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Subword Patterns in Smooth Words

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ABSTRACT: Let $\mathcal{U}_n$ denote the set of integral sequences $u_1 \cdots u_n$ such that $|u_{i+1} - u_i| \leq 1$ for $1 \leq i \leq n-1$, with $u_1 = 1$, which are referred to as *smooth* words. In this paper, we enumerate the members of $\mathcal{U}_n$ according to the number of occurrences of any subword pattern of length two or three. We consider, more generally, the joint distribution on $\mathcal{U}_n$ of several pairs (and one trio) of subword patterns, together with the final letter statistic, and compute the generating function of this joint distribution in each case. We make use of the *kernel method* to solve the functional equation satisfied by the generating function. In some instances, one or more auxiliary generating functions are needed giving rise to a system of functional equations. As particular cases of our results, we obtain the generating function of the univariate distribution on $\mathcal{U}_n$ for $n \geq 1$ for each subword pattern of length two or three. Special attention is paid to the subset $\mathcal{V}_n$ consisting of those members of $\mathcal{U}_n$ ending in 1, which are enumerated by the Motzkin number M_{n-1} for $n \geq 1$. Explicit formulas for the total number of occurrences of a pattern on $\mathcal{U}_n$ or $\mathcal{V}_n$ can be found by differentiating the respective generating functions.

Keywords: Generating function; Kernel method; Motzkin path; Smooth word; Subword pattern
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1. Introduction

Let $\pi = \pi_1 \cdots \pi_n$ and $\rho = \rho_1 \cdots \rho_m$ denote positive integral sequences, where $n \geq m \geq 1$ and ρ contains each letter in $[\ell] = \{1, \ldots, \ell\}$ for some $\ell \geq 1$ at least once. Then π is said to *contain* ρ as a *subword* (or *consecutive pattern*) if there exists a string $\pi_a \pi_{a+1} \cdots \pi_{a+m-1}$ of letters of π where $1 \leq a \leq n-m+1$ that is order-isomorphic to ρ, and is said to *avoid* ρ (as a subword) otherwise. For example, the sequence $\pi = 532343445421$ contains two occurrences of 212 as a subword (as witnessed by 323 and 434) and three occurrences of 321 (the strings 532, 542, and 421). It avoids 231 as a subword, though it contains subsequences that are isomorphic to 231 not corresponding to a string. Note that two occurrences of a subword pattern need not be disjoint, as seen with the second and third occurrences of 321 in π. An occurrence of a 12, 21, or 11 subword is known as an *ascent*, *descent*, or *level*, respectively. Among the various kinds of integer sequences that have been enumerated according to the number of occurrences of subword patterns include k-ary words [8], compositions [19], and finite set partitions [18] (represented sequentially as restricted growth functions). Henceforth, when discussing the question of avoidance concerning a particular pattern or its distribution on a set, it will be in the context of subword containment.

A *smooth* word $w_1 \cdots w_n$ is one on the alphabet of positive integers satisfying $|w_{i+1} - w_i| \leq 1$ for all $i \in [n-1]$. Smooth words and compositions were studied initially by Mansour et al. in [14] subject to various restrictions. Smooth words satisfying $w_1 = 1$ (and more generally having a fixed, but arbitrary, first letter) were considered briefly in [17], where the generating function of the joint distribution for the statistics tracking the sums of ascent tops and level values was found. Here, we consider the distribution of certain parameters on the set of smooth words starting and/or ending in 1. This extends recent work (see, e.g., [3, 6, 20]) done on the set of right-smooth words satisfying $w_{i+1} - w_i \leq 1$ for all i with $w_1 = 1$, which are also known as *Catalan* words (being an object enumerated by the n-th Catalan number, see [26]). In these works, avoidance classes of Catalan words both in the classical and subword sense were enumerated with respect to the number of descents. See also [24] for the distribution of subword patterns of length two or three on the set of all Catalan words of length n. A further variant of the notion of smoothness was recently considered in [16], where the difference in the positions corresponding to two adjacent letter changes within an integral sequence (going from left to right) is bounded above by a fixed number.

