3) *and* (1.4).

The following is our first main result.

Theorem 2.1. *For any $r \in \mathbb{P}$, we have*

$$H_0^r = \frac{(1-y)\Big(\big(1-x\cosh(Mt;q)\big)\cosh_{G(r)}(Mt;a,q) + x\sinh(Mt;q)\sinh_{G(r)}(Mt;a,q)\Big)}{1-(x+y)\cosh(Mt;q) + xy\exp(Mt;q)\exp(-Mt;q)}, \tag{2.10a}$$

$$H_1^r = \frac{M\Big(\big(1-y\cosh(Mt;q)\big)\sinh_{G(r)}(Mt;a,q) + y\sinh(Mt;q)\cosh_{G(r)}(Mt;a,q)\Big)}{1-(x+y)\cosh(Mt;q) + xy\exp(Mt;q)\exp(-Mt;q)}, \tag{2.10b}$$

where $M = \sqrt{(1-x)(1-y)}$.

Remark 2.3. When $(a,n,r) = (1,n,1)$ or $(0,n,2)$ we recover a formula equivalent to Theorem A. When $r = 2$, the length $\ell_G(\gamma)$ coincides with the length in the Coxeter (hyperoctahedral) group $B_n = G(2,n)$ (see [5, 6]), namely

$$\mathrm{inv}_B(\gamma) := \mathrm{inv}(\gamma) + \sum_{c_i\neq 0} |\gamma(i)|.$$

Hence, when $(a,n,r) = (1,n,2)$, replacing odes_G (respectively, edes_G) by odes_B (respectively, edes_B), we recover Theorem B.

Remark 2.4. As observed in [14] for permutations of types A and B, among the four statistics easc, oasc, edes, odes over permutations, it is sufficient to consider two of them. This is still valid for colored permutations. Indeed, by (2.5), the distribution of the quadruple statistics $(\mathrm{easc}_G, \mathrm{oasc}_G, \mathrm{edes}_G, \mathrm{odes}_G)$ is completely determined by any pair of the statistics in $\{\mathrm{odes}_G, \mathrm{oasc}_G\} \times \{\mathrm{edes}_G, \mathrm{easc}_G\}$, in particular [*],

$$\sum_{\gamma\in G(r,n)} x_0^{\mathrm{easc}_G(\gamma)} x_1^{\mathrm{oasc}_G(\gamma)} y_0^{\mathrm{edes}_G(\gamma)} y_1^{\mathrm{odes}_G(\gamma)} a^{\mathrm{col}_G(\gamma)} q^{\ell_G(\gamma)} = x_0^{\lfloor (n+1)/2\rfloor} x_1^{\lfloor n/2\rfloor} G_{(r,n)}\left(\frac{y_1}{x_1}, \frac{y_0}{x_0}, a, q\right). \tag{2.11}$$

---

[*] Here we count an ascent at the beginning as position 0, which is not counted in [14].

When $r=2$, $x=y$, $q=1$, substituting $t \leftarrow 2t$, and adding odd and even indexed generating functions, we obtain Brenti's Theorem 3.4(iv) in [6],

$$\sum_{n\geq 0} G_{(2,n)}(x,x,a,1)\frac{t^n}{n!}=\frac{(1-x)\exp(t(1-x))}{1-x\exp(t(1-x)(1+a))}.$$

When $x=y$, adding odd and even indexed generating functions, we obtain a formula of Reiner [17, Corollary 4.4, formula (2)],

$$\sum_{n\geq 0} G_{(2,n)}(x,x,a,q)\frac{t^n}{(-aq,q)_n[n]_q!}=\frac{(1-x)\exp_{G(2)}(t(1-x);a,q)}{1-x\exp(t(1-x);q)}.$$

Given a permutation $\gamma\in G(r,n)$, an index $i\in\{0,1,\dots,n-1\}$ is called an *alternating descent* if i is an *odd descent* or *even ascent*. Let $\widehat{Des}_G(\gamma)$ be the set of alternating descents of γ, i.e.,

$$\widehat{Des}_G(\gamma)=\{2i:\gamma(2i)<\gamma(2i+1)\}\cup\{2i+1:\gamma(2i+1)>\gamma(2i+2)\},$$

and let its cardinality be denoted by $\widehat{des}_G(\gamma)$. We define the q-alternating descent polynomial over $G(r,n)$ is defined by

$$\mathrm{Alt}_n^{G(r)}(x,q):=\sum_{\gamma\in G(r,n)} x^{\widehat{des}_G(\gamma)}q^{\ell_G(\gamma)}, \tag{2.12}$$

where ℓ_G is the length function (2.4). As an application of Theorem 2.1 we shall evaluate $\mathrm{Alt}_n^{G(r)}(x,q)$ when $q=-1$. The following is our second main result.

Theorem 2.2. *For integer $n\geq 1$ the following identities hold.*

1. *If r is a positive even integer, then*

$$\mathrm{Alt}_n^{G(r)}(x,-1)=(-1)^{\lfloor (n+1)/2\rfloor}(1-x)^n. \tag{2.13}$$

2. *If r is a positive odd integer, then $\mathrm{Alt}_1^{G(r)}(x,-1)=x$ and for $n\geq 2$,*

$$\mathrm{Alt}_n^{G(r)}(x,-1)=\begin{cases} x(1-x)^m A_m(x), & \text{if } n=2m\ (m\in\mathbb{N}^*);\\ \frac{2x^2}{1+x}(1-x)^{2m}A_{2m}(x), & \text{if } n=4m+1\ (m\in\mathbb{N}^*);\\ 0, & \text{if } n=4m+3\ (m\in\mathbb{N}).\end{cases} \tag{2.14}$$

We make the following remarks.

(i) It is known [15, Chapter 4] that Eulerian polynomial $A_n(x):=\sum_{i=0}^{n-1}A_{n,i}x^i$ is monic, of degree $n-1$ and palindromic, so $A_{2n}(x)=\sum_{i=0}^{n-1}A_{2n,i}x^i(1+x^{2n-2i-1})$, which is clearly divisible by $1+x$.

(ii) Formula (2.14) reduces to [11, Theorem 2] when $r=1$ † and (2.13) reduces to [11, Theorem 13] when $r=2$.

(iii) Our proof of the above theorem is à la Désarménien and Foata [9] using generating functions and q-calculus. When $r=1,2$, Dey and Sivasubramanian [11] gave a different proof. It would be interesting to find a combinatorial proof à la Wachs [21].

3. Counting colored permutations by the parity of descents

The aim of this section is to prove Theorem 2.1. Throughout this section, we assume that n and r are positive integers. For $0\leq m\leq n$, let $\binom{[n]}{m}$ be the set of m-subsets of $[n]$, that is, $\binom{[n]}{m}:=\{A\subseteq[n]:|A|=m\}$ and let $[n]^r:=\{i^{c_i}:i\in[n],c_i\in[0,r-1]\}$ be the set of colored integers. The set of m-subsets A^r of $[n]^r$ such that $A\in\binom{[n]}{m}$ is denoted by $\binom{[n]}{m}^r$.

Let A be a finite ordered set. We write $A=\{a_1,\dots,a_m\}_<$ to mean $a_1<\dots<a_m$ and denote by $[A]:=[a_1,\dots,a_m]$ the increasing sequence of its elements. In particular, if $A^r=\{a_1^{c_1},a_2^{c_2},\dots,a_m^{c_m}\}_<\in\binom{[n]}{m}^r$, then $[A^r]$ is the increasing permutation of A^r by the linear order (2.3), that is $[A^r]=[a_1^{c_1},a_2^{c_2},\dots,a_m^{c_m}]$.

---

† When $r=1$, the position 0 is not counted as an even ascent in [11, Theorem 2].

