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Associative-commutative Spectra for Some Varieties of Groupoids

Jia Huang[†] and Erkko Lehtonen[‡]

[†] *Department of Mathematics and Statistics, University of Nebraska, Kearney, NE 68849, USA*
Email: huangj2@unk.edu

[‡] *Department of Mathematics, Khalifa University of Science and Technology, P.O. Box 127788, Abu Dhabi, United Arab Emirates*
Email: erkko.lehtonen@ku.ac.ae

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Abstract: The associative spectrum of a groupoid (i.e., a set with a binary operation) measures its nonassociativity while the associative-commutative spectrum measures both nonassociativity and noncommutativity of the groupoid. The two spectra are also the coefficients of the Hilbert series of certain operads. We establish upper bounds for the two spectra of various varieties of groupoids defined by different sets of identities and provide examples (often groupoids with three elements) for which the upper bounds are achieved. Our results have connections to many interesting combinatorial objects and integer sequences and naturally lead to some questions for future studies.

Keywords: Associative-commutative spectrum; Associative spectrum; Binary operation; Tree; 3-element groupoid
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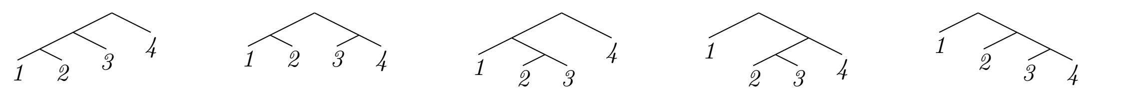
1. Introduction

A *groupoid* $(G, *)$ is a basic algebraic structure that consists of a set G together with a binary operation $*$ defined on G. Associativity and commutativity are common properties that could be satisfied by a groupoid. Csákány and Waldhauser [3] defined the *associative spectrum* (also called the *subassociativity type* by Braitt and Silberger [2]) to measure the failure of a groupoid to be associative, and we introduced the *associative-commutative spectrum*, or simply *ac-spectrum*, to measure both nonassociativity and noncommutativity of a groupoid in earlier work [6]; see the definition below.

Definition 1.1. *Fix a countable list of distinct variables $x_1, x_2, \ldots$. Let $\mathcal{B}_n$ denote the set of all* bracketings *of $x_1, \ldots, x_n$, which are terms in the language of groupoids obtained by inserting pairs of parentheses into the word $x_1x_2\cdots x_n$ in all valid ways. Let $\mathcal{F}_n$ denote the set of* full linear terms *over $x_1, \ldots, x_n$, which are obtained by permuting the variables in the bracketings of $x_1, \ldots, x_n$. We can view $\mathcal{B}_n$ as a subset of $\mathcal{F}_n$. Every term $t \in \mathcal{F}_n$ induces an n-ary operation t^* on a groupoid $(G, *)$. It is often convenient to think about the terms in $\mathcal{F}_n$ or the n-ary operations induced by them in terms of the corresponding* (ordered, full) binary trees *with n labeled leaves; see the example below for $\mathcal{B}_4$, which can give $\mathcal{F}_4$ if the variables are permuted in all possible ways.*



$((x_1{*}x_2){*}x_3){*}x_4$ $\quad (x_1{*}x_2){*}(x_3{*}x_4)$ $\quad (x_1{*}(x_2{*}x_3)){*}x_4$ $\quad x_1{*}((x_2{*}x_3){*}x_4)$ $\quad x_1{*}(x_2{*}(x_3{*}x_4))$

The associative spectrum (*resp.*, ac-spectrum) *of a groupoid $(G, *)$, or of its binary operation $*$, is a sequence whose nth term is $s_n^{\mathrm{a}}(*) := |P_n(*)|$ (resp., $s_n^{\mathrm{ac}}(*) := |\overline{P}_n(*)|$), where $P_n(*) := \{t^* : t \in \mathcal{B}_n\}$ (resp., $\overline{P}_n(*) := \{t^* : t \in \mathcal{F}_n\}$), for $n = 1, 2, \ldots$. It turns out that $\{P_n(*)\}_{n \geqslant 1}$ (resp., $\{\overline{P}_n(*)\}_{n \geqslant 1}$) together with a composition function becomes a* nonsymmetric operad (*resp.*, symmetric operad) *that satisfies certain coherence axioms [11], and the* Hilbert series *of this operad is the generating function (resp., exponential generating function) of the associative spectrum (resp., ac-spectrum) of $(G, *)$.*

By the above definition, we have (1) $s_n^{\mathrm{a}}(*) = 1$ for $n = 1, 2$, (2) $s_1^{\mathrm{ac}}(*) = 1$, and (3) $s_2^{\mathrm{ac}}(*)$ is either 1 or 2, depending on whether $*$ is commutative. Thus we may assume $n \geqslant 3$ when necessary. It is easy to see that isomorphic or anti-isomorphic groupoids have the same associative spectrum and the same ac-spectrum, where two groupoids $(G, *)$ and $(H, \otimes)$ are said to be *anti-isomorphic*, denoted by $G \simeq H^{\mathrm{op}}$, if there is a bijection $f : G \to H$ such that $f(a * b) = f(b) \otimes f(a)$ for all $a, b \in G$.

It is clear that $s_n^{\mathrm{a}}(*) = 1$ for all $n \in \mathbb{N}$ if and only if $*$ associative and that $s_n^{\mathrm{ac}}(*) = 1$ for all $n \in \mathbb{N}$ if and only if $*$ is associative and commutative, where $\mathbb{N} := \{1, 2, \ldots\}$. On the other hand, we have $s_n^{\mathrm{a}}(*) \leqslant C_{n-1}$, where $C_n := \frac{1}{n+1}\binom{2n}{n}$ is the ubiquitous *Catalan number*, and thus $s_n^{\mathrm{ac}}(*) \leqslant n!C_{n-1}$. We showed in previous work [6] that a commutative groupoid $(G, *)$ must have $s_n^{\mathrm{ac}}(*) \leqslant D_{n-1}$, where $D_n := (2n!)/(2^n n!)$ is the solution to Schröder's third problem [13, A001147], and that an associative groupoid $(G, *)$ must have $s_n^{\mathrm{ac}}(*) \leqslant n!$, which holds as an equality if the groupoid is noncommutative and has an identity element (see Theorem 7.1 for a generalization).

In addition, the precise values of the associative spectrum and ac-spectrum have been determined for various groupoids [3–6,8,9], including 2-element groupoids, generalizations of addition and subtraction, exponentiation, arithmetic/geometric/harmonic mean, cross product, Lie algebras with an $\mathfrak{sl}_2$-triple, graph algebras, and so on. The results show connections with interesting combinatorial objects, avoided patterns, and integer sequences. However, the ac-spectra of 3-element groupoids are largely undetermined.

$*$	0	1	2
0	0	0	0
1	1	1	0
2	2	2	2

SC271 $\simeq$ SC1610$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	0
1	2	1	1
2	1	2	2

SC356 $\simeq$ SC2032$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	0
1	0	0	0
2	1	0	0

SC10 $=$ SC367$^{\mathrm{op}}$

$*$	0	1	2
0	1	1	1
1	2	2	2
2	0	0	0

SC3242 $=$ SC3302$^{\mathrm{op}}$

$*$	0	1	2
0	0	1	1
1	0	1	2
2	0	1	2

SC1610

$*$	0	1	2
0	0	1	2
1	0	1	2
2	1	0	2

SC2032

$*$	0	1	2
0	0	0	1
1	0	0	0
2	0	0	0

SC367

$*$	0	1	2
0	1	2	0
1	1	2	0
2	1	2	0

SC3302

$*$	0	1	2
0	0	0	2
1	0	0	2
2	2	2	1

SC1066 $=$ SC1066$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	1
1	0	0	1
2	1	1	0

SC405 $=$ SC405$^{\mathrm{op}}$

$*$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	1

SC79 $=$ SC79$^{\mathrm{op}}$

Table 1: Some 3-element groupoids

According to the Siena Catalog [1], there are 3330 non-isomorphic 3-element groupoids, which are indexed from 1 to 3330. Each of these groupoids is determined by a binary operation $*$ defined on the set $\{0, 1, 2\}$. We write them as SC1, SC2, ..., SC3330. There are 729 *idempotent* 3-element groupoids, which can be labeled in a different way: ID0, ID1, ..., ID728. Csákány and Waldhauser [3] showed the following (see Table 1).

- Both ID35 = SC271($\simeq$ SC1610$^{\mathrm{op}}$) and ID68 = SC356($\simeq$ SC2032$^{\mathrm{op}}$) have associative spectrum $s_n^{\mathrm{a}}(*) = 2^{n-2}$ for $n \geqslant 2$.
- Both SC1066 and SC10($\simeq$ SC367$^{\mathrm{op}}$) have associative spectrum $s_n^{\mathrm{a}}(*) = n - 1$ for $n \geqslant 1$.
- Both SC405 and SC3242($\simeq$ SC3302$^{\mathrm{op}}$) have associative spectrum $s_n^{\mathrm{a}}(*) = 3$ for $n > 3$ (it is easy to check that $s_n^{\mathrm{a}}(*) = 1$ for $n = 1, 2$ and $s_n^{\mathrm{a}}(*) = 2$ for $n = 3$).
- The groupoid SC79 has associative spectrum $s_n^{\mathrm{a}}(*) = F_{n+1} - 1$ for $n \geqslant 2$, where F_{n+1} is the *Fibonacci number* defined by $F_{n+1} := F_n + F_{n-1}$ for $n \geqslant 1$ and $F_i = i$ for $i = 0, 1$,

Our original motivation for this work was to determine the ac-spectra of the above 3-element groupoids, whose Cayley tables are given in Table 1. However, we are able to establish more general results on various varieties of groupoids, where a *variety* of groupoids axiomatized by a set Σ of identities is the family of all groupoids satisfying the identities in Σ. For each variety of groupoids considered in this paper, we establish an upper bound for the associative spectra and an upper bound for the ac-spectra of the groupoids belonging to this variety; if the latter upper bound is reached by a member of the variety, so is the former. Moreover, we show that both upper bounds are attained by at least one 3-element groupoid.

For example, we showed in earlier work [6] that a commutative groupoid must have $s_n^{\mathrm{ac}}(*) \leqslant D_{n-1}$ and if the equality in this upper bound holds, so does the equality in the upper bound $s_n^{\mathrm{a}}(*) \leqslant C_{n-1}$. In the same paper, we showed that $s_n^{\mathrm{ac}}(*) = D_{n-1}$ for a 3-element groupoid called the *rock-paper-scissors groupoid*, which turns out to be isomorphic to SC1108, and the proof is also valid for SC2407 and SC3093.

$*$	0	1	2
0	0	0	2
1	0	1	1
2	2	1	2

SC1108

$*$	0	1	2
0	1	0	0
1	0	2	0
2	0	0	0

SC2407

$*$	0	1	2
0	1	1	0
1	1	2	0
2	0	0	1

SC3093

Therefore, we have the following result.

Theorem 1.1 ([6]). *A groupoid $(G,*)$ satisfying the identity $xy \approx yx$ must have $s_n^{\mathrm{a}}(*) \leqslant C_{n-1}$ and $s_n^{\mathrm{ac}}(*) \leqslant D_{n-1}$ for $n = 1, 2, \ldots$, where the first inequality holds as an equality whenever the second does and both equalities hold for the 3-element groupoids* SC1108, SC2407, *and* SC3093.

In this paper, we provide a series of results that are similar to the above one. A summary of our results is given by Table 2, where we use the well-known *Bell number* B_n counting partitions of the set $\{1, 2, \ldots, n\}$ into unordered nonempty blocks, the *restricted Bell number* $B_{n,m}$ counting partitions of $\{1, 2, \ldots, n\}$ into unordered nonempty blocks of size at most m [12], and the *ordered Bell number* or *Fubini number* B'_n counting partitions of $\{1, 2, \ldots, n\}$ into ordered nonempty blocks [13, A000670]. The "$n \geqslant$" column in Table 2 gives the smallest values of n for which the upper bounds of $s_n^{\mathrm{a}}(*)$ and $s_n^{\mathrm{ac}}(*)$ are valid and sharp. Note that different varieties of groupoids in the table may have the same associative spectrum upper bound but different ac-spectrum upper bounds (the upper bounds for $s_n^{\mathrm{ac}}(*)$ in Prop. 3.2 and Prop. 3.3 are different when $n = 3$). Therefore, the ac-spectrum may offer a finer distinction between groupoids than the associative spectrum.