Let $\mathcal{U}_n$ denote the set of smooth words of length n starting with 1, i.e., the set of positive integral sequences $u_1 \cdots u_n$ satisfying $|u_{i+1} - u_i| \leq 1$ for all i, with $u_1 = 1$. For example, we have

$$\mathcal{U}_4 = \{1111, 1112, 1121, 1122, 1123, 1211, 1212, 1221, 1222, 1223, 1232, 1233, 1234\}.$$

The cardinality of $\mathcal{U}_n$ for each $n \geq 1$ is given by sequence $A005773(n)$ in [25], which we will denote here by L_n. Recall that $L_n = |\mathcal{L}_n|$, where $\mathcal{L}_n$ is the set of lattice paths from $(0,0)$ to the line $x = n-1$ using $u = (1,1)$, $d = (1,-1)$, and $h = (1,0)$ steps that never go below the x-axis. Members of $\mathcal{L}_n$ are known as *Motzkin left factors* (see, e.g., [2, p. 111]). Let $\mathcal{V}_n$ denote the subset of $\mathcal{U}_n$ whose members satisfy $u_n = 1$; note that $|\mathcal{V}_n| = M_{n-1}$ for $n \geq 1$, where M_n is the n-th Motzkin number (see, e.g., [1] or [25, A001006]).

A simple bijection $\jmath$ between $\mathcal{U}_n$ and $\mathcal{L}_n$ is obtained by putting u, d, or h according to if the difference $u_{i+1} - u_i$ for $i \in [n-1]$ is 1, -1, or 0, respectively, and considering the resulting lattice path. For example, if $w = 12322112 \in \mathcal{U}_8$, then $\jmath(w) = uudhdhu \in \mathcal{L}_8$. Note that the final height of $\jmath(w)$ is equal to one less than the final letter of w for all w. Let $\mathcal{M}_n$ denote the subset of $\mathcal{L}_{n+1}$ whose members terminate at the point $(n,0)$ (i.e., have final height zero). Members of $\mathcal{M}_n$ are known as *Motzkin* paths and are enumerated by M_n for all $n \geq 0$. Under $\jmath$, the subset $\mathcal{V}_n$ of $\mathcal{U}_n$ corresponds to $\mathcal{M}_{n-1}$. At times, we will identify members of $\mathcal{U}_n$ or $\mathcal{V}_n$ with their corresponding lattice paths in $\mathcal{L}_n$ or $\mathcal{M}_{n-1}$ under $\jmath$.

In this paper, we study the distribution of statistics on $\mathcal{U}_n$ and $\mathcal{V}_n$ recording the number of occurrences of subword patterns of length two or three. Let $\sigma(\pi)$ denote the final letter of $\pi \in \mathcal{U}_n$ and $\mu_\rho(\pi)$ the number of occurrences of the subword ρ in π. We consider, more generally, a joint distribution of the form

$$\alpha_n(v,p,q) = \sum_{\pi \in \mathcal{U}_n} v^{\sigma(\pi)-1} p^{\mu_\rho(\pi)} q^{\mu_\tau(\pi)}, \qquad n \geq 1,$$

for a fixed pair of subword patterns ρ and τ. Note that $\alpha_n(v,p,q)$ reduces to L_n when $v = p = q = 1$ and to M_{n-1} when $v = 0$ and $p = q = 1$. Hence, the distribution $\alpha_n(v,p,q)$ will yield some new polynomial generalizations of the sequences L_n and M_{n-1} as ρ and τ vary. For other extensions of the Motzkin number sequence, see, e.g., [4, 5, 7, 10, 22, 23, 27, 29]. We will find explicit formulas for the generating function $h(x) = h(x; v, p, q)$, defined by

$$h(x) = \sum_{n \geq 1} \alpha_n(v,p,q) x^n,$$

for several pairs of patterns ρ and τ as well as for one triple of patterns.

Let

$$f_\tau(x;q) = \sum_{n \geq 1} \left(\sum_{\pi \in \mathcal{U}_n} q^{\mu_\tau(\pi)} \right) x^n$$

and

$$g_\tau(x;q) = \sum_{n \geq 1} \left(\sum_{\pi \in \mathcal{V}_n} q^{\mu_\tau(\pi)} \right) x^n$$

denote the generating functions for the respective univariate distributions of the subword pattern τ on $\mathcal{U}_n$ and $\mathcal{V}_n$. Note that $f_\tau(x;q) = h(x;1,1,q)$ and $g_\tau(x;q) = h(x;0,1,q)$. As corollaries of our main results, one obtains simple explicit formulas for $f_\tau(x;q)$ and $g_\tau(x;q)$ for all τ of length two or three. See Tables 1 and 2 at the end of the fourth section. Taking $q = 1$ in these formulas for $f_\tau(x;q)$ and $g_\tau(x;q)$ is seen to recover in each case the generating function for the sequence L_n or M_{n-1}, respectively. Note that we need not deal with the patterns 132, 213, 231, or 312 since a smooth word clearly must avoid each one as a subword.