Observation 3.1. *Let $\gamma = [\sigma_1^{c_1}, \ldots, \sigma_n^{c_n}] \in G(r,n)$. If $i, j \in [n]$ with $i \neq j$, then $\gamma(i) < \gamma(j)$ if and only if one of the following conditions hold,*

1. *$c_i = 0$, $c_j = 0$ and $|\gamma(i)| < |\gamma(j)|$;*
2. *$c_i > 0$, $c_j > 0$ and $|\gamma(i)| > |\gamma(j)|$;*
3. *$c_i > 0$, $c_j = 0$ and $|\gamma(i)| < |\gamma(j)|$ or $|\gamma(i)| > |\gamma(j)|$.*

For $\gamma = [\gamma(1), \ldots, \gamma(n)] = [\sigma_1^{c_1}, \ldots, \sigma_n^{c_n}] \in G(r,n)$, let

$$\operatorname{csum}(\gamma) = \sum_{c_i \neq 0} c_i$$

and

$$\operatorname{inv}_c(\gamma) = \sum_{1 \le i < j \le n} |\{(i,j) : \gamma_c(i) > \gamma(j)\}|,$$

where $\gamma_c(i)$ is defined by

$$\gamma_c(i) = \begin{cases} \sigma_i, & \text{if } c_i \neq 0; \\ \sigma_i^1, & \text{if } c_i = 0. \end{cases} \tag{3.1}$$

We now give an alternative characterization of the length function ℓ_G in (2.2) and (2.4).

Lemma 3.1. *For $\gamma = [\gamma(1), \ldots, \gamma(n)] = [\sigma_1^{c_1}, \ldots, \sigma_n^{c_n}] \in G(r,n)$, we have*

$$\ell_G(\gamma) = \operatorname{inv}(\gamma) + \operatorname{inv}_c(\gamma) + \operatorname{csum}(\gamma).$$

Proof. By definition (3.1) and Observation 3.1, we have $(c_i, c_j) \neq (0,0)$ if $\gamma_c(i) > \gamma(j)$. Hence

$$\begin{aligned} \operatorname{inv}_c(\gamma) &= \sum_{1 \le i < j \le n} |\{(i,j) : \gamma_c(i) > \gamma(j)\}| \\ &= \sum_{i<j} |\{i < j : |\gamma(i)| > |\gamma(j)|,\, c_i \neq 0\}| + \sum_{i<j} |\{j > i : |\gamma(i)| < |\gamma(j)|,\, c_j \neq 0\}| \\ &= \sum_{c_k \neq 0} (\sigma_k - 1). \end{aligned}$$

Comparing with (2.4), we are done. ☐

For $0 \le m \le n$, let A^r and B^r be disjoint subsets of $[n]^r$ with $|A^r| = m$ and $|B^r| = n - m$. If π (respectively, σ) is a permutation of A^r (respectively, B^r), in other words, A^r (respectively, B^r) is the set of letters in π (respectively, σ), we define the *between-permutation inversion (respectively, c-inversion)* as in the following:

$$\operatorname{inv}(\pi, \sigma) = |\{(\pi(i), \sigma(j)) \in A^r \times B^r : \pi(i) > \sigma(j), i \in [m] \text{ and } j \in [n-m]\}|; \tag{3.2a}$$

$$\operatorname{inv}_c(\pi, \sigma) = |\{(\pi(i), \sigma(j)) \in A^r \times B^r : \pi_c(i) > \sigma(j), i \in [m] \text{ and } j \in [n-m]\}|, \tag{3.2b}$$

where $\pi_c(i)$ is defined by (3.1).

For $0 \le i \le n-1$, let $G_i(r,n)$ be the set of colored permutations in $G(r,n)$ with the last $n-i$ elements being increasing from left-to-right, that is,

$$G_i(r,n) := \{\gamma \in G(r,n) : \gamma(i+1) < \gamma(i+2) < \cdots < \gamma(n-1) < \gamma(n)\}. \tag{3.3}$$

Note that $G_{n-1}(r,n) = G(r,n)$ and $|G_i(r,n)| = r^n \binom{n}{i} i!$. Define $G(r,0) = \{\varepsilon\}$, where ε is the empty word. The concatenation operator $*$ of two words u and v is defined by $u * v := uv$ with $\varepsilon * u = u * \varepsilon = u$ for any word u. For $0 \le i \le n-1$, let $\gamma = [\sigma_1^{c_1}, \sigma_2^{c_2}, \ldots, \sigma_i^{c_i}] \in G(r,i)$, $A^r \in \binom{[n]}{n-i}^r$ with $[n] \setminus A = \{s_1, s_2, \ldots, s_i\}_<$ and

$$\gamma|_{[n]\setminus A} := [s_{\sigma_1}^{c_1}, s_{\sigma_2}^{c_2}, \ldots, s_{\sigma_i}^{c_i}]. \tag{3.4}$$

We define

$$f(\gamma, A^r) = \gamma|_{[n]\setminus A} * [A^r]. \tag{3.5}$$

It is easy to see that the mapping $f : G(r,i) \times \binom{[n]}{n-i}^r \to G_i(r,n)$ is a bijection.

Lemma 3.2. *For* $0 \le i \le n-1$, *let* $\gamma = [\sigma_1^{c_1}, \sigma_2^{c_2}, \dots, \sigma_i^{c_i}] \in G(r,i)$, $A^r \in \binom{[n]}{n-i}^r$ *and* $[n] \setminus A = \{s_1, s_2, \dots, s_i\}_<$. *The mapping* $f : G(r,i) \times \binom{[n]}{n-i}^r \to G_i(r,n)$ *satisfies*

$$\operatorname{csum}(f(\gamma, A^r)) = \operatorname{csum}(\gamma) + \operatorname{csum}([s_1, s_2, \dots, s_i] * [A^r]), \tag{3.6a}$$

$$\operatorname{col}(f(\gamma, A^r)) = \operatorname{col}(\gamma) + \operatorname{col}([s_1, s_2, \dots, s_i] * [A^r]), \tag{3.6b}$$

$$\ell_G(f(\gamma, A^r)) = \ell_G(\gamma) + \ell_G([s_1, s_2, \dots, s_i] * [A^r]). \tag{3.6c}$$

Proof. By definition (3.5), we have

$$f(\gamma, A^r) = \gamma|_{[n]\setminus A} * [A^r] = [s_{\sigma_1}^{c_1}, s_{\sigma_2}^{c_2}, \dots, s_{\sigma_i}^{c_i}] * [A^r]. \tag{3.7}$$

So, it is easy to verify the first two identities (3.6a) and (3.6b). By Lemma 3.1 we have

$$\ell_G(f(\gamma, A^r)) = \operatorname{inv}(f(\gamma, A^r)) + \operatorname{inv}_c(f(\gamma, A^r)) + \operatorname{csum}(f(\gamma, A^r)).$$

The factorisation (3.7) of $f(\gamma, A^r)$ implies that

$$\operatorname{inv}(f(\gamma, A^r)) = \operatorname{inv}(\gamma|_{[n]\setminus A}) + \operatorname{inv}([A^r]) + \operatorname{inv}(\gamma|_{[n]\setminus A}, [A^r]),$$
$$\operatorname{inv}_c(f(\gamma, A^r)) = \operatorname{inv}_c(\gamma|_{[n]\setminus A}) + \operatorname{inv}_c([A^r]) + \operatorname{inv}_c(\gamma|_{[n]\setminus A}, [A^r]).$$

Note that γ acts on an ordered set (see (3.4)) preserving the inversion (respectively, c-inversion) number, i.e.,

$$\operatorname{inv}(\gamma|_{[n]\setminus A}) = \operatorname{inv}(\gamma) \quad \text{and} \quad \operatorname{inv}_c(\gamma|_{[n]\setminus A}) = \operatorname{inv}_c(\gamma).$$