Identities satisfied by $(G,*)$	$n \geqslant$	$s_n^{\mathrm{a}}(*) \leqslant$	$s_n^{\mathrm{ac}}(*) \leqslant$	Examples for =	reference
(1)	1, 1	1	n	SC275($\simeq$ SC2029$^{\mathrm{op}}$)	Prop. 3.1
(3), (4), (5), (7)	3, 3	2	$n+1$	SC7($\simeq$SC4$^{\mathrm{op}}$) SC28($\simeq$SC5$^{\mathrm{op}}$)	Prop. 3.2
(2), (7), (15)	4, 4	3	$n+1$	SC405	Prop. 3.3
(3), (5), (7), (8), (9)	3, 3	2	$2n$	SC189($\simeq$ SC170$^{\mathrm{op}}$)	Prop. 3.4
(5), (7), (10), (11), (12), (16)	4, 4	3	$3n$	SC3242($\simeq$ SC3302$^{\mathrm{op}}$)	Prop. 3.5
(5), (7), (11), (13), (17), (18)	4, 4	4	$2n^2$	SC3162($\simeq$ SC2467$^{\mathrm{op}}$)	Thm. 3.1
(2), (7)	2, 2	$n-1$	$2^{n-1}-1$	SC1066	Prop. 4.1
(4), (5), (7)	2, 1	$n-1$	$n! + \sum_{k=0}^{n-3} k!\binom{n}{k}$	SC367($\simeq$ SC10$^{\mathrm{op}}$)	Prop. 4.2
(3), (6), (14)	2, 2	2^{n-2}	2^n-2	SC2302($\simeq$ SC2155$^{\mathrm{op}}$)	Prop. 4.3
(3), (7), (12)	2, 2	2^{n-2}	$n(2^{n-1}-1)$	SC271($\simeq$SC1610$^{\mathrm{op}}$) SC356($\simeq$SC2032$^{\mathrm{op}}$)	Thm. 4.1
(2), (11)	2, 2	$F_{n+1}-1$	$B_{n,2}-1$	SC79, SC1701	Prop. 5.1
(3), (5)	2, 1	2^{n-2}	nB_{n-1}	SC41($\simeq$SC398$^{\mathrm{op}}$) SC96($\simeq$SC1069$^{\mathrm{op}}$)	Thm. 5.1
(5), (7)	2, 1	2^{n-2}	nB'_{n-1}	SC262($\simeq$SC1441$^{\mathrm{op}}$) SC1812($\simeq$SC1793$^{\mathrm{op}}$) SC2446($\simeq$SC2430$^{\mathrm{op}}$)	Thm. 5.2

(1) $xy \approx x$ (2) $xy \approx yx$ (3) $(xy)z \approx (xz)y$ (4) $x(yz) \approx y(xz)$ (5) $x(yz) \approx x(zy)$ (6) $x(yz) \approx z(yx)$

(7) $w(x(yz)) \approx w((xy)z)$ (8) $(wx)(yz) \approx (w(xy))z$ (9) $w(x(yz)) \approx ((wx)y)z$

(10) $((wx)y)z \approx ((wy)x)z$ (11) $((wx)y)z \approx ((wx)z)y$ (12) $(wx)(yz) \approx (wy)(xz)$

(13) $(w(xy))z \approx (w(xz))y$ (14) $w(x(yz)) \approx (w(xy))z$ (15) $(v(wx))(yz) \approx (vw)(x(yz))$

(16) $((vw)x)y)z \approx v(w(x(yz)))$ (17) $v(w(x(yz))) \approx ((v(wx))y)z$ (18) $((vw)(x(yz)) \approx (((vw)x)y)z$

Table 2: Summary of results

It is sometimes convenient to use not only identities but other conditions to describe a family of groupoids satisfying certain upper bounds for their spectra. Recall that every term $t \in \mathcal{F}_n$ corresponds to a binary tree with n leaves labeled by $1, \ldots, n$. Each leaf i has its *depth* $d_i(t)$ (resp. *left depth* $\delta_i(t)$ or *right depth* $\rho_i(t)$) defined as the number of edges (resp., left/right edges) in the unique path to the root of t. By abuse of notation, we also speak of these three kinds of depths for the variables in t. Previous work [4, 6] used the congruence modulo m relation on depths to study the associative spectra and ac-spectra of certain groupoids, and some of the results there can be rephrased to include Proposition 4.3 as a special case. We can also generalize Proposition 3.4 and Proposition 3.5 in a similar way.

The paper is structured as follows. We give some basic definitions and properties on the associative spectrum and ac-spectrum in Section 2. We establish some polynomial upper bounds and exponential upper bounds in

Section 3 and Section 4, respectively. We provide more upper bounds related to set partitions in Section 5. We use congruence on leaf depths in binary trees to provide generalizations of some of our results in Section 6. Finally, we make some remarks and pose some questions for future research in Section 7.

2. Preliminaries

We first give some notation and terminology. A *term** t over a set of variables X (we often use $X_n := \{x_1, \ldots, x_n\}$) is a bracketing of a word $x_{i_1} \cdots x_{i_k}$, where $x_{i_1}, \ldots, x_{i_k} \in X$; let $\mathrm{var}(t)$ denote the set of all variables in t. If $i_1, \ldots, i_k$ are distinct, then t is called a *linear term* with $|t| := k$. Define the *leftmost bracketing* $[t_1, \ldots, t_k]$ of terms $t_1, \ldots, t_k$ recursively by $[t_1] := t_1$ and $[t_1, \ldots, t_{n+1}] := ([t_1, \ldots, t_n]t_{n+1}])$ for $n \geqslant 1$. Similarly, define the *rightmost bracketing* $\langle t_1, \ldots, t_k\rangle$ recursively by $\langle t_1\rangle := t_1$ and $\langle t_1, \ldots, t_{n+1}\rangle := (t_1\langle t_2, \ldots, t_{n+1}\rangle)$ for $n \geqslant 1$. We can write every term as $t = [t_0, t_1, \ldots, t_m]$ with $|t_0| = 1$ for some $m \in \mathbb{N}$; this is known as the *leftmost decomposition* [6, Definition 6.1.2], which can also be obtained by writing $t = (t_{\mathrm{L}}t_{\mathrm{R}})$ if t is not a variable, then further writing $t_{\mathrm{L}} = (t'_{\mathrm{L}}t'_{\mathrm{R}})$ if the left subterm t_{L} is not a variable, and continuing in this way to decompose left subterms until we reach one that is a single variable.

Terms can be evaluated in a groupoid $(G, *)$ as follows. Given an *assignment* $h\colon X \to G$ of values from G for the variables in X, we can extend h to a map $\overline{h}$ defined on the set of all terms over X with the following recursive definition. We have $\overline{h}(x) := h(x)$ for every variable $x \in X$ (because $\overline{h}$ extends h), and if $t = (t_1t_2)$ for subterms t_1 and t_2, then we define $\overline{h}(t) := \overline{h}(t_1) * \overline{h}(t_2)$. In this way, every term t over X_n *induces* an n-ary operation t^* on $(G, *)$ (called a *term function*): $t^*(a_1, \ldots, a_n) := \overline{h}_{\mathbf{a}}(t)$, where $\overline{h}_{\mathbf{a}}$ is the extension of the assignment $h_{\mathbf{a}}\colon X_n \to G$ that maps x_i to a_i for all $i \in \{1, \ldots, n\}$. For notational simplicity, we will denote the extension $\overline{h}$ of an assignment h also by h.

An *identity* is a pair of terms, usually written as $s \approx t$. A groupoid $(G, *)$ *satisfies* an identity $s \approx t$ if $s^* = t^*$. (Here we have assumed that s and t are terms over X_n for some $n \in \mathbb{N}$ – this can always be done.)

In the subsequent sections, we will prove several results, each of which provides upper bounds for the ac-spectrum and the associative spectrum of a *variety* of groupoids axiomatized by a set Σ of identities, i.e., the family of all groupoids satisfying the identities in Σ. We will employ the following proof technique. We assume that a groupoid $(G, *)$ satisfies certain identities. Using these identities, we transform each full linear term t into an equivalent term t' that is in "standard form" (terms t and t' are *equivalent* if $(G, *)$ satisfies $t \approx t'$, i.e., $t^* = (t')^*$). It thus follows that $s_n^{\mathrm{ac}}(*)$, i.e., the number of term functions induced by full linear terms with n variables on $(G, *)$, is bounded above by the number of terms in standard form, so it is then a matter of counting the possible standard forms. Similarly, finding $s_n^{\mathrm{a}}(*)$ amounts to counting the standard forms that can be obtained from bracketings.

Let t be a linear term. Assume that $\mathrm{var}(t) = \{x_{i_1}, \ldots, x_{i_m}\}$ and that x_{i_k} occurs to the left from x_{i_ℓ} in t if and only if $k < \ell$. Assume further that $\{j_1, \ldots, j_m\} = \{i_1, \ldots, i_m\}$ and $j_1 < j_2 < \cdots < j_m$. Let

$$t^{\mathrm{L}} := [x_{i_1}, \ldots, x_{i_m}], \qquad t^{\mathrm{L}<} := [x_{j_1}, \ldots, x_{j_m}],$$
$$t^{\mathrm{R}} := \langle x_{i_1}, \ldots, x_{i_m}\rangle, \qquad t^{\mathrm{R}<} := \langle x_{j_1}, \ldots, x_{j_m}\rangle,$$

i.e., t^{L} and $t^{\mathrm{L}<}$ (t^{R} and $t^{\mathrm{R}<}$, resp.) are leftmost (rightmost, resp.) bracketings of the variables of t; in the former, the variables occur in the same order as in t, while in the latter, the variables occur in the increasing order of the indices.

The next lemma will be frequently used to establish our main results.

Lemma 2.1. *Let $(G, *)$ be a groupoid, and write an arbitrary term in $\mathcal{F}_n$ as $t = [t_0, t_1, \ldots, t_m]$ with $|t_0| = 1$ (leftmost decomposition).*

(i) *If $(G, *)$ satisfies the identity $w(x(yz)) \approx w((xy)z)$, then $(G, *)$ also satisfies the identities $t \approx [t_0, t_1^{\mathrm{L}}, \ldots, t_m^{\mathrm{L}}]$ and $t \approx [t_0, t_1^{\mathrm{R}}, \ldots, t_m^{\mathrm{R}}]$.*

(ii) *If $(G, *)$ satisfies the identities $w(x(yz)) \approx w((xy)z)$ and either $x(yz) \approx x(zy)$ or $xy \approx yx$, then $(G, *)$ also satisfies the identities $t \approx [t_0, t_1^{\mathrm{L}<}, \ldots, t_m^{\mathrm{L}<}]$ and $t \approx [t_0, t_1^{\mathrm{R}<}, \ldots, t_m^{\mathrm{R}<}]$.*

(iii) *If $(G, *)$ satisfies the identity $(xy)z \approx (xz)y$, then $(G, *)$ also satisfies the identity $t \approx [t_0, t_{\sigma(1)}, \ldots, t_{\sigma(m)}]$ for every permutation $\sigma \in \mathfrak{S}_m$.*

(iv) *If $(G, *)$ satisfies the identities $x(yz) \approx x(zy)$ and $(xy)z \approx (xz)y$, then $(G, *)$ also satisfies the identity $t \approx [t_0, t_1^{\mathrm{L}<}, \ldots, t_m^{\mathrm{L}<}]$.*

*More specifically, we are speaking about terms in the language of groupoids, i.e., terms of type (2). Since our language contains only one operation symbol, which is binary, we may simply omit it from terms. Variables and brackets are sufficient for writing down terms unambiguously in this language.

Proof. (i) We can use the identity $w(x(yz)) \approx w((xy)z)$ repeatedly to transform each t_i to the form $x_j s$, where x_j is the leftmost variable of t_i, and then apply the same procedure to s to eventually transform t_i into t_i^{R}. A similar argument shows that each t_i can be transformed into t_i^{L}.

(ii) By (i), $(G, *)$ satisfies $t \approx [t_0, t_1^{\mathrm{R}}, \dots, t_m^{\mathrm{R}}]$. We may arbitrarily permute the variables in each t_i^{R}, $i \in \{1, \dots, m\}$, thanks to the identities

$$w(x(zy)) \approx w(x(yz)) \approx w((xy)z) \approx w(z(xy)) \approx w(z(yx)) \approx w((yx)z) \approx w(y(xz)) \approx w(y(zx)).$$

Thus $(G, *)$ satisfies $t \approx [t_0, t_1^{\mathrm{R}<}, \dots, t_m^{\mathrm{R}<}]$. A similar argument shows that $(G, *)$ satisfies $t \approx [t_0, t_1^{\mathrm{L}<}, \dots, t_m^{\mathrm{L}<}]$.

(iii) We can use the identity $(xy)z \approx (xz)y$ to swap the subterms t_i and t_{i+1} in $[t_0, t_1, \dots, t_m]$, for any $i \in \{1, \dots, m-1\}$. Since the adjacent transpositions generate $\mathfrak{S}_m$, it follows that $(G, *)$ satisfies $t \approx [t_0, t_{\sigma(1)}, \dots, t_{\sigma(m)}]$ for every $\sigma \in \mathfrak{S}_m$.

(iv) By (iii), we can permute the subterms $t_1, \dots, t_m$, so it suffices to prove that $(G, *)$ satisfies $xs \approx x(s^{\mathrm{L}<})$ for any linear term s with $x \notin \operatorname{var}(s)$. We prove this by induction on $|s|$. This is trivial when $|s| = 1$, and this holds for $|s| = 2$ by the identity $x(yz) \approx x(zy)$. Let now $k \geqslant 3$, assume that the claim holds whenever $|s| < k$, and consider the case when $|s| = k$. We have the leftmost decomposition $s = [s_0, s_1, \dots, s_\ell]$. By the inductive hypothesis and (iii), we may assume that $s_j = s_j^{\mathrm{L}<}$ for all $j \in \{1, \dots, \ell\}$. Consequently, $(G, *)$ satisfies $xs \approx x(s_\ell^{\mathrm{L}<} u)$, where $u := [s_0, s_1^{\mathrm{L}<}, \dots, s_{\ell-1}^{\mathrm{L}<}]$, and by the inductive hypothesis, this is equivalent to $x(s_\ell^{\mathrm{L}<} u^{\mathrm{L}<})$. By the identity $x(yz) \approx x(zy)$, we may swap $s_\ell^{\mathrm{L}<}$ and $u^{\mathrm{L}<}$ if necessary to obtain a term of the form $x([x_{i_{k+1}}, \dots, x_{i_m}][x_{i_1}, \dots, x_{i_k}])$, where $i_1 < \dots < i_k$, $i_{k+1} < \dots < i_m$ and $i_1 < i_{k+1}$. Using the identities $x(yz) \approx x(zy)$ and $(xy)z \approx (xz)y$, we obtain

$$\begin{aligned}
xs &\approx \langle x, [x_{i_{k+1}}, \dots, x_{i_m}][x_{i_1}, \dots, x_{i_k}] \rangle = \langle x, ([x_{i_{k+1}}, \dots, x_{i_{m-1}}]x_{i_m})[x_{i_1}, \dots, x_{i_k}] \rangle \\
&\approx \langle x, ([x_{i_{k+1}}, \dots, x_{i_{m-1}}][x_{i_1}, \dots, x_{i_k}])x_{i_m} \rangle \approx \langle x, x_{i_m}([x_{i_{k+1}}, \dots, x_{i_{m-1}}][x_{i_1}, \dots, x_{i_k}]) \rangle \\
&= \langle x, x_{i_m}, [x_{i_{k+1}}, \dots, x_{i_{m-1}}][x_{i_1}, \dots, x_{i_k}] \rangle \approx \langle x, x_{i_m}, ([x_{i_{k+1}}, \dots, x_{i_{m-2}}][x_{i_1}, \dots, x_{i_k}])x_{i_{m-1}} \rangle \\
&\approx \langle x, x_{i_m}, x_{i_{m-1}}, [x_{i_{k+1}}, \dots, x_{i_{m-2}}][x_{i_1}, \dots, x_{i_k}] \rangle \approx \dots \approx \langle x, x_{i_m}, x_{i_{m-1}}, \dots, x_{i_{k+2}}, x_{i_{k+1}}, [x_{i_1}, \dots, x_{i_k}] \rangle \\
&\approx \langle x, x_{i_m}, x_{i_{m-1}}, \dots, x_{i_{k+2}}, [x_{i_1}, \dots, x_{i_k}]x_{i_{k+1}} \rangle = \langle x, x_{i_m}, x_{i_{m-1}}, \dots, x_{i_{k+2}}, [x_{i_1}, \dots, x_{i_k}, x_{i_{k+1}}] \rangle \\
&\approx \langle x, x_{i_m}, x_{i_{m-1}}, \dots, x_{i_{k+3}}, [x_{i_1}, \dots, x_{i_k}, x_{i_{k+1}}, x_{i_{k+2}}] \rangle \approx \dots \approx \langle x, [x_{i_1}, \dots, x_{i_k}, x_{i_{k+1}}, \dots, x_{i_m}] \rangle.
\end{aligned}$$

Since i_1 is the smallest of the indices $i_1, \dots, i_m$, we can then apply the identity $(xy)z \approx (xz)y$ and part (iii) to sort the variables in the subterm $[x_{i_1}, \dots, x_{i_k}, x_{i_{k+1}}, \dots, x_{i_m}]$ in the increasing order of indices, and we obtain $x(s^{\mathrm{L}<})$, as desired. ☐

3. Polynomial upper bounds

In this section, we establish some polynomial upper bounds for the ac-spectra of groupoids belonging to certain varieties of groupoids; in contrast, their associative spectra all have constant upper bounds.