To determine $h(x)$, we derive a functional equation in each case which may be solved explicitly for all v, p, and q in general, if desired, using the *kernel method* [13]. In several instances, it is useful to define one or more auxiliary sequences representing various restrictions of the joint distribution $\alpha_n = \alpha_n(v,p,q)$. This allows one to write a system of linear recurrences involving α_n and the auxiliary sequence(s), which leads to a system of functional equations satisfied by the corresponding generating functions. In solving these systems using the kernel method, we remark that the special case $v = 0$ plays a pivotal role. Indeed, it is for this reason that the v variable in α_n marks the parameter value $\sigma - 1$, instead of σ.

Further, via $\jmath$, these joint distributions on $\mathcal{U}_n$ and $\mathcal{V}_n$ may be viewed equivalently as distributions on $\mathcal{L}_n$ and $\mathcal{M}_{n-1}$ of lattice path statistics which track the number of occurrences of various kinds of step patterns (taken together with the final height parameter). For example, the polynomial $\alpha_n(v,p,q)$ when $\rho = 123$ and $\tau = 321$ (see Section 3) would correspond to the joint distribution for the parameters on $\mathcal{L}_n$ tracking the final height, occurrences of uu, and occurrences of dd (marked by v, p, and q, respectively). Thus, one obtains new formulas for the generating functions of several joint distributions of such statistics on $\mathcal{L}_n$ and $\mathcal{M}_{n-1}$ for $n \geq 1$. For comparable results on Dyck paths involving a single step pattern, see, e.g., [11, 15, 21], and for Motzkin paths, see [7, 23].

The organization of this paper is as follows. In the next section, we consider the joint distribution of the final letter, ascents, and descents statistics on $\mathcal{U}_n$ and compute the associated generating function (see Theorem 2.1 below). As a consequence, one can readily obtain by differentiation explicit formulas for the total number of ascents, descents, or levels in all the members of $\mathcal{U}_n$ or $\mathcal{V}_n$. One may subsequently explain these formulas (and others in later sections) bijectively in several instances. Further, finding alternative expressions for these totals yields identities relating L_n and M_n. Also, the ideas from this section lead to a new combinatorial proof of the two-term recurrence for M_n. For other combinatorial proofs of this recurrence, see [27–29]. In the third section, we provide a comparable treatment of the 123 and 321 patterns. In the fourth section, we deal with the remaining patterns of length three, each of which contains a repeated letter. For this, it is convenient to group the remaining patterns according to if they end in an ascent, descent, or level. Hence, we consider in turn the joint distributions for the three pattern sets 112/212, 121/221, and 111/122/211. In the final section, some further subword equivalences on $\mathcal{U}_n$ and $\mathcal{V}_n$ are noted.

We will make use of the following well-known generating function formulas:

$$\sum_{n\geq1} L_n x^n = \frac{3x-1+\sqrt{1-2x-3x^2}}{2(1-3x)}$$

and

$$\sum_{n\geq1} M_{n-1} x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x},$$

which will be denoted by $L(x)$ and $M(x)$, respectively. Also, let $\mathcal{G}_n$ be the set of lattice paths from $(0,0)$ to $(n,0)$ using u, d, and h steps with no restriction concerning going below the x-axis, the members of which are referred to as *grand Motzkin* paths. The terms of the sequence $G_n = |\mathcal{G}_n|$ are known as grand Motzkin numbers (see [25, A002426]) and have generating function given by