By definition (3.2), we have

$$\operatorname{inv}(\gamma|_{[n]\setminus A}, [A^r]) + \operatorname{inv}_c(\gamma|_{[n]\setminus A}, [A^r]) = \operatorname{inv}([s_1, s_2, \dots, s_i], [A^r]) + \operatorname{inv}_c([s_1, s_2, \dots, s_i], [A^r]),$$

which is independent from γ. Combining the above results with Lemma 3.1 results in

$$\ell_G(f(\gamma, A^r)) - \ell_G(\gamma) - \ell_G([s_1, s_2, \dots, s_i] * [A^r]) = 0,$$

which proves (3.6c). ☐

Example 3.1. *Let* $n = 8$, $r = 4$ *and* $i = 4$. *If* $\gamma = [2^1, 4, 3^1, 1^3] \in G(4,4)$, $A^4 = \{6^1, 3^2, 1^1, 5\}_< \in \binom{[8]}{4}^4$, *then* $[8] \setminus A = \{s_1, \dots, s_4\}_< = \{2, 4, 7, 8\}$, $\gamma|_{[8]\setminus A} = [4^1, 8, 7^1, 2^3]$, $[A^4] = [6^1, 3^2, 1^1, 5]$ *and*

$$\gamma' := [s_1, \dots, s_4] * [A^4] = [2,4,7,8] * [6^1, 3^2, 1^1, 5] = [2,4,7,8,6^1,3^2,1^1,5],$$

and $f(\gamma, A^4) = [4^1, 8, 7^1, 2^3, 6^1, 3^2, 1^1, 5]$. *By* (2.4) *we have*

$$\begin{aligned}
\ell_G(\gamma) &= \operatorname{inv}(\gamma) + \sum_{c_i \neq 0} (|\gamma(i)| + c_i - 1) \\
&= |\{(2^1, 3^1),\ (4, 3^1),\ (4, 1^3)\}| + (2+1-1) + (3+1-1) + (1+3-1) = 11, \\
\ell_G(\gamma') &= \operatorname{inv}(\gamma') + \sum_{c_i \neq 0} (|\gamma'(i)| + c_i - 1) \\
&= |\{(2,6^1),(2,3^2),(2,1^1),(4,6^1),(4,3^2),(4,1^1),(7,6^1),(7,3^2),(7,1^1), \\
&\qquad (7,5),(8,6^1),(8,3^2),(8,1^1),(8,5)\}| + (6+1-1) + (3+2-1) + (1+1-1) = 25.
\end{aligned}$$

In the same manner, we obtain $\ell_G(f(\gamma, A^4)) = 36$.

By convention, for any $n \in \mathbb{P}$ we denote by $\mathbf{1}$ the identity permutation in $G(r,n)$. Thus

$$f(\mathbf{1}, A^r) = [s_1, s_2, \dots, s_i] * [A^r]. \tag{3.9}$$

The q-binomial coefficients are defined by

$$\binom{n}{m}_q := \frac{[n]_q!}{[m]_q![n-m]_q!} \quad (0 \le m \le n). \tag{3.10}$$

Lemma 3.3. *Let* $0 \le i \le n-1$. *For any* $\gamma \in G(r,i)$, *we have*

$$\sum_{A^r \in \binom{[n]}{n-i}^r} a^{\operatorname{col}(f(\gamma,A^r))} q^{\ell_G(f(\gamma,A^r))} = a^{\operatorname{col}(\gamma)} q^{\ell_G(\gamma)} \binom{n}{n-i}_q (-aq^{i+1}[r-1]_q; q)_{n-i}. \tag{3.11}$$

Proof. By (3.6b) and (3.6c) in Lemma 3.2 and (3.9), it suffices to prove the $\gamma = \mathbf{1}$ case of (3.11), which is equivalent to the following identity

$$[i]_q!(-aq[r-1]_q;q)_i \times \sum_{A^r \in \binom{[n]}{n-i}^r} a^{\mathrm{col}(f(\mathbf{1},A^r))} q^{\ell_G(f(\mathbf{1},A^r))} \times [n-i]_q! = [n]_q!(-aq[r-1]_q;q)_n. \tag{3.12}$$

To this end, for any $\tau = \tau_1\tau_2\cdots\tau_{n-i} \in \mathfrak{S}_{n-i}$, we construct a mapping $f_\tau : G(r,i) \times \binom{[n]}{n-i}^r \to G(r,n)$ by

$$f_\tau(\gamma, A^r) := \gamma|_{[n]\setminus A} * \tau[A^r], \tag{3.13a}$$

where $[n] \setminus A = \{s_1, s_2, \dots, s_i\}_<$ and

$$\gamma = [\sigma_1^{c_1}, \sigma_2^{c_2}, \dots, \sigma_i^{c_i}], \qquad A^r = \{a_1^{c'_1}, a_2^{c'_2}, \dots, a_{n-i}^{c'_{n-i}}\}_<, \tag{3.13b}$$

$$\gamma|_{[n]\setminus A} = [s_{\sigma_1}^{c_1}, s_{\sigma_2}^{c_2}, \dots, s_{\sigma_i}^{c_i}], \qquad \tau[A^r] = [a_{\tau_1}^{c'_{\tau_1}}, a_{\tau_2}^{c'_{\tau_2}}, \dots, a_{\tau_{n-i}}^{c'_{\tau_{n-i}}}]. \tag{3.13c}$$

Note that

$$f(\gamma, A^r) = \gamma_{[n]\setminus A} * [A^r] = [s_{\sigma_1}^{c_1}, s_{\sigma_2}^{c_2}, \dots, s_{\sigma_i}^{c_i}] * [a_1^{c'_1}, a_2^{c'_2}, \dots, a_{n-i}^{c'_{n-i}}], \tag{3.14a}$$

$$f_\tau(\gamma, A^r) = \gamma|_{[n]\setminus A} * \tau[A^r] = [s_{\sigma_1}^{c_1}, s_{\sigma_2}^{c_2}, \dots, s_{\sigma_i}^{c_i}] * [a_{\tau_1}^{c'_{\tau_1}}, a_{\tau_2}^{c'_{\tau_2}}, \dots, a_{\tau_{n-i}}^{c'_{\tau_{n-i}}}]. \tag{3.14b}$$

It is clear that f_τ is a bijection. We show that f_τ satisfies the following properties:

$$\mathrm{col}\big(f(\mathbf{1}, A^r)\big) + \mathrm{col}(\gamma) = \mathrm{col}(f_\tau(\gamma, A^r)), \tag{3.15a}$$

$$\ell_G\big(f(\mathbf{1}, A^r)\big) + \ell_G(\gamma) + \mathrm{inv}(\tau) = \ell_G(f_\tau(\gamma, A^r)). \tag{3.15b}$$

By (3.9), (3.13), and (3.15a) is obvious. It remains to prove (3.15b). By definition (2.4) and (3.14), we have

$$\begin{aligned}\ell_G(f(\gamma, A^r)) + \mathrm{inv}(\tau) =& \,\mathrm{inv}(\gamma|_{[n]\setminus A}) + \mathrm{inv}([A^r]) + \mathrm{inv}(\gamma|_{[n]\setminus A}, [A^r]) + \mathrm{inv}(\tau)\\ &+ \sum_{c_j \neq 0} (s_{\sigma_j} + c_j - 1) + \sum_{c'_k \neq 0} (a_k + c'_k - 1),\end{aligned} \tag{3.16}$$

$$\begin{aligned}\ell_G(f_\tau(\gamma, A^r)) =& \,\mathrm{inv}(\gamma|_{[n]\setminus A}) + \mathrm{inv}(\tau[A^r]) + \mathrm{inv}(\gamma|_{[n]\setminus A}, \tau[A^r])\\ &+ \sum_{c_j \neq 0} (s_{\sigma_j} + c_j - 1) + \sum_{c'_{\tau_k} \neq 0} (a_{\tau_k} + c'_{\tau_k} - 1).\end{aligned} \tag{3.17}$$

We observe the following facts:

- $\mathrm{inv}(\tau) = 0$ and $\mathrm{inv}(\tau[A^r]) = \mathrm{inv}(\tau)$ because $[A^r]$ is an increasing word;
- $\ell_G\big(f(\mathbf{1}, A^r)\big) + \ell_G(\gamma) = \ell_G(f(\gamma, A^r))$, see (3.6c);
- $\mathrm{inv}(\gamma|_{[n]\setminus A}, [A^r]) = \mathrm{inv}(\gamma|_{[n]\setminus A}, \tau[A^r])$, see definition (3.2a).