For our first variety of groupoids, we can actually determine their associative spectrum and ac-spectrum.

Proposition 3.1. *A groupoid $(G, *)$ with at least two elements satisfying the identity $xy \approx x$ must have $s_n^{\mathrm{a}}(*) = 1$ and $s_n^{\mathrm{ac}}(*) = n$ for $n \geqslant 1$. In particular, the above two equalities hold for the 2-element groupoid $(\{0, 1\}, *)$ defined by $x * y := x$ for all $x, y \in \{0, 1\}$ and the 3-element groupoids* SC275 *and* SC2029.

$*$	0	0	0
0	0	0	0
1	1	1	1
2	2	2	2

SC275

$*$	0	1	2
0	0	1	2
1	0	1	2
2	0	1	2

SC2029

Proof. If $(G, *)$ is a groupoid with at least two elements satisfying the identity $xy \approx x$, then $s_n^{\mathrm{a}}(*) = 1$ and $s_n^{\mathrm{ac}}(*) = n$ for all $n \geqslant 1$ since the n-ary operation t^* induced by every term $t \in \mathcal{F}_n$ is determined by the leftmost variable of t and distinct variables induce distinct operations.

In earlier work [6, Example 4.1.2], we showed that the 2-element groupoid $(\{0, 1\}, *)$ with $x * y := x$ for all $x, y \in \{0, 1\}$ has $s_n^{\mathrm{a}}(*) = 1$ and $s_n^{\mathrm{ac}}(*) = n$ for $n \geqslant 1$. One can check that SC275 satisfies the identity $xy \approx x$ and that SC2029 is anti-isomorphic to SC275. Thus their associative spectrum and ac-spectrum are also given as above. ☐

The upper bounds in the next result are achieved by the 3-element groupoids SC7 and SC28, which are anti-isomorphic to SC4 (by swapping 1 and 2) and SC5, respectively.

$*$	0	1	2
0	0	0	0
1	0	0	0
2	0	1	0

SC4

$*$	0	1	2
0	0	0	0
1	0	0	0
2	0	1	1

SC5

$*$	0	1	2
0	0	0	0
1	0	0	0
2	0	2	0

SC7

$*$	0	1	2
0	0	0	0
1	0	0	1
2	0	0	1

SC28

Proposition 3.2. *A groupoid $(G,*)$ satisfying the identities below must have $s_n^{\mathrm{a}}(*) \leqslant 2$ and $s_n^{\mathrm{ac}}(*) \leqslant n+1$ for $n = 3, 4, \ldots$, where the first inequality holds as an equality if so does the second and both equalities hold for* SC7 *and* SC28*.*

$$\text{(i) } (xy)z \approx (xz)y, \quad \text{(ii) } x(yz) \approx x(zy) \approx y(xz), \quad \text{(iii) } w(x(yz)) \approx w((xy)z)$$

Proof. Let t be an arbitrary term in $\mathcal{F}_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $t_0 = x_a$ for some $a \in \{1, 2, \ldots, n\}$. By Lemma 2.1, we may assume that $t_i = t_i^{\mathrm{R}<}$ for all $i \in \{1, \ldots, m\}$ and $|t_1| \leqslant \cdots \leqslant |t_m|$. We then distinguish the two cases below.

- If $|t_m| > 1$, then $t_m = \langle x_{b_1}, \ldots, x_{b_\ell} \rangle$ and we can apply (ii) to swap the leftmost variable of t_m and $[x_a, t_1, \ldots, t_{m-1}]$. The resulting term $x_{b_1} \langle [x_a, t_1, \ldots, t_{m-1}], x_{b_2}, \ldots, x_{b_\ell} \rangle$ can be transformed to
$$\langle x_{b_1}, x_1, \ldots, x_{b_1-1}, x_{b_1+1}, \ldots, x_n \rangle$$
by Lemma 2.1. Then we can apply (ii) to swap x_{b_1} with x_1, and finally we can turn the term into $\langle x_1, \ldots, x_n \rangle$.
- If $|t_m| = 1$, then $t = [x_a, x_{b_1}, \ldots, x_{b_{n-1}}]$, and we can apply (i) to make sure $b_1 < \cdots < b_{n-1}$.

It follows that $s_n^{\mathrm{ac}}(*) \leqslant n+1$ since there are n possibilities for a in the second case. If the variables $x_1, \ldots, x_n$ are ordered increasingly in t, then we must have $a = 1$ in the second case. Thus $s_n^{\mathrm{a}}(*) \leqslant 2$. If $s_n^{\mathrm{ac}}(*) = n+1$, then the two cases above cannot yield identical n-ary operations on $(G,*)$, and thus $s_n^{\mathrm{a}}(*) = 2$.

Now we determine $s_n^{\mathrm{a}}(*)$ and $s_n^{\mathrm{ac}}(*)$ for SC7. It is routine to check that SC7 satisfies the identities (i), (ii), and (iii). Let t be an arbitrary term in $\mathcal{F}_n$. We may assume that $t = \langle x_1, \ldots, x_n \rangle$ or $t = [x_a, x_{b_1}, \ldots, x_{b_{n-1}}]$ with $b_1 < \cdots < b_{n-1}$ by the above argument. For the former, we have $h(t) = 0$ for all $h : X_n \to \{0, 1, 2\}$. For the latter, we have $h(t) = 2$ if $h(x_a) = 2$ and $h(x_{b_1}) = \cdots = h(x_{b_{n-1}}) = 1$ or $h(t) = 0$ otherwise. Therefore $s_n^{\mathrm{ac}}(*) = n+1$, which implies $s_n^{\mathrm{a}}(*) = 2$.

In a similar way, we can determine $s_n^{\mathrm{a}}(*)$ and $s_n^{\mathrm{ac}}(*)$ for SC28, which also satisfies the identities (i), (ii), and (iii). If $t = \langle x_1, \ldots, x_n \rangle$, then $h(t) = 0$ for all $h : X_n \to \{0, 1, 2\}$. If $t = [x_a, x_{b_1}, \ldots, b_{n-1}]$, then $h(t) = 1$ if $h(x_a) \in \{1, 2\}$ and $h(x_{b_i}) = 2$ for $i = 1, \ldots, n-1$, or $h(t) = 0$ otherwise. It follows that $s_n^{\mathrm{ac}}(*) = n+1$, which implies $s_n^{\mathrm{a}}(*) = 2$. ☐

The upper bounds in the next result are very close to but not the same as those in Proposition 3.2.

Proposition 3.3. *A groupoid $(G,*)$ satisfying the identities below must have upper bounds $s_n^{\mathrm{a}}(*) \leqslant 2$ and $s_n^{\mathrm{ac}}(*) \leqslant 3$ for $n = 3$ and $s_n^{\mathrm{a}}(*) \leqslant 3$ and $s_n^{\mathrm{ac}}(*) \leqslant n+1$ for $n = 4, 5, \ldots$.*

$$\text{(i) } xy \approx yx, \quad \text{(ii) } w(x(yz)) \approx w((xy)z), \quad \text{(iii) } (v(wx))(yz) \approx (vw)(x(yz))$$

If $s_n^{\mathrm{ac}}(*)$ reaches its upper bound, so does $s_n^{\mathrm{a}}(*)$, and both upper bounds are reached by SC405 *(see Table 1).*

Proof. Let t be an arbitrary term in $\mathcal{F}_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $|t_0| = 1$. By (i), (ii), and Lemma 2.1, we can assume that $t_i = t_i^{\mathrm{R}<}$ for all $i \in \{1, \ldots, m\}$. We can use (i) to swap $u := [t_0, t_1, \ldots, t_{m-1}]$ and t_m. By Lemma 2.1, the resulting term $t_m u$ is equivalent to $t_m u^{\mathrm{R}<}$. Because $t_m = t_m^{\mathrm{R}<}$, we can apply (i) again to transform this into

$$u^{\mathrm{R}<} t_m^{\mathrm{R}<} = \langle x_{i_1}, \ldots, x_{i_k} \rangle \langle x_{i_{k+1}}, \ldots, x_{i_n} \rangle,$$

where $\{x_{i_1}, \ldots, x_{i_k}\} = \mathrm{var}([t_0, t_1, \ldots, t_{m-1}])$ and $\{x_{i_{k+1}}, \ldots, x_{i_n}\} = \mathrm{var}(t_m)$ with $i_1 < \cdots < i_k$ and $i_{k+1} < \cdots < i_n$. Note that $i_j = j$ for $j = 1, \ldots, n$ if $t \in \mathcal{B}_n$. If $k = 2$, then we can show that

$$(\langle x_{i_1}, x_{i_2} \rangle \langle x_{i_3}, \ldots, x_{i_n} \rangle)^* = (\langle x_1, x_2 \rangle \langle x_3, \ldots, x_n \rangle)^*.$$

We have either $i_1 = 1$ or $i_3 = 1$. If $i_3 = 1$, then we can do the following transformations to make the leftmost index 1.

$$\langle i_1, i_2 \rangle \langle i_3, \ldots, i_n \rangle \xrightarrow{\text{(iii)}} \langle i_1, i_2, i_3 \rangle \langle i_4, \ldots, i_n \rangle \xrightarrow{\text{(i)}} \langle i_4, \ldots, i_n \rangle \langle i_1, i_2, i_3 \rangle$$
$$\xrightarrow{\text{Lemma 2.1}} \langle i_4, \ldots, i_n \rangle \langle i_3, i_1, i_2 \rangle \xrightarrow{\text{(i)}} \langle i_3, i_1, i_2 \rangle \langle i_4, \ldots, i_n \rangle \xrightarrow{\text{(iii)}} \langle i_3, i_1 \rangle \langle i_2, i_4, \ldots, i_n \rangle$$

Here we drop x for ease of notation and represent an application of an identity by an arrow with the label of the identity above it. Similarly, we can make the second leftmost index 2 and then make the rest $3, \ldots, n$. If $3 \leqslant k \leqslant n-2$, then we have

$$\langle i_1, \ldots, i_k \rangle\langle i_{k+1}, \ldots, i_n \rangle \xrightarrow{\text{(iii)}} \langle i_1, i_2 \rangle(\langle i_3, \ldots, i_k \rangle\langle i_{k+1}, \ldots, i_n \rangle) \xrightarrow{\text{Lemma 2.1}} \langle i_1, i_2 \rangle\langle i_3, \ldots, i_n \rangle.$$

Here the application of (iii) uses $v = x_{i_1}$, $w = x_{i_2}$, $x = \langle x_{i_3}, \ldots, x_{i_k} \rangle$, $y = x_{i_{k+1}}$, and $z = \langle x_{i_{k+2}}, \ldots, x_{i_n} \rangle$. Thus t induces the same n-ary operation on $(G, *)$ as one of the following "standard" terms

$$x_{i_1}\langle x_{i_2}, \ldots, x_{i_n} \rangle, \quad \langle x_{i_1}, \ldots, x_{i_{n-1}} \rangle x_{i_n}, \quad \langle x_1, x_2 \rangle\langle x_3, \ldots, x_n \rangle.$$

The first standard term is determined by i_1 since $i_2 < \cdots < i_n$, and the second is determined by i_n since $i_1 < \cdots < i_{n-1}$. Moreover, $x_{i_1}\langle x_{i_2}, \ldots, x_{i_n} \rangle$ and $\langle x_{i_1}, \ldots, x_{i_{n-1}} \rangle x_{i_n}$ induce the same n-ary operation on $(G, *)$ if $i_1 = i_n$ by (i). Thus there are n possibilities in total for the first two standard terms. On the other hand, the last standard term $\langle x_1, x_2 \rangle\langle x_3, \ldots, x_n \rangle$ does not occur when $n = 3$. Thus $s_n^{\text{ac}}(*) \leqslant 3$ when $n = 3$ and $s_n^{\text{ac}}(*) \leqslant n+1$ for $n \geqslant 4$.

If $t \in \mathcal{B}_n$ is a bracketing of $x_1, \ldots, x_n$, then by the above argument, it induces the same n-ary operation on $(G, *)$ as one of $x_1\langle x_2, \ldots, x_n \rangle$, $\langle x_1, \ldots, x_{n-1} \rangle x_n$, or $\langle x_1, x_2 \rangle\langle x_3, \ldots, x_n \rangle$. Thus $s_n^{\text{a}}(*) \leqslant 2$ for $n = 3$ and $s_n^{\text{a}}(*) \leqslant 3$ for $n \geqslant 4$. It is easy to see that the equality holds in the upper bound for $s_n^{\text{a}}(*)$ when the equality holds in the upper bound for $s_n^{\text{ac}}(*)$.

Now we consider SC405. Write an arbitrary term $t \in \mathcal{F}_n$ as $t = (t_L)(t_R)$, where t_L and t_R are linear terms. Also view t as a bracketing of $x_{i_1}, \ldots, x_{i_n}$. We distinguish the following cases on $|t_L|$ and $|t_R|$.

(i) If $|t_L| = 1 < |t_R|$ then t_R^* evaluates to 0 or 1, so $t^*(a_1, \ldots, a_n) = \lfloor a_{i_1}/2 \rfloor$.

(ii) If $|t_L| > 1 = |t_R|$ then $t^*(a_1, \ldots, a_n) = \lfloor a_{i_n}/2 \rfloor$ for the same reason as above.

(iii) If $|t_L| \geqslant 2$ and $|t_R| \geqslant 2$ then t_L^* and t_R^* both evaluate to 0 or 1, so t^* is always zero.

For $n = 3$ we must have (i) or (ii), so $t^*(a_1, a_2, a_3) = \lfloor a_i/2 \rfloor$, where i varies in $\{1, 2, 3\}$. Thus $s_n^{\text{ac}}(*) = 3$ for $n = 3$. For $n \geqslant 4$, we have $t^*(a_1, \ldots, a_n) = \lfloor a_i/2 \rfloor$, where i varies in $\{1, 2, \ldots, n\}$, or $t^* = 0$. Thus $s_n^{\text{ac}}(*) = n+1$ for $n \geqslant 4$. □

The next result involves the 3-element groupoid SC189, which is anti-isomorphic to SC170.

*	0	1	2
0	0	0	0
1	0	2	1
2	0	2	1

SC170

*	0	1	2
0	0	0	0
1	0	2	2
2	0	1	1

SC189

Proposition 3.4. *A groupoid $(G, *)$ satisfying the identities below must have $s_n^{\text{a}}(*) \leqslant 2$ and $s_n^{\text{ac}}(*) \leqslant 2n$ for $n = 3, 4, \ldots$, where the first inequality holds as an equality whenever the second does and both hold for the 2-element groupoid $(\{0, 1\}, *)$ defined by $x * y := x + 1 \pmod 2$ for all $x, y \in \{0, 1\}$ and the 3-element groupoids* SC170 *and* SC189.

$$\text{(i) } x(yz) \approx x(zy), \quad \text{(ii) } (xy)z \approx (xz)y, \quad \text{(iii) } w(x(yz)) \approx w((xy)z),$$

$$\text{(iv) } (wx)(yz) \approx (w(xy))z, \quad \text{(v) } w(x(yz)) \approx ((wx)y)z$$

Proof. Let t be an arbitrary term in $\mathcal{F}_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $|t_0| = 1$. By (i), (iii), and Lemma 2.1, we may assume that $t_i = t_i^{\text{R}<}$ for all $i \in \{1, \ldots, m\}$. By (v), we may assume $m \leqslant 2$. If $m = 1$ then $t^* = \langle x_{i_1}, \ldots, x_{i_n} \rangle^*$ with $i_2 < \cdots < i_n$. If $m = 2$ then we can further use (iv) to obtain $t^* = [x_{i_1}, x_{i_2}, \langle x_{i_3}, \ldots, x_{i_n} \rangle]^*$ and make sure $i_2 < \cdots < i_n$ by (ii) and Lemma 2.1. Thus $s_n^{\text{ac}}(*) \leqslant 2n$.

If t is a bracketing of $x_1, \ldots, x_n$, then $t^* = \langle x_1, \ldots, x_n \rangle^*$ or $t^* = [x_1, x_2, \langle x_3, \ldots, x_n \rangle]^*$ by a similar argument. Thus $s_n^{\text{a}}(*) \leqslant 2$, and the equality must hold if $s_n^{\text{ac}}(*) = 2n$.

It is routine to check that the 2-element groupoid $(\{0, 1\}, *)$ defined by $x * y := x + 1 \pmod 2$ for all $x, y \in \{0, 1\}$ satisfies the identities (i)–(v). It has $s_n^{\text{a}}(*) = 2$ for $n \geqslant 2$ by Csákány and Waldhauser [3, §4.1] and $s_n^{\text{ac}}(*) = 2n$ for $n \geqslant 3$ by our earlier work [6, Example 4.1.2]. It is easy to see that SC189 is obtained from this 2-element groupoid by adding an absorbing element; hence the term operations behave in essentially the same ways in both groupoids. □

Our next result is similar to Proposition 3.4, and we will use leaf depths to generalize them in Section 6. The result here involves two anti-isomorphic groupoids:

$*$	0	1	2
0	1	1	1
1	2	2	2
2	0	0	0

SC3242

$*$	0	1	2
0	1	2	0
1	1	2	0
2	1	2	0

SC3302

Proposition 3.5. *A groupoid $(G,*)$ satisfying the identities below must have*

$$s_n^{\mathrm{a}}(*) \leqslant \begin{cases} 1 & n=1,2 \\ 2 & n=3 \\ 3 & n=4,5,\ldots \end{cases} \quad \text{and} \quad s_n^{\mathrm{ac}}(*) \leqslant \begin{cases} n & n=1,2 \\ 2n & n=3 \\ 3n & n=4,5,\ldots \end{cases}$$

where the first inequality holds as an equality if so does the second and both hold for SC3242 *and the anti-isomorphic* SC3302*.*

$$\text{(i) } x(yz) \approx x(zy), \quad \text{(ii) } w(x(yz)) \approx w((xy)z), \quad \text{(iii) } (xy)z \approx (xz)y,$$

$$\text{(iv) } (wx)(yz) \approx (wy)(xz), \quad \text{(v) } (((vw)x)y)z \approx v(w(x(yz)))$$

Proof. The result is trivial when $n=1,2$; assume $n \geqslant 3$ below. Let t be an arbitrary term in $\mathcal{F}_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $|t_0| = 1$. By (i), (ii), and Lemma 2.1, we may assume that $t_i = t_i^{\mathrm{R}<}$ for all $i \in \{1, \ldots, m\}$. By (iii) and Lemma 2.1, we may assume that $|t_1| \leqslant \cdots \leqslant |t_m|$. If $|t_i| > 1$ for some $i \in \{1, \ldots, m-1\}$, then we apply (iv) to make sure $|t_i| = 1$. Thus we may assume that $|t_1| = \cdots = |t_{m-1}| = 1$. Therefore, t induces the same n-ary operation on $(G,*)$ as $[x_{i_1}, \ldots, x_{i_k}, \langle x_{i_{k+1}}, \ldots, x_{i_n} \rangle]$, where we may further assume that $x_{i_2} < \cdots < x_{i_n}$ by (iii) and (iv) and that $k \in \{1,2,3\}$ by (v). It follows that $s_n^{\mathrm{ac}}(*) \leqslant 2n$ for $n=3$ (in this case $k \in \{1,2\}$) and $s_n^{\mathrm{ac}}(*) \leqslant 3n$ for $n = 4,5,\ldots$.

If $t \in \mathcal{B}_n$ is a bracketing of $x_1 x_2 \cdots x_n$, then we must have $i_1 = 1$ since the above argument does not alter the leftmost variable. Thus $s_n^{\mathrm{a}}(*) \leqslant 2$ for $n=3$ and $s_n^{\mathrm{a}}(*) \leqslant 3$ for $n = 4,5,\ldots$. It is clear that if the upper bound of $s_n^{\mathrm{ac}}(*)$ is reached, so is the upper bound of $s_n^{\mathrm{a}}(*)$.

For SC3242, we have $t^*(a_1, \ldots, a_n) = (a_{i_1} + d) \bmod 3$ whenever the binary tree corresponding to $t \in \mathcal{F}_n$ has leftmost leaf i_1 of left depth d. The number of possibilities for i_1 is n, and the number of possibilities for d (mod 3) is 1 when $n \in \{1,2\}$, 2 when $n=3$, and 3 when $n = 4,5,\ldots$. The proof is now complete. ☐

We next present a family of groupoids whose associative spectrum and ac-spectrum are bounded above by $1,1,2,4,4,4,4,\ldots$ and $1,2,9,32,50,72,98,\ldots$ and show that both upper bounds are reached by the 3-element groupoid SC3162, which is anti-isomorphic to SC2467.

$*$	0	1	2
0	1	0	0
1	1	0	0
2	1	0	1

SC2467

$*$	0	1	2
0	1	1	1
1	0	0	0
2	0	0	1

SC3162

Theorem 3.1. *A groupoid $(G,*)$ satisfying the identities below must have $s_n^{\mathrm{a}}(*) \leqslant 2$ and $s_n^{\mathrm{ac}}(*) \leqslant n^2$ for $n=3$ and $s_n^{\mathrm{a}}(*) \leqslant 4$ and $s_n^{\mathrm{ac}}(*) \leqslant 2n^2$ for $n = 4,5,\ldots$, where the upper bound for $s_n^{\mathrm{a}}(*)$ is reached if the upper bound for $s_n^{\mathrm{ac}}(*)$ is reached and both upper bounds are reached by* SC3162 *and the anti-isomorphic* SC2467*.*

$$\text{(i) } x(yz) \approx x(zy), \quad \text{(ii) } w(x(yz)) \approx w((xy)z), \quad \text{(iii) } ((wx)y)z \approx ((wx)z)y,$$

$$\text{(iv) } (w(xy))z \approx (w(xz))y, \quad \text{(v) } v(w(x(yz))) \approx ((v(wx))y)z, \quad \text{(vi) } (vw)(x(yz)) \approx (((vw)x)y)z$$

Proof. Let t be an arbitrary term in $\mathcal{F}_n$ with leftmost decomposition $t = [x_a, t_1, t_2, \ldots, t_m]$. By (i), (ii), and Lemma 2.1, we may assume that $t_i = t_i^{\mathrm{L}<}$ for all $i \in \{1, \ldots, m\}$. By (vi), we may assume that $m \leqslant 3$. Consequently, t induces the same n-ary operation on $(G,*)$ as one of the following four types of standard terms.

Type 1: $m=1$. Then $t^* = (x_a[x_{b_1}, \ldots, x_{b_{n-1}}])^*$, where $b_1 < \cdots < b_{n-1}$.

Type 2: $m=2$ and $|t_1| = 1$. Then $t^* = ([x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-2}}]])^*$, where $c_1 < \cdots < c_{n-2}$.

Type 3: $m=2$ and $|t_1| \geqslant 2$. Then $t^* = ([x_a, [x_{b_1}, \ldots, x_{b_{n-2}}], x_{b_{n-1}}])^*$, where $b_1 < \cdots < b_{n-1}$, thanks to (iv).

Type 4: $m=3$. We may assume that $|t_1| = 1$ by the identity (v) and that $|t_2| \geqslant |t_3|$ by the identity (iii). If $|t_2| \geqslant |t_3| > 1$, then we can write $t_2 = t_2' x$ for any variable $x \in \mathrm{var}(t_2)$ and switch x with t_3 by (iv). Thus we may also assume $|t_3| = 1$. It follows that $t^* = ([x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-3}}], x_{c_{n-2}}])^*$, where $c_1 < \cdots < c_{n-2}$.

Summing up the possibilities for the above four types of standard terms, we obtain that

$$s_n^{\mathrm{ac}}(*) \leqslant \begin{cases} n + n(n-1) = n^2 & \text{if } n = 3 \\ n + n(n-1) + n + n(n-1) = 2n^2 & \text{if } n = 4, 5, \ldots. \end{cases}$$

If $t \in \mathcal{B}_n$ is a bracketing of $x_1 x_2 \cdots x_n$, then there is only one possibility in each of the above four (two when $n = 3$) cases. This shows that $s_n^{\mathrm{a}}(*) \leqslant 2$ for $n = 3$ and $s_n^{\mathrm{a}}(*) \leqslant 4$ for $n = 4, 5, \ldots$. If $s_n^{\mathrm{ac}}(*) = 2n^2$ then the above four cases must induce distinct terms on $(G, *)$, and thus $s_n^{\mathrm{a}}(*) = 4$.

It is routine to check that SC3162 satisfies the identities (i)–(vi). It remains to show that any two distinct standard terms t and t' in $\mathcal{F}_n$ must induce distinct n-ary operations on SC3162. Assume that t is one of the following, where $b_1 < \cdots < b_{n-1}$ and $c_1 < \cdots < c_{n-2}$.

$$x_a[x_{b_1}, \ldots, x_{b_{n-1}}],\ [x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-2}}]],\ [x_a, [x_{b_1}, \ldots, x_{b_{n-2}}], x_{b_{n-1}}],\ [x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-3}}], x_{c_{n-2}}]$$

Similarly, assume that t' is one of the following, where $b'_1 < \cdots < b'_{n-1}$ and $c'_1 < \cdots < c'_{n-2}$.

$$x_{a'}[x_{b'_1}, \ldots, x_{b'_{n-1}}],\ [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-2}}]],\ [x_{a'}, [x_{b'_1}, \ldots, x_{b'_{n-2}}], x_{b'_{n-1}}],\ [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-3}}], x_{c'_{n-2}}]$$

It is clear that $[0, s_1, \ldots, s_\ell]$ gives 0 if ℓ is even or 1 if ℓ is odd, no matter what $s_1, \ldots, s_\ell$ are. Therefore, we only need to consider the following cases.

Case 1: $t = x_a[x_{b_1}, \ldots, x_{b_{n-1}}]$ and $t' = x_{a'}[x_{b'_1}, \ldots, x_{b'_{n-1}}]$, where $a \neq a'$. We have $h(t) = 0 \neq 1 = h(t')$, where $h(x_a) := 1$ and $h(x) := 0$ for all $x \neq x_a$.

Case 2: $t = x_a[x_{b_1}, \ldots, x_{b_{n-1}}]$ and $t' = [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-3}}], x_{c'_{n-2}}]$. We have $h(t) = 0 \neq 1 = h(t')$, where $h(x_a) = h(x_{a'}) = h(x_{b'}) := 2$ and $h(x) := 0$ for all $x \notin \{x_a, x_{a'}, x_{b'}\}$. Here a may coincide with a' or b'.

Case 3: $t = [x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-2}}]]$ and $t' = [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-2}}]]$, where $(a, b) \neq (a', b')$.
If $a \neq a'$ then $h(t) = 0 \neq 1 = h(t')$, where $h(x_a) := 0$ and $h(x) := 1$ for all $x \neq x_a$.
If $a = a'$ then $b \neq b'$ and $h(t) = 0 \neq 1 = h(t')$, where $h(x_a) = h(x_b) := 2$ and $h(x) := 0$ for all $x \notin \{x_a, x_b\}$.

Case 4: $t = [x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-2}}]]$ and $t' = [x_{a'}, [x_{b'_1}, \ldots, x_{b'_{n-2}}], x_{b'_{n-1}}]$.
If $a \neq a'$ then $h(t) = 0 \neq 1 = h(t')$, where $h(x_a) := 0$ and $h(x) := 1$ for all $x \neq x_a$.
If $a = a'$ then $h(t) = 0 \neq 1 = h(t')$, where $h(x_a) = h(x_b) := 2$ and $h(x) := 0$ for all $x \notin \{x_a, x_b\}$.