From (3.16), (3.17) and the above facts, we derive (3.15b). Finally, combining (3.13a), (3.15) and Proposition 2.1, we prove (3.12). ☐

Example 3.2. *Let* $n = 9$, $r = 4$ *and* $i = 5$. *If* $\gamma = [4^1, 5, 1^2, 3^1, 2^3] \in G(4,5)$, $A^4 = \{6^1, 4^3, 2^1, 1\}_< \in \binom{[9]}{4}^4$ *and* $\tau = 3412 \in \mathfrak{S}_4$, *then* $[9] \setminus A = \{3, 5, 7, 8, 9\}$,

$$\gamma_{[9]\setminus A} = [8^1, 9, 3^2, 7^1, 5^3], \quad \tau[A^4] = [2^1, 1, 6^1, 4^3].$$

Hence $f_\tau(\gamma, A^4) = [8^1, 9, 3^2, 7^1, 5^3, 2^1, 1, 6^1, 4^3]$.

Recall the enumerative polynomials see (2.6),

$$G_{(r,n)}(x, y, a, q) = \sum_{\gamma \in G(r,n)} x^{\mathrm{odes}_G(\gamma)} y^{\mathrm{edes}_G(\gamma)} a^{\mathrm{col}(\gamma)} q^{\ell_G(\gamma)}.$$

For convenience, we define the weight

$$w(\gamma) = x^{\mathrm{odes}_G(\gamma)} y^{\mathrm{edes}_G(\gamma)} a^{\mathrm{col}(\gamma)} q^{\ell_G(\gamma)}.$$

By convention, we set

$$G_{-1}(r,n) = \{[1, 2, \dots, n]\}.$$

Lemma 3.4. *Let $n, r \in \mathbb{P}$ and $0 \le i \le n-1$. For $G_i(r,n)$ in (3.3), the following identities hold.*

1. *If i is odd,*

$$\sum_{\gamma \in G_i(r,n)} w(\gamma) = x\frac{G_{(r,i)}(x,y,a,q)G_{(r,n)}(a,q)}{G_{(r,i)}(a,q)[n-i]_q!} + (1-x)\sum_{\gamma \in G_{i-1}(r,n)} w(\gamma). \tag{3.18}$$

2. *If i is even,*

$$\sum_{\gamma \in G_i(r,n)} w(\gamma) = y\frac{G_{(r,i)}(x,y,a,q)G_{(r,n)}(a,q)}{G_{(r,i)}(a,q)[n-i]_q!} + (1-y)\sum_{\gamma \in G_{i-1}(r,n)} w(\gamma). \tag{3.19}$$

Proof. Let i be a positive odd integer. Multiplying the two sides of (3.11) by $x^{\mathrm{odes}_G(\gamma)}y^{\mathrm{edes}_G(\gamma)}$ and summing over $\gamma \in G(r,i)$ we obtain the identity

$$\begin{aligned}
&\sum_{\gamma \in G(r,i)} \sum_{A^r \in \binom{[n]}{n-i}^r} x^{\mathrm{odes}_G(\gamma)}y^{\mathrm{edes}_G(\gamma)}a^{\mathrm{col}(f(\gamma,A^r))}q^{\ell_G(f(\gamma,A^r))}\\
&\qquad = G_{(r,i)}(x,y,a,q)\binom{n}{n-i}_q(-aq^{i+1}[r-1]_q;q)_{n-i}\\
&\qquad = G_{(r,i)}(x,y,a,q)\frac{G_{(r,n)}(a,q)}{G_{(r,i)}(a,q)[n-i]_q!},
\end{aligned} \tag{3.20}$$

where the last equality follows from Proposition 2.1.

Let $F_i(r,n)$ denote the subset of colored permutations $\gamma = [\gamma(1), \dots, \gamma(n)]$ in $G_i(r,n)$ such that $\gamma(i) > \gamma(i+1) < \gamma(i+2) < \cdots < \gamma(n)$. By definition (3.3), we have $F_i(r,n) = G_i(r,n) \setminus G_{i-1}(r,n)$. For $(\gamma, A^r) \in G(r,i) \times \binom{[n]}{n-i}^r$, let $\gamma' = f(\gamma, A^r)$, see (3.5). Then $f : G(r,i) \times \binom{[n]}{n-i}^r \to G_i(r,n)$ is a bijection satisfying the following properties:

- if $f(\gamma, A^r) = \gamma' \in F_i(r,n)$, then i is an odd descent of γ' (but i is clearly not a descent of γ), thus $\mathrm{odes}_G(\gamma') = \mathrm{odes}_G(\gamma) + 1$ and $\mathrm{edes}_G(\gamma') = \mathrm{edes}_G(\gamma)$;
- if $f(\gamma, A^r) = \gamma' \in G_{i-1}(r,n)$, then i is not a descent for neither γ' nor γ. Hence $\mathrm{odes}_G(\gamma') = \mathrm{odes}_G(\gamma)$ and $\mathrm{edes}_G(\gamma') = \mathrm{edes}_G(\gamma)$.

By the above arguments, we have

$$\begin{aligned}
&\sum_{(\gamma,A^r) \in G(r,i)\times\binom{[n]}{n-i}^r} x^{\mathrm{odes}_G(\gamma)}y^{\mathrm{edes}_G(\gamma)}a^{\mathrm{col}(f(\gamma,A^r))}q^{\ell_G(f(\gamma,A^r))}\\
&\qquad = \sum_{\gamma' \in G_{i-1}(r,n)} w(\gamma') + \frac{1}{x}\sum_{\gamma' \in G_i(r,n)} w(\gamma') - \frac{1}{x}\sum_{\gamma' \in G_{i-1}(r,n)} w(\gamma')\\
&\qquad = \left(1 - \frac{1}{x}\right)\sum_{\gamma' \in G_{i-1}(r,n)} w(\gamma') + \frac{1}{x}\sum_{\gamma' \in G_i(r,n)} w(\gamma').
\end{aligned} \tag{3.21}$$

Equating the right-hand-sides of (3.20) and (3.21) we obtain

$$x\frac{G_{(r,i)}(x,y,a,q)G_{(r,n)}(a,q)}{G_{(r,i)}(a,q)[n-i]_q!} = (x-1)\sum_{\gamma' \in G_{i-1}(r,n)} w(\gamma') + \sum_{\gamma' \in G_i(r,n)} w(\gamma'). \tag{3.22}$$

This completes the proof of (3.18).