Case 5: $t = [x_a, [x_{b_1}, \ldots, x_{b_{n-2}}], x_{b_{n-1}}]$ and $t' = [x_{a'}, [x_{b'_1}, \ldots, x_{b'_{n-2}}], x_{b'_{n-1}}]$, where $a \neq a'$. We have $h(t) = 0 \neq 1 = h(t')$, where $h(x_a) := 0$ and $h(x) := 1$ for all $x \neq x_a$.

Case 6: $t = [x_a, x_b, [x_{c_1}, \ldots, x_{c_{n-3}}], x_{c_{n-2}}]$ and $t' = [x_{a'}, x_{b'}, [x_{c'_1}, \ldots, x_{c'_{n-3}}], x_{c'_{n-2}}]$, where $(a, b) \neq (a', b')$.
If $a \neq a'$ then $h(t) = 1 \neq 0 = h(t')$, where $h(x_a) := 0$ and $h(x) := 1$ for all $x \neq x_a$.
If $a = a'$ then $b \neq b'$ and $h(t) = 1 \neq 0 = h(t')$, where $h(x_a) = h(x_b) := 2$ and $h(x) := 0$ for all $x \notin \{x_a, x_b\}$.

The proof is now complete. □

4. Exponential upper bounds

In this section, we establish some exponential upper bounds for the ac-spectra for a few varieties of groupoids; the respective associative spectra may have linear or exponential upper bounds.

Proposition 4.1. *Every groupoid $(G, *)$ satisfying the identities below must have $s_n^{\mathrm{a}}(*) \leqslant n - 1$ and $s_n^{\mathrm{ac}}(*) \leqslant 2^{n-1} - 1$ for $n = 2, 3, \ldots$, where the first inequality holds as an equality whenever the second does and both equalities hold for* SC1066 *(see Table 1).*

$$\text{(i) } xy \approx yx, \quad \text{(ii) } w(x(yz)) \approx w((xy)z)$$

Proof. Let t be an arbitrary term in $\mathcal{F}_n$ with leftmost decomposition $t = [x_a, t_1, t_2, \ldots, t_m]$. By Lemma 2.1, we may assume that $t_i = t_i^{\mathrm{L}<}$ for all $i \in \{1, \ldots, m\}$. Next, we use (i) to swap $[x_a, t_1, \ldots, t_{m-1}]$ and t_m. Then we transform $[x_a, t_1, \ldots, t_{m-1}]$ to a leftmost bracketing again by Lemma 2.1. It follows that t induces the same n-ary operation on $(G, *)$ as $[x_{j_1}, \ldots, x_{j_k}][x_{j_{k+1}}, \ldots, x_{j_n}]$, where $\{x_{j_1}, \ldots, x_{j_k}\} = \mathrm{var}(t_m)$ and $\{x_{j_{k+1}}, \ldots, x_{j_n}\} = X_n \setminus \mathrm{var}(t_m)$. The order of the elements of either set of variables does not affect t^* by the above, nor does the order of the two sets by (i). Thus $s_n^{\mathrm{ac}}(*)$ is bounded above by $(2^n - 2)/2 = 2^{n-1} - 1$, the number of partitions of $\{1, \ldots, n\}$ into two unordered nonempty blocks.

Restricting the above argument to bracketings of $x_1 x_2 \cdots x_n$ in $\mathcal{B}_n$ instead of full linear terms in $\mathcal{F}_n$, we have the variables in $\mathrm{var}(t_m)$ indexed by larger numbers than the other variables. Thus the partitions of $\{1, \ldots, n\}$

associated with these bracketings have two blocks $\{1,\ldots,k\}$ and $\{k+1,\ldots,n\}$ for some $k \in \{1,\ldots,n-1\}$. It follows that $s_n^{\mathrm{a}}(*) \leqslant n-1$.

If $s_n^{\mathrm{ac}}(*) = 2^{n-1}-1$, then distinct partitions of $\{1,\ldots,n\}$ into two unordered nonempty blocks correspond to distinct n-ary operations on $(G,*)$, and we can restrict this to partitions with two blocks $\{1,\ldots,k\}$ and $\{k+1,\ldots,n\}$ to conclude that $s_n^{\mathrm{a}}(*) = n-1$.

It remains to consider SC1066. Every full linear term $t \in \mathcal{F}_n$ can be written as $t = t_L t_R$. Let $h : X_n \to \{0,1,2\}$ be an assignment. We have that $h(t) = 1$ if and only if $h(t_L) = h(t_R) = 2$ and that $h(t) = 0$ if and only if $h(t_L) \neq 2$ and $h(t_R) \neq 2$. As observed by Csákány and Waldhauser [3], one can show by induction that $h(t) = 2$ if and only if h assigns 2 to an odd number of variables. Thus $h(t)$ is completely determined by how many variables in t_L and t_R take the value 2. In particular, if $s = [x_{i_1},\ldots,x_{i_\ell}][x_{i_{\ell+1}},\ldots,x_{i_n}]$ and $t = [x_{j_1},\ldots,x_{j_k}][x_{j_{k+1}},\ldots,x_{j_n}]$ with $1 \in \{i_1,\ldots,i_\ell\} \cap \{j_1,\ldots,j_k\}$ and $i \in \{i_1,\ldots,i_\ell\} \backslash \{j_1,\ldots,j_k\}$, then $s^* \neq t^*$ since $h(s) = 0 \neq 1 = h(t)$, where $h(x_1) = h(x_i) := 2$ and $h(x) := 0$ for all $x \notin \{x_1, x_i\}$. This implies that $s_n^{\mathrm{ac}}(*) = 2^{n-1}-1$, which in turn implies $s_n^{\mathrm{a}}(*) = n-1$. □

We study another variety of groupoids, for which the associative spectra have the same upper bound $n-1$ as in Proposition 4.1 but the ac-spectra have a different upper bound $1, 2, 7, 29, 146, \ldots$ [13, A185109]. We show that both upper bounds are reached by SC367, which is anti-isomorphic to SC10.

$*$	0	0	0
0	0	0	0
1	0	0	0
2	1	0	0

SC10

$*$	0	1	2
0	0	0	1
1	0	0	0
2	0	0	0

SC367

Proposition 4.2. *A groupoid $(G,*)$ must have $s_n^{\mathrm{a}}(*) \leqslant n-1$ for $n = 2, 3, \ldots$ and*

$$s_n^{\mathrm{ac}}(*) \leqslant n! + \sum_{k=0}^{n-3} n(n-1)\cdots(n-k+1) = n! + \sum_{k=0}^{n-3} k!\binom{n}{k}$$

for $n = 1, 2, \ldots$ if it satisfies the identities below, where the first inequality holds as an equality whenever the second does and both equalities hold for SC367 *and the anti-isomorphic* SC10.

$$\text{(i) } x(yz) \approx x(zy) \approx y(xz), \quad \text{(ii) } w(x(yz)) \approx w((xy)z)$$

Proof. We show that we can transform an arbitrary term $t \in \mathcal{F}_n$, whose leftmost decomposition is $t = [t_0, t_1, \ldots, t_m]$ with $|t_0| = 1$, to a "standard" term of the form $[\langle x_{i_1},\ldots,x_{i_\ell}\rangle, x_{i_{\ell+1}},\ldots,x_{i_n}]$, where $\ell \in \{0,3,4,\ldots,n\}$ and $i_1 < \cdots < i_\ell$.

If $|t_i| = 1$ for all $i = 1,\ldots,m$, then $t = [x_{i_1},\ldots,x_{i_n}]$ is already a standard term with $\ell = 0$. Here $i_1,\ldots,i_n$ form a permutation of $1,\ldots,n$, and we have $n!$ possibilities in this case.

Suppose $|t_j| > 1$ for some j, where j is as large as possible. Then $|t_{j+1}| = \cdots = |t_m| = 1$. We can transform t_j to the rightmost bracketing of its variables in any prescribed order by (i), (ii), and Lemma 2.1, then switch its leftmost variable x_{i_1} with $[t_0, t_1, \ldots, t_{j-1}]$ by (i), and use (i), (ii), and Lemma 2.1 again to transform t to the standard form $[\langle x_{i_1},\ldots,x_{i_\ell}\rangle, x_{i_{\ell+1}},\ldots,x_{i_n}]$, where $i_1,\ldots,i_\ell$ can be in any prescribed order, say the increasing one. There are $n(n-1)\cdots(\ell+1)$ possibilities for $i_{\ell+1},\ldots,i_n$, and we must have $3 \leqslant \ell \leqslant n$ since $|t_j| > 1$.

Summing the numbers of possibilities in the above two cases with $k = n-\ell$ in the second case gives the desired upper bound for $s_n^{\mathrm{ac}}(*)$. Restricting the above argument to bracketings of $x_1x_2\cdots x_n$ in $\mathcal{B}_n$, we obtain standard terms of the form $[\langle x_1,\ldots,x_\ell\rangle, x_{\ell+1},\ldots,x_n]$ with $\ell \in \{0,3,4,\ldots,n\}$. Thus $s_n^{\mathrm{a}}(*) \leqslant n-1$. It is easy to see that if the upper bound for $s_n^{\mathrm{ac}}(*)$ is reached, so is the upper bound for $s_n^{\mathrm{a}}(*)$.

It is clear that SC10 is anti-isomorphic to SC367. The latter satisfies the identities (i) and (ii). It remains to show that $s^* \neq t^*$ whenever s and t are distinct standard terms in $\mathcal{F}_n$. We may assume that $s = [\langle x_{i_1},\ldots,x_{i_\ell}\rangle, x_{i_{\ell+1}},\ldots,x_{i_n}]$ and $t = [\langle x_{j_1},\ldots,x_{j_m}\rangle, x_{j_{m+1}},\ldots,x_{j_n}]$ for some $\ell, m \in \{0,3,4,\ldots,n\}$, where $i_1 < \cdots < i_\ell$, $j_1 < \cdots < j_m$, and $\ell \leqslant m$.

First, assume that $i_k \neq j_k$ for some $k \in \{m+1,\ldots,n\}$. Let k be as large as possible. We have $h(s) = 0 \neq 1 = h(t)$ if $n-k$ is odd or $h(s) = 1 \neq 0 = h(t)$ if $n-k$ is even, where $h(x_{i_k}) = \cdots = h(x_{i_n}) := 2$ and $h(x) := 0$ for all $x \notin \{x_{i_k},\ldots,x_{i_n}\}$.

Next, assume that $i_k = j_k$ for all $k = m+1,\ldots,n$. This implies that $\ell < m$ (otherwise $s = t$). We have $h(s) = 0 \neq 1 = h(t)$ if $n-m$ is odd or $h(s) = 1 \neq 0 = h(t)$ if $n-m$ is even, where $h(x_{i_m}) = \cdots = h(x_{i_n}) := 2$ and $h(x) := 0$ for all $x \notin \{x_{i_m},\ldots,x_{i_n}\}$. □

The upper bounds in the next result are reached by the 3-element groupoid SC2302, which can be viewed as subtraction on a finite field of three elements, or more generally, reached by the subtraction on any commutative group $(G,+)$ of exponent greater than 2 (cf. [6, Example 7.1.4]). It is clear SC2302 is anti-isomorphic to SC2155.

$*$	0	1	2
0	0	1	2
1	2	0	1
2	1	2	0

SC2155

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

SC2302

Proposition 4.3. *A groupoid $(G, *)$ satisfying the identities below must have $s_n^{\mathrm{a}}(*) \leqslant 2^{n-2}$ and $s_n^{\mathrm{ac}}(*) \leqslant 2^n - 2$ for $n = 2, 3, \ldots$, where the second inequality holds as an equality whenever the second does and both equalities hold for the subtraction operation $-$ on any commutative group $(G, +)$ of exponent greater than 2, in particular, for* SC2302 *(hence the anti-isomorphic* SC2155*).*

$$\text{(i) } (xy)z \approx (xz)y, \quad \text{(ii) } x(yz) \approx z(yx), \quad \text{(iii) } w(x(yz)) \approx (w(xy))z.$$

Proof. Let t be an arbitrary term in $\mathcal{F}_n$. We show by induction on $|t|$ that t can be transformed to a "standard" term $[x_{i_1}, \ldots, x_{i_k}, [x_{i_{k+1}}, \ldots, x_{i_n}]]$ for some $k \in \{1, \ldots, n-1\}$, where the sets $\{i_1, i_{k+2}, \ldots, i_n\}$ and $\{i_2, \ldots, i_{k+1}\}$ respectively contain the indices of the leftmost two variables of t and either set of indices can be permuted arbitrarily. We first write $t = [t_0, t_1, \ldots, t_m]$ with $|t_0| = 1$. We may assume that $|t_1| \geqslant |t_2| \geqslant \cdots \geqslant |t_m|$, thanks to the identity (i) and Lemma 2.1. We distinguish some cases below.

Case 1: $m > 1$ and $|t_1| = 1$. Then $t = [x_{i_1}, \ldots, x_{i_n}]$, which is in standard form with $k = n - 1$. The leftmost two variables of t are indexed by $i_1 \in \{i_1\}$ and $i_2 \in \{i_2, \ldots, i_{k+1}\}$, and we can permute $i_2, \ldots, i_{k+1}$ by (i).

Case 2: $m > 1$ and $|t_1| > 1$. We can first apply (iii) repeatedly to transform t to $x_{i_n} t'$, where i_n is the index of the leftmost variable of t and the leftmost variable of t' is the second leftmost variable of t. By the induction hypothesis, we may assume that $t' = [x_{i_{k+1}}, \ldots, x_{i_{n-1}}, [x_{i_1}, \ldots, x_{i_k}]]$, where $i_{k+1}, i_2, \ldots, i_k$ can be permuted in all possible ways and so can be $i_{k+2}, \ldots, i_{n-1}, i_1$, and the leftmost variable of t' is indexed by one of $i_{k+1}, i_2, \ldots, i_k$. We then apply (ii) to switch x_{i_n} with $[x_{i_1}, \ldots, x_{i_k}]$ and get $[x_{i_1}, \ldots, x_{i_k}, [x_{i_{k+1}}, \ldots, x_{i_n}]]$. By (i), we can switch i_n and each of $i_{k+2}, \ldots, i_{n-1}$. Thus we are done for this case.

Case 3: $m = 1$. Similarly to the above case, we can apply the induction hypothesis to t_1 and then use (i) and (ii) to finish the argument for this case.

It follows that $s_n^{\mathrm{ac}}(*)$ is bounded above by the number of nonempty proper subsets of $\{1, \ldots, n\}$, which is clearly $2^n - 2$. Restricting the above argument to $t \in \mathcal{B}_n$, we must have $1 \in \{i_1, i_{k+2}, \ldots, i_n\}$ and $2 \in \{i_2, \ldots, i_{k+1}\}$. Thus $s_n^{\mathrm{a}}(*) \leqslant 2^{n-2}$; see also earlier work [4]. It is easy to see that $s_n^{\mathrm{ac}}(*) = 2^n - 2$ implies $s_n^{\mathrm{a}}(*) = 2^{n-2}$.

The usual subtraction $-$ on $\mathbb{R}$ or $\mathbb{C}$ satisfies the identities (i), (ii), and (iii). We have $s_n^{\mathrm{a}}(-) = 2^{n-2}$ and $s_n^{\mathrm{ac}}(-) = 2^n - 2$ by previous work [6, Example 7.1.4]. The same argument there is also valid for subtraction on any commutative group $(G, +)$ of exponent greater than 2 and in particular, for SC2302. ☐

Remark 4.1. *If $(G, +)$ is a commutative group of exponent at most 2, then the subtraction coincides with addition and $s_n^{\mathrm{ac}}(-) = 1$ for all $n \in \mathbb{N}_+$.*

We provide another variety of groupoids $(G, *)$ with the same associative spectrum upper bound 2^{n-2} as Proposition 4.3 but a different ac-spectrum upper bound $1, 2, 9, 28, 75, 186, \ldots$ [13, A058877]. We show that both upper bounds are reached by two 3-element groupoids SC271 and SC356, which are anti-isomorphic to SC1610 (by $0 \mapsto 1$, $1 \mapsto 2$, $2 \mapsto 0$) and SC2032 (by $0 \mapsto 2$, $1 \mapsto 0$, and $2 \mapsto 1$), respectively.

$*$	0	1	2
0	0	0	0
1	1	1	0
2	2	2	2

SC271

$*$	0	1	2
0	0	0	0
1	2	1	1
2	1	2	2

SC356

$*$	0	1	2
0	0	1	1
1	0	1	2
2	0	1	2

SC1610

$*$	0	1	2
0	0	1	2
1	0	1	2
2	1	0	2

SC2032

Theorem 4.1. *A groupoid $(G, *)$ satisfying the identities below must have $s_n^{\mathrm{a}}(*) \leqslant 2^{n-2}$ and $s_n^{\mathrm{ac}}(*) \leqslant n(2^{n-1} - 1)$ for $n = 2, 3, \ldots$, where the first upper bound is reached whenever the second one is.*

$$\text{(i) } (xy)z \approx (xz)y \quad \text{(ii) } w(x(yz)) \approx w((xy)z) \quad \text{(iii) } (wx)(yz) \approx (wy)(xz)$$

Moreover, both upper bounds are reached for the 3*-element groupoids* SC271 *and* SC356 *(hence the anti-isomorphic* SC1610 *and* SC2032*).*

Proof. We transform an arbitrary term $t \in \mathcal{F}_n$, whose leftmost decomposition is $t = [x_a, t_1, \ldots, t_m]$, to a "standard term" using (i), (ii), (iii), and Lemma 2.1. We may assume that $|t_1| \leqslant |t_2| \leqslant \cdots \leqslant |t_m|$, thanks to the identity (i). Let x_{a_i} be the leftmost variable of t_i for $i = 1, \ldots, m$. If there exists a positive integer $j < m$ such that $|t_j| > 1$, we can use the identity (ii) to transform t_{j+1} to $x_{a_{j+1}} t'_{j+1}$ and then use the identity (iii) to switch t_j and $x_{a_{j+1}}$. Repeating this, we obtain $[x_a, x_{b_1}, \ldots, x_{b_{m-1}}, (x_{b_m} t'_m)]$ from t, where $\{b_1, \ldots, b_m\} = \{a_1, \ldots, a_m\}$.

We can assume that $b_1 < b_2 < \cdots < b_m$, thanks to the identities (i) and (iii). Applying the identity (ii) repeatedly to $x_{b_m} t'_m$ gives $\langle x_{b_m}, x_{c_1}, x_{c_2}, \ldots, x_{c_{n-m-1}} \rangle$. We may assume that $c_1 < c_2 < \cdots < c_{n-m-1}$ by the identities $w(x(yz)) \approx w((xy)z) \approx w((xz)y) \approx w(x(zy))$.

It follows that every $t \in \mathcal{F}_n$ induces the same n-ary operation as a standard term

$$[x_a, x_{b_1}, \ldots, x_{b_{m-1}}]\langle x_{b_m}, x_{c_1}, \ldots, x_{c_{n-m-1}} \rangle,$$

where $b_1 < \cdots < b_m$ and $c_1 < \cdots < c_{n-m-1}$. This implies that $s_n^{\mathrm{ac}}(*) \leqslant n(2^{n-1}-1)$ since there are n possibilities for a and $2^{n-1} - 1$ possibilities for $(b_1, \ldots, b_m)$.

Restricting the above argument to $t \in \mathcal{B}_n$, we must have $a = 1$ and $b_1 = 2$ since $a_1 = 2 \in \{b_1, \ldots, b_m\}$. Thus $s_n^{\mathrm{a}}(*) \leqslant 2^{n-2}$, and it is easy to see that the equality must hold when $s_n^{\mathrm{ac}}(*) = n(2^{n-1} - 1)$.

One can check that SC271 and SC356 both satisfy the identities (i), (ii), and (iii). It remains to show that $h(s) \neq h(t)$ for some assignment $h : X_n \to \{0, 1, 2\}$, where s and t are terms in $\mathcal{F}_n$ corresponding to distinct standard terms

$$[x_a, x_{b_1}, \ldots, x_{b_{m-1}}]\langle x_{b_m}, x_{c_1}, \ldots, x_{c_{n-m-1}} \rangle \neq [x_{a'}, x_{b'_1}, \ldots, x_{b'_{\ell-1}}]\langle x_{b'_\ell}, x_{c'_1}, \ldots, x_{c'_{n-\ell-1}} \rangle.$$

Assume $m \leqslant \ell$, without loss of generality.

First suppose that $a \neq a'$. Define $h(x_a) := 0$ and $h(x) = 2$ for all $x \neq x_a$. For both SC271 and SC356, one can check that $h(s) = 0 \neq h(t)$.

Next, suppose $a = a'$. Then $\{c_1, \ldots, c_{n-m-1}\}$ and $\{c'_1, \ldots, c'_{n-\ell-1}\}$ must be different sets. Suppose some i belongs to the latter but not the former, without loss of generality. We must have $i \in \{b_1, \ldots, b_m\}$.

For SC271, we have $h(s) = 0 \neq 1 = h(t)$, where $h(x_i) := 2$ and $h(x) := 1$ for all $x \neq x_i$.

For SC356, we have $h(s) = 1 \neq 2 = h(t)$, where $h(x_i) := 0$ and $h(x) := 2$ for all $x \neq x_i$. □

5. Upper bounds related to set partitions

In this section, we present a few varieties of groupoids, whose ac-spectra are related to set partitions. Recall that the *restricted Bell number* $B_{n,m}$ counts partitions of the set $\{1, 2, \ldots, n\}$ into unordered nonempty blocks of size at most m [12]; it gives the well-known *Bell number* B_n when $m \geqslant n$. In particular, we have $B_{n,2} = 1$ for $n = 0, 1$ and $B_{n,2} = B_{n-1,2} + (n-1)B_{n-2,2}$ for $n \geqslant 2$; see the sequence A000085 in OEIS [13] for other interpretations and closed formulas for $B_{n,2}$. We also need the following definition by Csákány and Waldhauser [3].

Definition 5.1. *Define a term t to be a* nest *if either $|t| = 1$ (a* trivial nest*) or there exists a term t' together with a variable x such that $t = xt'$ or $t = t'x$, $|t'| = |t| - 1$, and t' is a nest. Each variable in t must be contained in a unique maximal nest, which is simply called a* nest *of t. Every nontrivial nest must have a unique subterm of the form $x_i x_j$, and the variables x_i and x_j are called the* eggs *of this nest.*

Our first result is concerned with a variety of groupoids including the following two 3-element groupoids.

$*$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	1

SC79

$*$	0	1	2
0	0	1	1
1	1	0	0
2	1	0	1

SC1701

Proposition 5.1. *A groupoid $(G, *)$ satisfying the identities below must have $s_n^{\mathrm{a}}(*) \leqslant F_{n+1} - 1$ and $s_n^{\mathrm{ac}}(*) \leqslant B_{n,2} - 1$ for $n = 2, 3, \ldots$, where the first inequality holds as an equality whenever the second does.*

$$\text{(i) } xy \approx yx, \quad \text{(ii) } ((wx)y)z \approx ((wx)z)y$$

Moreover, both upper bounds are reached by SC79 *and* SC1701*.*

Proof. Suppose $s, t \in \mathcal{F}_n$ have the same eggs of nests. We show by induction on n that $s^* = t^*$. Let x_i and x_j be the eggs of a nest of s; they must be the eggs of a nest of t. The case $n = 2$ is trivial; assume $n \geqslant 3$ below. Thanks to the identity (i), we may assume $s = [x_i, x_j, s_1, \ldots, s_\ell]$ and $t = [x_i, x_j, t_1, \ldots, t_m]$. We may also assume that $|s_1| \geqslant \cdots \geqslant |s_\ell|$ and $|t_1| \geqslant \cdots \geqslant |t_m|$ by (ii). Assume $|s_1| \leqslant |t_1|$, without loss of generality.

Case 1: $|t_1| \geqslant |s_1| > 1$. Replacing $x_i x_j$ with a new variable x_0 in both s and t gives full linear terms s' and t' in $n - 1$ variables that share the same eggs of nests. It follows from the induction hypothesis that $(s')^* = (t')^*$, and this implies $s^* = t^*$.

Case 2: $|s_1| = 1$. Then $|s_2| = \cdots = |s_\ell| = 1$ and s has only two eggs x_i and x_j. We must have $|t_1| = 1$ (otherwise t_1 contains eggs different from x_i and x_j) and thus $|t_2| = \cdots = |t_m| = 1$. We can use (ii) to make sure $s_1 = t_1 = x_k$ for some $k \notin \{i, j\}$. Replacing $x_i x_j$ with a new variable x_0 in both s and t gives s' and t' with eggs x_0 and x_k. By the induction hypothesis, we have $(s')^* = (t')^*$. This implies that $s^* = t^*$.

Therefore, $s_n^{\mathrm{ac}}(*)$ is bounded above by $B_{n,2} - 1$, which is the number of partitions of $\{1, \dots, n\}$ into blocks of size one or two with at least one block of size two (since there is at least one nest with two eggs).

Restricting the above argument to bracketings of $x_1, \dots, x_n$, we have $s_n^{\mathrm{a}}(*) \leqslant F_{n+1} - 1$ since the partitions associated with bracketings of $x_1, \dots, x_n$ must have two consecutive integers in each block of size two; see also Csákány and Waldhauser [3, §5.6]. It is easy to see that $s_n^{\mathrm{ac}}(*) = B_{n,2} - 1$ implies $s_n^{\mathrm{a}}(*) = F_{n+1} - 1$.

It is routine to verify that groupoids SC79 and SC1701 satisfy identities (i) and (ii). It remains to verify that if $s, t \in \mathcal{F}_n$ are terms whose eggs of nests are not the same, then s and t induce distinct operations on SC79 and on SC1701. Suppose that x_i and x_j are eggs of a nest in s but not eggs of any nest in t. For SC79, Csákány and Waldhauser [3] observed that $h(s) = 1 \neq 0 = h(t)$, where $h(x_i) = h(x_j) := 2$ and $h(x) := 1$ for all $x \notin \{x_i, x_j\}$. For SC1701, we have $h(s) = 1 \neq 0 = h(t)$, where $h(x_i) = h(x_j) := 2$ and $h(x) := 0$ for all $x \notin \{x_i, x_j\}$. Thus $s_n^{\mathrm{ac}}(*) = B_{n,2} - 1$ and $s_n^{\mathrm{a}}(*) = F_{n+1} - 1$ for SC79 and SC1701. □

A set partition is *rooted* if it has a distinguished singleton block called the *root*. The number of rooted partitions of $\{1, 2, \dots, n\}$ is $nB_{n-1} = 1, 2, 6, 20, 75, 312, \dots$ [13, A052889]. We show below that this number is the upper bound for the ac-spectra of a variety of groupoids and can be attained by the 3-element groupoids SC41 and SC96. The Cayley tables of these two groupoids together with the anti-isomorphic groupoids SC398 and SC1069 are given below.

$*$	0	1	2
0	0	0	0
1	0	0	1
2	1	1	2

SC41

$*$	0	1	2
0	0	0	0
1	0	1	0
2	2	0	2

SC96

$*$	0	1	2
0	0	0	1
1	0	0	1
2	0	1	2

SC398

$*$	0	1	2
0	0	0	2
1	0	1	0
2	0	0	2

SC1069

Theorem 5.1. *A groupoid $(G, *)$ satisfying the identities below must have $s_n^{\mathrm{a}}(*) \leqslant 2^{n-2}$ for $n = 2, 3, \dots$ and $s_n^{\mathrm{ac}}(*) \leqslant nB_{n-1}$ for $n = 1, 2, \dots$, where the second inequality holds as an equality whenever the second does.*

$$\text{(i) } x(yz) \approx x(zy), \quad \text{(ii) } (xy)z \approx (xz)y.$$

Moreover, both upper bounds are reached by SC41 *and* SC96 (*hence the anti-isomorphic* SC398 *and* SC1069).

Proof. Let t be an arbitrary term in $\mathcal{F}_n$ with leftmost decomposition $t = [x_a, t_1, \dots, t_m]$. Define a rooted set partition of $\{1, 2, \dots, n\}$ associated with t: we have a block consisting of the indices of the variables in t_j for all $j = 1, 2, \dots, m$ together with a singleton block $\{a\}$ that is the root of this partition. By (i), (ii), and Lemma 2.1, t induces on $(G, *)$ the same term operation as $[x_a, t_{\sigma(1)}^{\mathrm{L}<}, \dots, t_{\sigma(m)}^{\mathrm{L}<}]$ for any permutation $\sigma \in \mathfrak{S}_m$. It follows that terms in $\mathcal{F}_n$ associated with the same rooted partition must induce the same n-ary operation on $(G, *)$. Thus $s_n^{\mathrm{ac}}(*) \leqslant nB_{n-1}$.