For $n \ge 1$, and i is a nonnegative even integer, (3.19) can be proved similarly. We just verify the $i = 0$ case. Clearly we can construct any $\gamma \in G_0(r,n)$ as follows: choose $\gamma(i) = (i, c_i) \in [n] \times \{0, \dots, r-1\}$ for $i \in [n]$ and order $\gamma(1), \dots, \gamma(n)$ increasingly. As $\gamma = \mathbf{1}$ if and only if $c_i = 0$ for all $i \in [n]$, the index 0 is always a descent if $\gamma \ne \mathbf{1}$. Therefore

$$\begin{aligned}
\sum_{\gamma \in G_0(r,n)} w(\gamma) &= 1 + y\left(\prod_{i=1}^{n}\sum_{c_i=0}^{r-1} aq^{i+c_i-1} - 1\right)\\
&= 1 - y + y(-aq[r-1]_q;q)_n.
\end{aligned}$$

On the other hand, as $G_{-1}(r,n) = \{[1, \dots, n]\}$, (3.19) holds by Proposition 2.1. □

By definition (3.3), for $0 \le i \le n-1$, the elements of $G_i(r,n)$ are colored permutations $\gamma \in G(r,n)$ such that the last $n-i$ elements of γ are increasing. When $i = n-1$, we have $G_{n-1}(r,n) = G(r,n)$. Recall that (see (2.6))

$$G_{(r,n)}(x,y,a,q) = \sum_{\gamma \in G_{n-1}(r,n)} w(\gamma). \tag{3.23}$$

Lemma 3.5. *For any $n,k \in \mathbb{N}$, $r \in \mathbb{P}$, the polynomials $G_{(r,n)}(x,y,a,q)$ satisfy the following recurrences:*

1. *If $n = 2k$ is a nonnegative even integer, and $0 \le j \le k$, we have*

$$\begin{aligned}\frac{G_{(r,n)}(x,y,a,q)}{G_{(r,n)}(a,q)} = {} & \frac{(1-x)^j(1-y)^j}{G_{(r,n)}(a,q)} \sum_{\gamma \in G_{n-2j-1}(r,n)} w(\gamma)\\ & + \sum_{m=0}^{j-1} \frac{x(1-x)^m(1-y)^m}{[2m+1]_q!} \frac{G_{(r,n-2m-1)}(x,y,a,q)}{G_{(r,n-2m-1)}(a,q)}\\ & + \sum_{m=1}^{j} \frac{y(1-x)^m(1-y)^{m-1}}{[2m]_q!} \frac{G_{(r,n-2m)}(x,y,a,q)}{G_{(r,n-2m)}(a,q)}.\end{aligned} \tag{3.24a}$$

2. *If $n = 2k+1$ is a positive odd integer, and $0 \le j \le k$, we have*

$$\begin{aligned}\frac{G_{(r,n)}(x,y,a,q)}{G_{(r,n)}(a,q)} = {} & \frac{(1-x)^j(1-y)^{j+1}}{G_{(r,n)}(a,q)} \sum_{\gamma \in G_{n-2j-2}(r,n)} w(\gamma)\\ & + \sum_{m=0}^{j} \frac{y(1-x)^m(1-y)^m}{[2m+1]_q!} \frac{G_{(r,n-2m-1)}(x,y,a,q)}{G_{(r,n-2m-1)}(a,q)}\\ & + \sum_{m=1}^{j} \frac{x(1-x)^{m-1}(1-y)^m}{[2m]_q!} \frac{G_{(r,n-2m-2)}(x,y,a,q)}{G_{(r,n-2m-2)}(a,q)}.\end{aligned} \tag{3.24b}$$

Proof. It is clear that (3.24a) is valid for $n = 0$. Assuming that $n = 2k$ ($k \ge 1$), we prove (3.24a) by induction on j. The base case when $j = 0$ is obvious by (3.23). Assume that (3.24a) is true for j and show that it holds for $j+1$, that is,

$$\begin{aligned}G_{(r,n)}(x,y,a,q) = {} & (1-x)^{j+1}(1-y)^{j+1} \sum_{\gamma \in G_{n-2j-3}(r,n)} w(\gamma)\\ & + \sum_{m=0}^{j} \frac{x(1-x)^m(1-y)^m}{[2m+1]_q!} \frac{G_{(r,n-2m-1)}(x,y,a,q)G_{(r,n)}(a,q)}{G_{(r,n-2m-1)}(a,q)}\\ & + \sum_{m=1}^{j+1} \frac{y(1-x)^m(1-y)^{m-1}}{[2m]_q!} \frac{G_{(r,n-2m)}(x,y,a,q)G_{(r,n)}(a,q)}{G_{(r,n-2m)}(a,q)}.\end{aligned} \tag{3.25}$$

Equation (3.25) is easy to verify by applying (3.18) and (3.19) because

$$\begin{aligned}\sum_{\gamma \in G_{n-2j-1}(r,n)} w(\gamma) &= (1-x) \sum_{\gamma \in G_{n-2j-2}(r,n)} w(\gamma) + x \frac{G_{(r,n-2j-1)}(x,y,a,q)G_{(r,n)}(a,q)}{[2j+1]_q! G_{(r,n-2j-1)}(a,q)}\\ &= (1-x)(1-y) \sum_{\gamma \in G_{n-2j-3}(r,n)} w(\gamma) + x \frac{G_{(r,n-2j-1)}(x,y,a,q)G_{(r,n)}(a,q)}{[2j+1]_q! G_{(r,n-2j-1)}(a,q)}\\ &\quad + (1-x)y \frac{G_{(r,n-2j-2)}(x,y,a,q)G_{(r,n)}(a,q)}{[2j+2]_q! G_{(r,n-2j-2)}(a,q)}.\end{aligned} \tag{3.26}$$

Plugging (3.26) in (3.24a) and dividing both sides by $G_{(r,n)}(a,q)$, we derive (3.25). Formula (3.24b) can be proved similarly. $\square$

Proof of Theorem 2.1. As $\sum_{\gamma \in G_{-1}(r,n)} w(\gamma) = 1$, multiplying identity (3.24a) (respectively, (3.24b)) with $j = k$ by $(1-y)$ (respectively, $(1-x)$), and then adding $\frac{yG_{(r,n)}(x,y,a,q)}{G_{(r,n)}(a,q)}$ (respectively, $\frac{xG_{(r,n)}(x,y,a,q)}{G_{(r,n)}(a,q)}$) on both sides, we obtain the following recurrence relations:

- if $n=2k$ is even,

$$\frac{G_{(r,n)}(x,y,a,q)}{G_{(r,n)}(a,q)} = \frac{(1-x)^k(1-y)^{k+1}}{G_{(r,n)}(a,q)} + \sum_{m=0}^{k-1} \frac{x(1-x)^m(1-y)^{m+1}}{[2m+1]_q!} \frac{G_{(r,n-2m-1)}(x,y,a,q)}{G_{(r,n-2m-1)}(a,q)} + \sum_{m=0}^{k} \frac{y(1-x)^m(1-y)^m}{[2m]_q!} \frac{G_{(r,n-2m)}(x,y,a,q)}{G_{(r,n-2m)}(a,q)}; \tag{3.27}$$

- if $n=2k+1$ is odd

$$\frac{G_{(r,n)}(x,y,a,q)}{G_{(r,n)}(a,q)} = \frac{(1-x)^{k+1}(1-y)^{k+1}}{G_{(r,n)}(a,q)} + \sum_{m=0}^{k} \frac{y(1-x)^{m+1}(1-y)^m}{[2m+1]_q!} \frac{G_{(r,n-2m-1)}(x,y,a,q)}{G_{(r,n-2m-1)}(a,q)} + \sum_{m=0}^{k} \frac{x(1-x)^m(1-y)^m}{[2m]_q!} \frac{G_{(r,n-2m)}(x,y,a,q)}{G_{(r,n-2m)}(a,q)}. \tag{3.28}$$

Invoking (2.7), multiplying (3.27) (respectively, (3.28)) by t^{2k} (respectively, t^{2k+1}) and then summing over $k \geq 1$ (respectively, $k \geq 0$), we obtain the system

$$\begin{cases} (1-y\cosh(Mt;q))\,H_0^r - xL\sinh(Mt;q)\,H_1^r & = & (1-y)\cosh_{G(r)}(Mt;a,q), \\ -\frac{y}{L}\sinh(Mt;q)\,H_0^r + (1-x\cosh(Mt;q))\,H_1^r & = & M\sinh_{G(r)}(Mt;a,q), \end{cases}$$

where $M=\sqrt{(1-x)(1-y)}$ and $L=\sqrt{(1-y)/(1-x)}$.