The rooted set partition associated with a bracketing of $x_1 * \cdots * x_n$ must have $\{1\}$ as its root and the other blocks are intervals. The number of such “interval partitions” can be found by counting the number of ways of inserting bars into the $n-2$ spaces between $2, \dots, n$. Thus $s_n^{\mathrm{a}}(*) \leqslant 2^{n-2}$.

If $s_n^{\mathrm{ac}}(*) = nB_{n-1}$ for $n \geqslant 1$, then $s^* \neq t^*$ whenever $s, t \in \mathcal{F}_n$ are associated with distinct rooted set partitions, and restricting this to bracketings of $x_1 * \cdots * x_n$ gives $s_n^{\mathrm{a}}(*) = 2^{n-2}$.

It is routine to check that SC41 and SC96 both satisfy the identities (i) and (ii). It remains to show that $s^* \neq t^*$ whenever s and t are terms in $\mathcal{F}_n$ associated with distinct rooted set partitions. Suppose $s = [x_a, s_1, \dots, s_\ell]$ and $t = [x_b, t_1, \dots, t_m]$, where $s_1, \dots, s_\ell$ and $t_1, \dots, t_m$ are ordered according to the smallest index of the variables they contain. If $a \neq b$ then $s^* \neq t^*$ since

- $h(s) = 0 \neq 1 = h(t)$ if $(\{0, 1, 2\}, *) =$ SC41, $h(x_a) = 0$ and $h(x_i) = 2$ for all $i \neq a$, and
- $h(s) = 0 \neq 2 = h(t)$ if $(\{0, 1, 2\}, *) =$ SC96, $h(x_a) = 0$ and $h(x_i) = 2$ for all $i \neq a$.

Assume $a = b$ below. Let j be the smallest integer such that s_j and t_j do not contain the same set of variables. The least index c of the variables of s_j must agree with that of t_j. There exists another variable x_d in exactly one of s_j and t_j, say the former. Then x_d is in t_k for some $k > j$. We have

- $h(s) = 1 \neq 0 = h(t)$ if $(\{0, 1, 2\}, *) =$ SC41, $h(x_c) = h(x_d) = 0$, and $h(x_i) = 2$ for all $i \notin \{c, d\}$, and
- $h(s) = 2 \neq 0 = h(t)$ if $(\{0, 1, 2\}, *) =$ SC96, $h(x_a) = h(x_c) = 2$, and $h(x_i) = 1$ for all $i \notin \{a, c\}$.

Thus $s^* \neq t^*$. □

Next, we provide an ordered version of Theorem 5.1 that has the same associative spectrum upper bound but a different ac-spectrum upper bound. Recall that the *ordered Bell number* or *Fubini number* B'_n counts ordered partitions of the set $\{1,2,\ldots,n\}$ [13, A000670]. The number of rooted ordered set partitions of $\{1,\ldots,n\}$ is $nB'_{n-1} = 1,2,9,52,375,\ldots$ [13, A052882]. We show that nB'_{n-1} is also the upper bound for the ac-spectra of a variety of groupoids and can be reached by the 3-element groupoids SC262, SC1812, and SC2446, which are anti-isomorphic to SC1441 (by $0 \mapsto 2$, $1 \mapsto 0$, and $2 \mapsto 1$), SC1793 and SC2430, respectively.

SC262

$*$	0	1	2
0	0	0	0
1	1	1	0
2	1	1	2

SC1441

$*$	0	1	2
0	0	0	2
1	2	1	2
2	0	0	2

SC1793

$*$	0	1	2
0	0	1	1
1	1	2	1
2	1	2	1

SC1812

$*$	0	1	2
0	0	1	1
1	1	2	2
2	1	1	1

SC2430

$*$	0	1	2
0	1	0	0
1	0	2	1
2	0	2	1

SC2446

$*$	0	1	2
0	1	0	0
1	0	2	2
2	0	1	1

Theorem 5.2. *A groupoid $(G,*)$ satisfying the identities below must have $s^{\mathrm{a}}_n(*) \leqslant 2^{n-2}$ for $n = 2,3,\ldots$ and $s^{\mathrm{ac}}_n(*) \leqslant nB'_{n-1}$ for $n = 1,2,\ldots$, where the first inequality holds as an equality whenever the second does.*

$$\text{(i) } x(yz) \approx x(zy), \quad \text{(ii) } w(x(yz)) \approx w((xy)z)$$

Moreover, both equalities hold for SC262*,* SC1812*, and* SC2446 (*hence the anti-isomorphic* SC1441*,* SC1793*, and* SC2430).

Proof. By (i), (ii), and Lemma 2.1, we can transform an arbitrary term $t \in \mathcal{F}_n$ with leftmost decomposition $t = [t_0, t_1, \ldots, t_m]$, where $|t_0| = 1$, to $[t_0, t_1^{\mathrm{L}<}, \ldots, t_m^{\mathrm{L}<}]$. Thus terms in $\mathcal{F}_n$ induce the same n-ary operation if they are associated with the same rooted ordered set partitions. It follows that $s^{\mathrm{ac}}_n(*) \leqslant nB'_{n-1}$. Restricting the above argument to $\mathcal{B}_n$ gives $s^{\mathrm{a}}_n(*) \leqslant 2^{n-2}$, where the equality holds if $s^{\mathrm{ac}}_n(*) = nB'_{n-1}$.

It is routine to check that SC262, SC1812, and SC2446 all satisfy the identities (i) and (ii). It remains to show that $s^* \neq t^*$ whenever $s, t \in \mathcal{F}_n$ are associated with distinct rooted ordered set partitions of $\{1,2,\ldots,n\}$. We can write $s = [x_a, s_1, \ldots, s_\ell]$ and $t = [x_b, t_1, \ldots, t_m]$. If $a \neq b$ then $s^* \neq t^*$ by the following:

- For SC262, we have $h(s) = 1 \neq 0 = h(t)$, where $h(x_a) := 1$ and $h(x) := 0$ for all $x \neq x_a$.
- For SC1812 and SC2446, one of $h(s)$ and $h(t)$ is 1 and the other is 2, where $h(x) := 1$ for all x if ℓ and m have different parities or $h(x_a) := 1$ and $h(x) := 2$ for all $x \neq x_a$ otherwise.

Assume $a = b$ below. Let j be the smallest integer such that $\mathrm{var}(s_j) \neq \mathrm{var}(t_j)$.

For SC262, we distinguish two cases.

Case 1: $\mathrm{var}(t_i) \nsubseteq \mathrm{var}(s_j)$ for all i. Define $h(x) := 2$ for all $x \in \{x_a\} \cup \mathrm{var}(s_j)$ and $h(x) := 0$ for all $x \notin \mathrm{var}(s_j)$. Then $h(x_a) = 2$, $h(s_j) = 2$, $h(s_i) = 0$ for all $i \neq j$, and $h(t_i) \in \{0,1\}$ for all i. One can check that $h(s) = 1 \neq 0 = h(t)$ when $j = 1$ and $h(s) = 0 \neq 1 = h(t)$ when $j > 1$.

Case 2: $\mathrm{var}(t_k) \subseteq \mathrm{var}(s_j)$ for some k. If $\mathrm{var}(t_k) \subsetneq \mathrm{var}(s_j)$, then we are back to Case 1 by switching s and t and using t_k instead of s_j, since $\mathrm{var}(s_i) \nsubseteq \mathrm{var}(t_k)$ for all i. Thus we may assume that $\mathrm{var}(s_j) = \mathrm{var}(t_k)$, which implies $j < k$ since $\mathrm{var}(s_i) = \mathrm{var}(t_i)$ for all $i < j$. Define

$$h(x) := \begin{cases} 2, & \text{if } x \in \{x_a\} \cup \mathrm{var}(s_1) \cup \cdots \cup \mathrm{var}(s_j) = \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_{j-1}) \cup \mathrm{var}(t_k); \\ 0, & \text{if } x \notin \{x_a\} \cup \mathrm{var}(s_1) \cup \cdots \cup \mathrm{var}(s_j) = \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_{j-1}) \cup \mathrm{var}(t_k). \end{cases}$$

We have $h(s_1) = \cdots = h(s_j) = 2$, $h(s_i) = 0$ for all $i = j+1, \ldots, \ell$, and thus $h(s) = 1$. On the other hand, we have $h(t_1) = \cdots = h(t_{j-1}) = h(t_k) = 2$, $h(t_i) = 0$ for all $i \in \{j, \ldots, m\} \backslash \{k\}$, and thus $h(t) = 0 \neq h(s)$.

For SC1812, we may assume that ℓ and m have the same parity by the all-1 substitution as discussed earlier. We distinguish some cases below.

Case 1: $\mathrm{var}(t_i) \nsubseteq \mathrm{var}(s_j)$ for all i. We further distinguish two subcases below.

- Suppose that j is odd. Define $h(x) := 0$ for all $x \in \mathrm{var}(s_j)$ and $h(x) := 1$ for all $x \notin \mathrm{var}(s_j)$. Then $h(x_a) = 1$, $h(s_j) = 0$, $h(s_i) \in \{1,2\}$ for all $i \neq j$, and $h(t_i) \in \{1,2\}$ for all i. One can check that $h(s) = 1$ if ℓ is odd or $h(s) = 2$ if ℓ is even. On the other hand, we have $h(t) = 1$ if m is even or $h(t) = 2$ otherwise. Since ℓ and m have the same parity, it follows that $h(s) \neq h(t)$.
- Suppose that j is even. Defined by $h(x) := 0$ for all $x \in \mathrm{var}(s_j)$ and $h(x) := 2$ for all $x \notin \mathrm{var}(s_j)$. Then $h(x_a) = 1$, $h(s_j) = 0$, $h(s_i) \in \{1,2\}$ for all $i \neq j$, and $h(t_i) \in \{1,2\}$ for all i. One can check that $h(s) = 1$ if ℓ is even or $h(s) = 2$ if ℓ is odd. On the other hand, we have $h(t) = 1$ if m is odd or $h(t) = 2$ if m is even. Since ℓ and m have the same parity, we must have $h(s) \neq h(t)$.

Case 2: $\mathrm{var}(t_k) \subseteq \mathrm{var}(s_j)$ for some k. If $\mathrm{var}(t_k) \subsetneq \mathrm{var}(s_j)$, then we are back to Case 1 by switching s and t and using t_k instead of s_j, since $\mathrm{var}(s_i) \nsubseteq \mathrm{var}(t_k)$ for all i. Thus we may assume that $\mathrm{var}(s_j) = \mathrm{var}(t_k)$, which implies $j < k$. We further distinguish two subcases below.

- Suppose that j and k have different parities. Define $h(x) := 0$ for all $x \in \mathrm{var}(s_j)$ and $h(x) := 2$ for all $x \notin \mathrm{var}(s_j)$. Then $h(x_a) = 2$, $h(s_j) = h(t_k) = 0$, $h(s_i) \in \{1, 2\}$ for all $i \neq j$, and $h(t_i) \in \{1, 2\}$ for all $i \neq k$. One can check that $h(s) = 1$ if j has the same parity as ℓ or $h(s) = 2$ otherwise. Similarly, $h(t) = 1$ if k has the same parity as m or $h(t) = 2$ otherwise. Since ℓ and m have the same parity, we must have $h(s) \neq h(t)$.

- Suppose that j and k have the same parity. Define
$$h(x) := \begin{cases} 0, & \text{if } x \in \{x_a\} \cup \mathrm{var}(s_1) \cup \cdots \cup \mathrm{var}(s_j) = \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_{j-1}) \cup \mathrm{var}(t_k); \\ 1, & \text{if } x \notin \{x_a\} \cup \mathrm{var}(s_1) \cup \cdots \cup \mathrm{var}(s_j) = \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_{j-1}) \cup \mathrm{var}(t_k). \end{cases}$$
Then $h(x_a) = h(s_j) = h(t_k) = 0$, $h(s_i) = h(t_i) = 0$ for all $i = 1, \ldots, j-1$, $h(s_i) \in \{1, 2\}$ for all $i = j+1, \ldots, \ell$, and $h(t_i) \in \{1, 2\}$ for all $i \in \{j, \ldots, m\} \setminus \{k\}$. One can check that $h(s) = 1$ if j and ℓ have different parities or $h(s) = 2$ otherwise (note that $j < \ell$). Similarly, $h(t) = 1$ if k and m have the same parity or $h(t) = 2$ otherwise. Since ℓ and m have the same parity, we must have $h(s) \neq h(t)$.

For SC2446, we may again assume that ℓ and m have the same parity by the all-1 substitution. There exists a variable x_c in exactly one of s_j and t_j, say the former. Then x_c is in t_k for some $k > j$. We distinguish two cases below.

Case 1: j and k have different parities. Define $h(x_a) = h(x_c) := 0$ and $h(x) := 1$ for all $x \notin \{x_a, x_c\}$. We have $h(s_j) = 0$ and $h(s_i) \in \{1, 2\}$ for all $i \neq j$. Thus $h(s) = 1$ if j has the same parity as ℓ or $h(s) = 2$ otherwise. Similarly, we have $h(t_k) = 0$ and $h(t_i) \in \{1, 2\}$ for all $i \neq k$. Thus $h(t) = 1$ if k has the same parity as m or $h(t) = 2$ otherwise. Then $h(s) \neq h(t)$ since ℓ and m have the same parity.

Case 2: j and k have the same parity. Pick any variable x_d in t_{k-1}, which must be in $s_{j'}$ for some $j' \geqslant j$. The argument in the above paragraph is valid for j' and $k-1$ if they have different parities. Otherwise j' and k must have different parities, and it follows that $j' > j$. Define $h(x_c) = h(x_d) := 0$ and $h(x) := 1$ for all $x \notin \{x_c, x_d\}$. We have $h(s_j) = h(s_{j'}) = 0$ and $h(s_i) \in \{1, 2\}$ for all $i \notin \{j, j'\}$. Thus $h(s) = 1$ if j' has the same parity as ℓ, or $h(s) = 2$ otherwise. Similarly, we have $h(t_{k-1}) = h(t_k) = 0$ and $h(s_i) \in \{1, 2\}$ for all $i \notin \{k-1, k\}$. Thus $h(t) = 1$ if k has the same parity as m, or $h(t) = 2$ otherwise. Then $h(s) \neq h(t)$ since ℓ and m have the same parity but j' and k have different parities. ☐

6. Congruence on depths

In this section we discuss the natural occurrence of leaf depths in the study of associative and ac-spectra of groupoids and how it can help us generalize some of our results.