Solving the above system using Cramer's rule results in (2.10). □

4. Counting colored permutations by signed alternating descents

The aim of this section is to prove Theorem 2.2 by applying Theorem 2.1. By the wreath product analogue of exponential series $\exp_{G(r)}(t;1,q)$ (see (2.8)) we define the q-trigonometric series over wreath product by

$$\cos_{G(r)}(t;q) := \sum_{n\geq 0} \frac{(-1)^n}{(-q[r-1]_q;q)_{2n}} \cdot \frac{t^{2n}}{[2n]_q!},$$
$$\sin_{G(r)}(t;q) := \sum_{n\geq 0} \frac{(-1)^n}{(-q[r-1]_q;q)_{2n+1}} \cdot \frac{t^{2n+1}}{[2n+1]_q!}.$$

It follows from (2.6), (2.11) and (2.12) that

$$\mathrm{Alt}_n^{G(r)}(x,q) = x^{\lfloor (n+1)/2 \rfloor} G_{(r,n)}(x,1/x,1,q).$$

Combining with Theorem 2.1 we obtain the following generating functions.

Lemma 4.1. *Let* $\mathrm{Alt}_0^{G(r)}(x;q)=1$. *We have*

$$\sum_{n\geq 0} \frac{\mathrm{Alt}_{2n}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2n}} \cdot \frac{t^{2n}}{[2n]_q!} = (x-1) \times \frac{\big(1-x\cos((1-x)t;q)\big)\cos_{G(r)}((1-x)t;q) - x\sin((1-x)t;q)\sin_{G(r)}((1-x)t;q)}{x-(x^2+1)\cos((1-x)t;q) + x\exp(i(1-x)t;q)\exp(-i(1-x)t;q)}, \tag{4.2a}$$

$$\sum_{n\geq 0} \frac{\mathrm{Alt}_{2n+1}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2n+1}} \cdot \frac{t^{2n+1}}{[2n+1]_q!} = (x-1) \times \frac{\big(x-\cos((1-x)t;q)\big)\sin_{G(r)}((1-x)t;q) + \sin((1-x)t;q)\cos_{G(r)}((1-x)t;q)}{x-(x^2+1)\cos((1-x)t;q) + x\exp(i(1-x)t;q)\exp(-i(1-x)t;q)}. \tag{4.2b}$$

Recall the following q-binomial idenity [1, p. 37]

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}_q=\begin{cases}(q;q^2)_n, & \text{if} \quad m=2n;\\ 0, & \text{if} \quad m \text{ is odd},\end{cases} \tag{4.3}$$

and the limits of q-binomial coefficients (3.10) when $q\to -1$:

$$\lim_{q\to-1}\binom{2n}{2m+1}_q=0, \tag{4.4a}$$

$$\lim_{q\to-1}\binom{2n}{2m}_q=\lim_{q\to-1}\binom{2n+1}{2m}_q=\lim_{q\to-1}\binom{2n+1}{2m+1}_q=\binom{n}{m}, \tag{4.4b}$$

$$\lim_{q\to-1}[r-1]_q=\lim_{q\to-1}\frac{1-q^{r-1}}{1-q}=\begin{cases}0, & \text{if } r \text{ is odd};\\ 1, & \text{if } r \text{ is even}.\end{cases} \tag{4.4c}$$

By (4.4c), if r is even and $n-k\geq 2$, then

$$\lim_{q\to-1}\frac{(-q[r-1]_q;q)_n}{(-q[r-1]_q;q)_k}=((-1)^k;-1)_{n-k}=0. \tag{4.4d}$$

Now, we are ready to prove Theorem 2.2 in the next three sections. First, we shall prove that for any even integer $r\geq 2$,

$$\mathrm{Alt}_n^{G(r)}(x,-1)=(-1)^{\lfloor (n+1)/2\rfloor}(1-x)^n \quad \text{for } n\geq 0. \tag{4.5}$$

4.1 Proof of Theorem 2.2 when r is even

Multiplying the two sides of (4.2a) by

$$x-(x^2+1)\cos((1-x)t;q)+x\exp(i(1-x)t;q)\cdot\exp(-i(1-x)t;q),$$

and then comparing the coefficients of $\frac{t^{2n}}{[2n]_q!}$ $(n\geq 0)$, we derive the recurrence relation, after simplification using (4.3), for even indices:

$$\begin{aligned}\frac{-(1-x)^2\mathrm{Alt}_{2n}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2n}}&+\sum_{k=0}^{n-1}\binom{2n}{2k}_q\frac{(-1)^{n-k}\mathrm{Alt}_{2k}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2k}}\big(x(q;q^2)_{n-k}-x^2-1\big)(1-x)^{2n-2k}\\ &=\frac{(-1)^{n+1}(1-x)^{2n+2}}{(-q[r-1]_q;q)_{2n}}+x(1-x)\sum_{k=0}^{2n-1}\binom{2n}{k}_q\frac{(-1)^n(1-x)^{2n}}{(-q[r-1]_q;q)_k}.\end{aligned} \tag{4.6}$$

In the same vein, from (4.2b) we derive the recurrence relation for odd indices $(n\geq 0)$:

$$\begin{aligned}\frac{-(1-x)^2\mathrm{Alt}_{2n+1}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2n+1}}&+\sum_{k=0}^{n-1}\binom{2n+1}{2k+1}_q\frac{(-1)^{n-k}\mathrm{Alt}_{2k+1}^{G(r)}(x,q)}{(-q[r-1]_q;q)_{2k+1}}\big(x(q;q^2)_{n-k}-x^2-1\big)(1-x)^{2n-2k}\\ &=\frac{(-1)^{n}(1-x)^{2n+3}}{(-q[r-1]_q;q)_{2n+1}}+(x-1)\sum_{k=0}^{2n}\binom{2n+1}{k}_q\frac{(-1)^{n-k}(1-x)^{2n+1}}{(-q[r-1]_q;q)_k}.\end{aligned} \tag{4.7}$$

Clearing the fractions in (4.6) and (4.7) by multiplying $(-q[r-1]_q;q)_{2n}$ and $(-q[r-1]_q;q)_{2n+1}$, respectively, and then taking the limit $q\to -1$, we obtain by invoking (4.4),

$$\begin{aligned}-(1-x)^2\mathrm{Alt}_{2n}^{G(r)}(x,-1)&=(-1)^{n+1}(1-x)^{2n+2},\\ -(1-x)^2\mathrm{Alt}_{2n+1}^{G(r)}(x,-1)&=(-1)^{n}(1-x)^{2n+3},\end{aligned}$$

which are equivalent to (4.5).

In the next two sections, we shall prove the remaining part of Theorem 2.2, i.e., for $n\geq 2$, if r is a positive odd integer, then

$$\mathrm{Alt}_n^{G(r)}(x,-1)=\begin{cases}x(1-x)^mA_m(x), & \text{if} \quad n=2m \ (m\in\mathbb{N}^*);\\ \frac{2x^2}{1+x}(1-x)^{2m}A_{2m}(x), & \text{if} \quad n=4m+1 \ (m\in\mathbb{N}^*);\\ 0, & \text{if} \quad n=4m+3 \ (m\in\mathbb{N}),\end{cases} \tag{4.9}$$

where $A_m(x)$ are the classical Eulerian polynomials, see (1.1).

4.2 Proof of Theorem 2.2 when r is odd and $n \neq 4m+3$

For $n \geq 1$, clearing the fractions in (4.6) and (4.7) by multiplying $(-q[r-1]_q;q)_{2n}$ and $(-q[r-1]_q;q)_{2n+1}$, respectively, then taking $q=-1$ results in

$$\begin{aligned}\mathrm{Alt}_{2n}^{G(r)}(x,-1) =& \sum_{k=0}^{n-1}\binom{n}{k}\mathrm{Alt}_{2k}^{G(r)}(x,-1)(-1)^{n-k}(1-x)^{2n-2k-2}(2^{n-k}x-x^2-1)\\ &+(-1)^n(1-x)^{2n-1}(1-2^n x),\end{aligned} \tag{4.10a}$$

$$\begin{aligned}\mathrm{Alt}_{2n+1}^{G(r)}(x,-1) =& \sum_{k=0}^{n-1}\binom{n}{k}\mathrm{Alt}_{2k+1}^{G(r)}(x,-1)(-1)^{n-k}(1-x)^{2n-2k-2}(2^{n-k}x-x^2-1)\\ &+(-1)^n x(1-x)^{2n}.\end{aligned} \tag{4.10b}$$

Now, we prove that $\mathrm{Alt}_{2m}^{G(r)}(x,-1)=x(1-x)^m A_m(x)$ for $m \geq 1$. By (4.10a), this is clear for $m=1$. It remains to show that $x(1-x)^m A_m(x)$ satisfies recurrence relation (4.10a), namely,

$$A_m(x)=\sum_{k=0}^{m-1}\binom{m}{k}(1-x)^{m-k-2}A_k(x)(-1)^{m-k}(2^{m-k}x-x^2-1)-x(-1)^m(1-x)^{m-1}.$$

Multiplying the above identity by $t^m/m!$ and summing over $m \geq 1$ yields

$$1+\sum_{m\geq 1}A_m(x)\frac{t^m}{m!}=\frac{x(1-x)\exp((x-1)t)-x}{(x^2+1)\exp((x-1)t)-x-x\exp(2(x-1)t)},$$

which is equivalent to (1.1). Thus, (4.9) holds true if n is even.

By definition we have $\mathrm{Alt}_1^{G(r)}(x,q)=x(1+q+\cdots+q^{r-1})$, hence $\mathrm{Alt}_1^{G(r)}(x,-1)=x$. For the time being, we admit (4.9) for $n=4m+3$, namely, assume that $\mathrm{Alt}_{4k+3}^{G(r)}(x,-1)=0$ for $k \in \mathbb{N}$, see the proof in the next section. Thus, replacing n by $2m$ in (4.10b) results in

$$\mathrm{Alt}_{4m+1}^{G(r)}(x,-1)=\sum_{k=0}^{m-1}\binom{2m}{2k}\mathrm{Alt}_{4k+1}^{G(r)}(x,-1)(1-x)^{4m-4k-2}(2^{2m-2k}x-x^2-1)+x(1-x)^{4m}. \tag{4.10c}$$

It remains to show that $\mathrm{Alt}_{4m+1}^{G(r)}(x,-1)=\frac{2x^2(1-x)^{2m}}{1+x}A_{2m}(x)$ for $m \geq 1$. As a check, setting $m=1$ in (4.10c) yields $\mathrm{Alt}_5^{G(r)}(x,-1)=2x^2(1-x)^2$, since $A_2(x)=1+x$, (4.9) is valid for $n=5$. Now we prove that $\frac{2x^2(1-x)^{2m}}{1+x}A_{2m}(x)$ satisfy recurrence relation (4.10c), namely,

$$\begin{aligned}\frac{2x^2(1-x)^{2m}}{1+x}A_{2m}(x)=&\sum_{k=1}^{m-1}\binom{2m}{2k}\frac{2x^2(1-x)^{4m-2k-2}A_{2k}(x)}{1+x}(2^{2m-2k}x-x^2-1)\\ &+x(1-x)^{4m-2}(2^{2m}x-2x).\end{aligned}$$

Multiplying the above identity by $t^{2m}/(2m)!$ and summing over $m \geq 1$ yields

$$1+\sum_{m\geq 1}A_{2m}(x)\frac{t^{2m}}{(2m)!}=\frac{(x-1)(\exp((1-x)t)+\exp((x-1)t))}{2x(\exp((1-x)t)+\exp((x-1)t))-2x^2-2},$$

which can be verified straightforwardly by (1.1). □

4.3 Proof of Theorem 2.2 when r is odd and $n=4m+3$

Recall that the elements of the set $\{0,1,\ldots,n,1^1,\ldots,n^1,\ldots,1^{r-1},\ldots,n^{r-1}\}$ are ordered as in the following (see (2.3)),

$$n^{r-1}<\cdots<n^1<\cdots<1^{r-1}<\cdots<1^1<0<1<\cdots<n.$$

For $1 \leq i \leq n$, we define an operator ϕ_i over $G(r,n)$ by

$$\phi_i(\gamma)=\begin{cases}(c_1,\ldots,c_i+1,\ldots,c_n;\sigma), & \text{if } c_i \text{ is odd;}\\ (c_1,\ldots,c_i-1,\ldots,c_n;\sigma), & \text{if } c_i \neq 0 \text{ is even;}\\ \gamma, & \text{if } c_i=0,\end{cases}$$

where $\gamma=(c_1,\ldots,c_n;\sigma)\in G(r,n)$, see (2.1).

Lemma 4.2. *For $1 \le i \le n$, the alternating descent set is invariant under operator ϕ_i, i.e.,*

$$\widehat{Des}_G(\phi_i(\gamma)) = \widehat{Des}_G(\gamma) \quad \text{for} \quad \gamma \in G(r,n). \tag{4.11}$$

Hence, we have $\widehat{des}_G(\phi_i(\gamma)) = \widehat{des}_G(\gamma)$.

Proof. Let $\gamma = (c_1, \dots, c_n; \sigma) \in G(r,n)$ and $\phi_i(\gamma) = (c'_1, \dots, c'_n; \sigma)$ for a fixed $i \in [n]$. Then, $c'_j = c_j$ if $j \neq i$ for $j \in [n]$ and $c'_i = c_i \pm 1$. Equation (4.11) is obvious if $c_i = 0$. Since operator ϕ_i only acts on $\gamma(i)$, we need only to check the nature of positions $i-1$ and i through ϕ_i. In what follows we assume that $c_i > 0$ for some odd index i.

(i) *$i \in \widehat{Des}_G(\gamma)$ if and only if (iff) $i \in \widehat{Des}_G(\phi_i(\gamma))$.*
Indeed, $i \in \widehat{Des}_G(\gamma)$ (with $c_i > 0$) iff $c_{i+1} > 0$ and $|\gamma(i)| < |\gamma(i+1)|$ iff $c'_i = c_i \pm 1 > 0$, $c'_{i+1} = c_{i+1} > 0$ and $|\gamma(i)| < |\gamma(i+1)|$, which is equivalent to $i \in \widehat{Des}_G(\phi_i(\gamma))$.

(ii) *$i-1 \in \widehat{Des}_G(\gamma)$ if and only if $i-1 \in \widehat{Des}_G(\phi_i(\gamma))$.*
Indeed, $\gamma(i-1) > \gamma(i)$ (with $c_i > 0$) iff $c_{i-1} = 0$ or $c_{i-1} > 0$ and $|\gamma(i-1)| < |\gamma(i)|$. As $c'_i = c_i \pm 1 > 0$ and $c'_{i-1} = c_{i-1}$, the latter statement is equivalent to $i-1 \in \widehat{Des}_G(\phi_i(\gamma))$.

Thus, (4.11) is valid for odd $i \in [n]$. The proof for even $i \in [n]$ is similar. ☐

In what follows, for a permutation $\sigma \in \mathfrak{S}_n$, we write $\ell_G(\sigma) = \mathrm{inv}(\sigma)$ and denote the number of alternating descents of σ by $\widehat{des}(\sigma)$ as in [8]. Note that $\mathrm{sgn}(\sigma) = (-1)^{\mathrm{inv}(\sigma)}$ is the signature of σ.