Using both identities and the left/right depth, Hein and the first author [4] determined the associative spectrum of a generalization of addition and subtraction to be the *modular Catalan number*

$$C_{k,n} := \sum_{0 \leqslant j \leqslant (n-1)/k} \frac{(-1)^j}{n} \binom{n}{j} \binom{2n-jk}{n+1},$$

and we determined its ac-spectrum in our previous work [6]. These results are rephrased below to include Proposition 4.3 as a special case (using right depth instead of identities).

Theorem 6.1 ([4,6]). *Let $(G, *)$ be a groupoid such that for all $s, t \in \mathcal{F}_n$, we have $s^* = t^*$ whenever $\rho_i(s) \equiv \rho_i(t) \pmod{k}$ for $i = 1, \ldots, n$. Then $s_n^{\mathrm{a}}(*) \leqslant C_{k,n-1}$ and*

$$s_n^{\mathrm{ac}}(*) \leqslant k!S(n,k) + n \sum_{0 \leqslant i \leqslant k-2} i!S(n-1,i)$$

for $n = 1, 2, \ldots$, where the first equality holds as an equality if the second one does. Moreover, both upper bounds are reached if "whenever" can be replaced with "if and only if" in the above condition. In particular, both upper bounds are attained by $(\mathbb{C}, *)$, where $a * b := a + e^{2\pi i/k} b$ for all $a, b \in \mathbb{C}$.

Now we use the left depth to generalize Proposition 3.4 and Proposition 3.5 as follows.

Theorem 6.2. *Let $(G, *)$ be a groupoid such that for all $s, t \in \mathcal{F}_n$, we have $s^* = t^*$ whenever s and t have the same leftmost variable x_i, whose left depths in s and t are congruent modulo k. Then $s_n^{\mathrm{a}}(*) \leqslant k$ and $s_n^{\mathrm{ac}}(*) \leqslant kn$ for $n = k+1, \ldots$, where the first inequality holds as an equality if the second does. Moreover, both upper bounds are reached if "whenever" can be replaced with "if and only if" in the above condition.*

Proof. First, suppose that $s^* = t^*$ whenever s and t have the same leftmost variable x_i and the left depths of x_i in s and t are congruent modulo k. Then every term in $\mathcal{F}_n$ induces the same n-ary operation on $(G, *)$ as a standard term $[x_i, x_{i_1}, \dots, x_{i_m}, \langle x_{i_{m+1}}, \dots, x_{i_{n-1}} \rangle]$, where $i_1 < \cdots < i_{n-1}$ and $m \in \{0, \dots, k-1\}$. The above standard term is determined by x_i and m, for which there are n and k possibilities, respectively (the latter requires $n \geqslant k+1$). Thus $s_n^{\mathrm{ac}}(*) \leqslant kn$. Similarly, the standard term of each bracketing in $\mathcal{B}_n$ must begin with x_1. Thus $s_n^{\mathrm{a}}(*) \leqslant k$. It is easy to see that $s_n^{\mathrm{ac}}(*) = kn$ implies $s_n^{\mathrm{a}}(*) = k$.

Now suppose that $s^* = t^*$ if and only if s and t have the same leftmost variable x_i and the left depths of x_i in s and t are congruent modulo k. The "only if" part implies that $s^* \neq t^*$ if s and t correspond to different standard terms. Thus $s_n^{\mathrm{a}}(*) = k$ and $s_n^{\mathrm{ac}}(*) = kn$. □

Remark 6.1. *Hein and the first author [4] observed that the congruence relation modulo k on the left depths of the bracketings in $\mathcal{B}_n$ is characterized by the identity $s_0[s_1, \dots, s_{k+1}] \approx [s_0, s_1, \dots, s_{k+1}]$ and showed that $C_{k,n-1}$ is the number of terms in $\mathcal{B}_n$ avoiding subterms of the form $s_0[s_1, \dots, s_{k+1}]$. We also have $s_n^{\mathrm{a}}(*) \leqslant C_{k,n-1}$ for a groupoid $(G, *)$ satisfying a different identity*

$$s_0[s_1, \dots, s_{k+1}] \approx s_0(s_1[s_2, \dots, s_{k+1}]) \tag{1}$$

since we can still use this identity to transform every bracketing in $\mathcal{B}_n$ to some bracketing in $\mathcal{B}_n$ that avoids subterms of the form $s_0[s_1, \dots, s_{k+1}]$. Although not needed for the proof of the upper bound $s_n^{\mathrm{a}}(*) \leqslant C_{k,n-1}$, we can even show that distinct bracketings $t, t' \in \mathcal{B}_n$ both avoiding $s_0[s_1, \dots, s_{k+1}]$ cannot be obtained from each other by the identity (1), using the technique due to Hein and the first author [4]. In fact, we know that t and t' correspond to two binary trees with n leaves labeled $1, \dots, n$ from left to right, which in turn correspond to two rooted plane trees *T and T' with n vertices labeled $1, \dots, n$ in the* preorder *by contracting each northeast southwest "long edge" in the drawings of t and t'. If t can be obtained from t' by the identity (1), then a non-root vertex in T must have its* degree *(the number of children) less than k and congruent to the degree of the vertex with the same label in T' modulo $k-1$, and the leaves (degree-zero vertices) in T must correspond to the leaves in T'. Thus the degrees of the vertices of T must agree with those of T', and this forces $T = T'$.*

For $k = 3$, we suspect that $s_n^{\mathrm{a}}(*) = C_{k,n-1}$ holds for SC64*, which is anti-isomorphic to* SC399*.*

$*$	0	1	2
0	0	0	0
1	0	0	2
2	1	1	0

SC64

$*$	0	1	2
0	0	0	1
1	0	0	1
2	0	2	0

SC399

In fact, our computations show that the initial terms of the associative spectrum and ac-spectrum of SC64 *are* $1, 1, 2, 5, 13, 35, 96, 267$ *and* $1, 2, 12, 84, 710$*, respectively; the former sequence coincides with $C_{3,n-1}$ while the latter differs from the upper bound of $s_n^{\mathrm{ac}}(*)$ for $k = 3$ in Theorem 6.1, whose initial terms are* 1, 2, 9, 40, 155, 546, 1813, 5804, 18159*. One can check that* SC64 *satisfies at least the four identities below.*

$$w(x(yz)) \approx w(y(xz)), \quad w((xy)z) \approx w((zy)x), \quad ((wx)y)z \approx ((wz)y)x, \quad v(w((xy)z)) \approx v(((wx)y)z)$$

But these identities seem unrelated to the left/right depth modulo $k = 3$.

The first author, Mickey, and Xu [7] used the depth to find the associative spectrum of the *double minus* operation $a * b := -a - b$, and we determined the ac-spectrum of this operation in previous work [6]. Both proofs are valid for any field with at least three elements, giving the following result.

Theorem 6.3 ([7]). *Suppose that two terms $s, t \in \mathcal{F}_n$ induce the same n-ary operation on a groupoid $(G, *)$ whenever $d_i(s) \equiv d_i(t) \pmod 2$ for $i = 1, \dots, n$. Then $s_n^{\mathrm{a}}(*) \leqslant \lfloor 2^n/3 \rfloor$ and $s_n^{\mathrm{ac}}(*) \leqslant (2^n - (-1)^n)/3$ for $n = 1, 2, \dots$, where the first equality holds as an equality if the second one does. Moreover, both upper bounds are reached if "whenever" can be replaced with "if and only if" in the above condition. In particular, both upper bounds are achieved by the double minus operation on any field with at least three elements.*

The two upper bounds in the above theorem are both well studied [13, A000975, A001045] from many other perspectives; the latter is known as the *Jacobsthal sequence*. The double minus operation on a field of three elements is actually the 3-element groupoid SC2346.

$*$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

SC2346

To generalize the above theorem, one could use a primitive root of unity $\omega := e^{2\pi i/k}$ to define an operation $a * b := \omega a + \omega b$ on the field of complex numbers, which reduces to the double minus operation when $k = 2$; for $k \geqslant 3$, the n-th term of the associative spectrum was shown in [10] to coincide with the number of equivalence classes of the equivalence relation on n-leaf binary trees that relates two trees if the depths of corresponding leaves are congruent modulo k. Closed formulas for the associative spectrum and the ac-spectrum of this operation are yet to be determined.

7. Questions and remarks

We have some more questions other than those in the last section. Our computations suggest that a majority of the 3330 non-isomorphic 3-element groupoids have their ac-spectrum reaching the upper bound $n!C_{n-1}$ and thus have their associative spectrum reaching the upper bound C_{n-1}. Some other 3-element groupoids have smaller spectra, including those given earlier in this paper as examples for various upper bounds to be sharp. We also have computational data on the spectra of several other 3-element groupoids but do not have any general result on them.

For instance, our computations show that the first several terms of the associative spectrum and ac-spectrum of each of the following groupoids are $1, 1, 2, 5, 12, 28, 65, 151, 351$ and $1, 2, 12, 96, 880$, respectively; the former agrees with the initial terms of a trisection of the Padovan sequence [13, A034943].

*	0	1	2
0	0	0	0
1	1	1	0
2	1	0	1

SC258

*	0	1	2
0	0	0	1
1	1	1	0
2	1	0	0

SC685

*	0	1	2
0	0	1	1
1	0	1	0
2	0	0	1

SC1594

*	0	1	2
0	0	1	1
1	0	1	0
2	1	0	0

SC1600

It is clear that SC258 and SC685 are anti-isomorphic to SC1594, SC1600, respectively. One can check that SC258 and SC685 both satisfy at least the following identities.

$$(wx)(yz) \approx (wx)(zy), \quad ((wx)y)z \approx ((wx)z)y, \quad (vw)(x(yz)) \approx (vw)((xy)z), \quad v((wx)(yz)) \approx (v(wx))(yz)$$

Next, consider the following 3-element groupoids.

*	0	1	2
0	0	0	2
1	2	0	2
2	2	2	0

SC1414

*	0	1	2
0	0	0	2
1	2	2	0
2	2	0	0

SC1477

*	0	1	2
0	0	1	1
1	1	0	0
2	0	0	1

SC1693

*	0	1	2
0	0	1	1
1	1	0	1
2	0	1	0

SC1717

There is an anti-isomorphism between SC1414 and SC1717 and between SC1477 and SC1693 by swapping 1 and 2. It is routine to check that SC1414 and SC1693 both satisfy the identities $(wx)(yz) \approx (yz)(wx)$ and $((wx)y)z \approx ((wx)z)y$. Computations show that the first several terms of its associative spectrum and ac-spectrum are $1, 1, 2, 5, 13, 35, 97, 275, 794, 2327$ and $1, 2, 12, 96, 980$; the former matches with the initial terms of a *generalized Catalan number* [13, A025242], which counts Dyck paths of length $2n$ avoiding $UUDD$.

Computations also show that the first several terms of the associative spectrum and ac-spectrum of the following two anti-isomorphic groupoids are $1, 1, 2, 5, 14, 42, 132, 429, 1430$ and $1, 2, 12, 108, 1340$; the former agrees with C_{n-1} while the latter is less than $n!C_{n-1}$.

*	0	1	2
0	0	0	0
1	1	0	1
2	1	1	1

SC229

*	0	1	2
0	0	1	1
1	0	0	1
2	0	1	1

SC1553

One can check that SC229 satisfies the identity $((wx)y)z \approx ((wx)z)y$.

It would be nice if the associative spectra and ac-spectra of the above 3-element groupoids (or even better, groupoids satisfying the same identities as the above groupoids) could be determined.

Another question is about the arithmetic mean on $\mathbb{R}$. Csákány and Waldhauser [3] showed that its associative spectrum is C_{n-1}. In previous work [6], we showed that its ac-spectrum is the number of ways to write 1 as an ordered sum of n powers of 2 [13, A007178]. It would be interesting to find the identities that could be used to characterize all the groupoids whose associative spectra and ac-spectra are bounded by the above and if possible, find a 3-element groupoid to achieve the upper bounds.

Lastly, we provide a generalization of a result in our earlier work [6], which asserts that an associative groupoid $(G, *)$ must have $s_n^{\mathrm{ac}}(*) \leqslant n!$ and this upper bound holds as an equality if $(G, *)$ is noncommutative and has an identity element.

Theorem 7.1. *For any groupoid $(G, *)$, we have $s_n^{\mathrm{ac}}(*) \leqslant n! \cdot s_n^{\mathrm{a}}(*)$. Moreover, this inequality holds as an equality if $(G, *)$ is noncommutative and has an identity element.*

Proof. For a bracketing $t \in \mathcal{B}_n$ and a permutation $\sigma \in \mathfrak{S}_n$, let t denote the full linear term obtained by replacing the variable x_i with $x_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$. Consider two full linear terms in $\mathcal{F}_n$; they can be written as s_σ and t_τ, where $s, t \in \mathcal{B}_n$ and $\sigma, \tau \in \mathfrak{S}_n$. It is clear that if $\sigma = \tau$, then $(s_\sigma)^* = (t_\tau)^*$ if and only if $s^* = t^*$. The inequality $s_n^{\mathrm{ac}}(*) \leqslant n! \cdot s_n^{\mathrm{a}}(*)$ follows immediately from this fact.

Assume now that $(G, *)$ is noncommutative and has a neutral element 0. Then there are elements $a, b \in G$ such that $a * b \neq b * a$. Assume that $\sigma \neq \tau$. Then there exist $i, j \in \{1, \ldots, n\}$ such that $\sigma^{-1}(i) < \sigma^{-1}(j)$ and $\tau^{-1}(i) > \tau^{-1}(j)$. Let $h\colon X_n \to G$ be the assignment $x_i \mapsto a$, $x_j \mapsto b$ and $x \mapsto 0$ for all $x \in X_n \backslash \{x_i, x_j\}$. It is easy to see that $h(s_\sigma) = a * b$ and $h(t_\tau) = b * a$; hence $(s_\sigma)^* \neq (t_\tau)^*$. We conclude that $(s_\sigma)^* = (t_\tau)^*$ if and only if $s^* = t^*$ and $\sigma = \tau$, and the equality $s_n^{\mathrm{ac}}(*) = n! \cdot s_n^{\mathrm{a}}(*)$ follows. □

It would be nice to find a sufficient and necessary condition for the upper bound in Theorem 7.1 to hold as an equality.

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