Lemma 4.3. *Let $n = 4m+3$ with $m \ge 0$. Then*

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{inv}(\sigma)} x^{\widehat{des}(\sigma)} = 0. \tag{4.12}$$

Proof. Recall the reversing operator R on $\mathfrak{S}_n$, which maps $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ to $R(\sigma) = \sigma_n\sigma_{n-1}\cdots\sigma_1$. Clearly, an index $i \in [n-1]$ is an even ascent (respectively, odd descent) in σ if and only if $n-i$ is an odd descent (respectively, even ascent) in $R(\sigma)$. Also, position 0 is an even ascent in both σ and $R(\sigma)$. Thus

$$\widehat{des}(\sigma) = \widehat{des}(R(\sigma)). \tag{4.13}$$

Since $\mathrm{inv}(\sigma) + \mathrm{inv}(R(\sigma)) = \binom{n}{2}$ and $\binom{n}{2} = (2m+1)(4m+3)$ is odd, we have $\mathrm{sgn}(R(\sigma)) = -\mathrm{sgn}(\sigma)$. Combining with (4.13), we see that R is a *weight preserving and sign reversing involution* or *killing involution* over $\mathfrak{S}_n$, which yields (4.12). ☐

Let $\mathfrak{S}^c_{r,n}$ be the subset of $G(r,n)$ consisting of permutations $\gamma = (c_1, \dots, c_n; \sigma)$ such that $c_i > 0$ for some index $i \ge 1$.

Lemma 4.4. *Let r be an odd positive integer and $n \in \mathbb{N}^*$. Then*

$$\sum_{\gamma \in \mathfrak{S}^c_{r,n}} (-1)^{\ell_G(\gamma)} x^{\widehat{des}_G(\gamma)} = 0. \tag{4.14}$$

Proof. We construct a *killing involution* Φ on $\mathfrak{S}^c_{r,n}$ such that if $\Phi(\gamma) = \gamma'$ for $\gamma \in \mathfrak{S}^c_{r,n}$, then $\widehat{des}_G(\gamma') = \widehat{des}_G(\gamma)$ and

$$\ell_G(\gamma') = \ell_G(\gamma) \pm 1. \tag{4.15}$$

For $\gamma \in \mathfrak{S}^c_{r,n}$ with $\gamma = (c_1, \dots, c_n; \sigma)$, we define $\Phi(\gamma)$ as follows: let $\Phi(\gamma) = \phi_i(\gamma)$ where i is the smallest index such that $c_i > 0$. It is obvious that Φ is an involution on $\mathfrak{S}^c_{r,n}$.

By Lemma 4.2, it is clear that $\widehat{des}_G(\gamma') = \widehat{des}_G(\gamma)$. It remains to verify (4.15). We first show that each inversion pair is invariant through Φ.

(1) For $1 \le j < i$, pair (j,i) is an inversion of $\gamma \in \mathfrak{S}^c_{r,n}$ if and only if it is an inversion of $\Phi(\gamma) \in \mathfrak{S}^c_{r,n}$.
Indeed, pair (j,i) is an inversion: $\sigma(j)^{c_j} > \sigma(i)^{c_i}$ with $c_i > 0$ iff $c_j = 0$, or $c_j > 0$ and $|\gamma(j)| < |\gamma(i)|$; as $c'_i = c_i \pm 1 > 0$ and $c'_j = c_j$, the previous statement shows that (j,i) is an inversion of $\Phi(\gamma)$.

(2) For $i < l \le n$, pair (i,l) is an inversion of $\gamma \in \mathfrak{S}^c_{r,n}$ if and only if it is an inversion of $\Phi(\gamma) \in \mathfrak{S}^c_{r,n}$.
Indeed, pair (i,l) is an inversion of γ: $\sigma(i)^{c_i} > \sigma(l)^{c_l}$ (with $c_i > 0$) iff $c_l > 0$, and $|\gamma(i)| < |\gamma(l)|$; as $c'_i = c_i \pm 1 > 0$, $c'_l = c_l > 0$, the previous statement means that (i,l) is an inversion of $\Phi(\gamma)$.

As ϕ_i fixes all $\gamma(k)$ for $k \neq i$, it follows that $\mathrm{inv}(\Phi(\gamma)) = \mathrm{inv}(\gamma)$. Therefore, by (2.4),

$$\begin{aligned}\ell_G(\Phi(\gamma)) &= \mathrm{inv}(\Phi(\gamma)) + \sum_{c_k \neq 0} (|\gamma(k)| + c_k - 1) \pm 1 \\ &= \ell_G(\gamma) \pm 1,\end{aligned}$$

which is (4.15). Hence (4.14) is proved. ☐

Acknowledgement

We thank the anonymous reviewer for his/her careful reading of the manuscript and his/her many insightful comments and suggestions.

The first author's work was supported by the National Natural Science Foundation of China grant 12201468. The second author was supported by the China Scholarship Council (No. 202206220034).

References

[1] G. E. Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its Applications, Vol. **2**, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

[2] E. Bagno, D. Garber, and T. Mansour, *Counting descent pairs with prescribed colors in the colored permutation groups*, Sém. Lothar. Combin. 60 (2008/09), Article B60e.

[3] E. Bagno, *Euler-Mahonian parameters on colored permutation groups*, Sém. Lothar. Combin. 51 (2004), Article B51f.

[4] R. Biagioli and J. Zeng, *Enumerating wreath products via Garsia-Gessel Bijections*, European J. Combin. 32 (2011), 538–553.

[5] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, G.T.M 231, Springer-Verlag, New York, 2005.

[6] F. Brenti, *q-Eulerian polynomials arising from Coxeter groups*, European J. Combin. 15 (1994), 417–441.

[7] L. Carlitz and R. Scoville, *Enumeration of rises and falls by position*, Discrete Math. 5 (1973), 45–59.

[8] D. Chebikin, *Variations on descents and inversions in permutations*, Electron. J. Comb. 15 (2008), R132.

[9] J. Désarménien and D. Foata, *The signed Eulerian numbers*, Discrete Math. 99:1-3 (1992), 49–58.

[10] H. K. Dey, U. Shankar, and S. Sivasubramanian, *q-enumeration of type B and D Eulerian Polynomials based on parity of descents*, ECA 4:1 (2024), Article S2R3.

[11] H. K. Dey and S. Sivasubramanian, *Signed alternating descent enumeration in classical Weyl groups*, Discrete Math. 346:10 (2023), 113540.

[12] S.-P. Eu, Z.-C. Lin, and Y.-H. Lo, *Signed Euler-Mahonian identities*, European J. Combin. 91 (2021), 103209.

[13] J. L. Loday, *Opérations sur l'homologie cyclique des algèbres commutatives*, Invent. Math. 96 (1989), 205–230.

[14] Q. Q. Pan and J. Zeng, *Enumeration of permutations by the parity of descent position*, Discrete Math. 346:10 (2023), 113575.

[15] T. K. Petersen, *Eulerian Numbers, With a foreword by Richard Stanley*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser/Springer, New York, 2015.

[16] V. Reiner, *Signed permutation statistics*, European J. Combin. 14 (1993), 553–567.

[17] V. Reiner, *Descents and one-dimensional characters for classical Weyl groups*, Discrete Math. 140 (1995), 129–140.

[18] R. P. Stanley, *Binomial posets, Möbius inversion, and permutation enumeration*, J. Combin. Theory Ser. A, 20 (1976), 336–356.

[19] R. P. Stanley, *Enumerative combinatorics*, Volume 1, Second edition, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, Cambridge, 2012.

[20] E. Steingrímsson, *Permutation statistics of indexed permutations*, European J. Combin. 15 (1994), 187–205.

[21] M. Wachs, *An involution for signed Eulerian numbers*, Discrete Math. 99:1-3 (1992), 59–62.