$$\sum_{n\geq0} G_n x^n = \frac{1}{\sqrt{1-2x-3x^2}}.$$

2. Distribution of ascents and descents

Given $n\geq2$, $i\in[n-1]$, and $j\in[n]$, let $\mathcal{U}_{n,i,j}$ denote the subset of $\mathcal{U}_n$ whose members have last two letters i and j in that order. If $n\geq2$ and $1\leq j\leq n$, let $\mathcal{U}_{n,j}=\cup_{i=1}^{n-1}\mathcal{U}_{n,i,j}$, with $\mathcal{U}_{1,1}=\{1\}$. Let $a_{n,i,j}=a_{n,i,j}(p,q)$ denote the joint distribution for the statistics on $\mathcal{U}_{n,i,j}$ recording the number the ascents and descents (marked by p and q, respectively). Note that $a_{n,i,j}=0$ if it is not the case $i\in[n-1]$ with $j\in[i-1,i+1]$, as the underlying set $\mathcal{U}_{n,i,j}$ is empty for such i and j. If $n\geq2$ and $1\leq j\leq n$, let $a_{n,j}=a_{n,j-1,j}+a_{n,j,j}+a_{n,j+1,j}$, with $a_{1,1}=1$. Define the joint distribution polynomial

$$a_n(v)=\sum_{j=1}^{n} a_{n,j}v^{j-1}, \qquad n\geq1,$$

where v is an indeterminate.

The $a_n(v)$ for $n\geq2$ satisfy the following recursion.

Lemma 2.1. *If $n\geq2$, then*

$$a_n(v)=\left(pv+1+\frac{q}{v}\right)a_{n-1}(v)-\frac{q}{v}a_{n-1}(0), \tag{1}$$

with $a_1(v)=1$.

Proof. By the definitions, we have $a_{n,j-1,j}=pa_{n-1,j-1}$ for $2\leq j\leq n$, $a_{n,j,j}=a_{n-1,j}$ for $1\leq j\leq n-1$, and $a_{n,j+1,j}=qa_{n-1,j+1}$ for $1\leq j\leq n-2$. If $n\geq3$ and $2\leq j\leq n-1$, then

$$a_{n,j}=a_{n,j-1,j}+a_{n,j,j}+a_{n,j+1,j}=pa_{n-1,j-1}+a_{n-1,j}+qa_{n-1,j+1},$$

which is also seen to hold in the $j=1$ and $j=n$ cases for all $n\geq2$, upon assuming $a_{n,m}=0$ if $m<1$ or $m>n$. Thus, we have

$$a_n(v)=\sum_{j=1}^{n}a_{n,j}v^{j-1}=p\sum_{j=2}^{n}a_{n-1,j-1}v^{j-1}+\sum_{j=1}^{n-1}a_{n-1,j}v^{j-1}+q\sum_{j=1}^{n-2}a_{n-1,j+1}v^{j-1}$$

$$
\begin{aligned}
&= pva_{n-1}(v) + a_{n-1}(v) + q\sum_{j=2}^{n-1} a_{n-1,j}v^{j-2}\\
&= pva_{n-1}(v) + a_{n-1}(v) + \frac{q}{v}(a_{n-1}(v) - a_{n-1,1})\\
&= \left(pv + 1 + \frac{q}{v}\right) a_{n-1}(v) - \frac{q}{v}a_{n-1}(0),
\end{aligned}
$$

as desired. □

Let $A(x;v) = A(x;v,p,q)$ be defined by $A(x;v) = \sum_{n\geq 1} a_n(v)x^n$. Multiplying both sides of (1) by x^n, and summing over all $n \geq 2$, implies

$$A(x;v) = x + x\left(pv + 1 + \frac{q}{v}\right) A(x;v) - \frac{qx}{v}A(x;0),$$

which yields the following functional equation.

Lemma 2.2. *We have*

$$\left(1 - (pv+1)x - \frac{qx}{v}\right) A(x;v) = x - \frac{qx}{v}A(x;0). \tag{2}$$

We have the following explicit formulas for $A(x;v)$ in the cases $v = 1$ and $v = 0$.

Theorem 2.1. *The generating functions for the joint distributions of the ascents and descents statistics on $\mathcal{U}_n$ and $\mathcal{V}_n$ for $n \geq 1$ are given respectively by*

$$A(x;1) = \frac{(2p+1)x - 1 + \sqrt{1-2x+(1-4pq)x^2}}{2p(1-(p+q+1)x)} \tag{3}$$

and

$$A(x;0) = \frac{1 - x - \sqrt{1-2x+(1-4pq)x^2}}{2pqx}. \tag{4}$$

Proof. We apply the kernel method and let $v_0 = v_0(x;p,q)$ satisfy $1-(pv+1)x-\frac{qx}{v} = 0$, i.e., $qx-(1-x)v+pxv^2 = 0$, and hence

$$v_0 = \frac{1 - x \pm \sqrt{1-2x+(1-4pq)x^2}}{2px}.$$

We select the negative root since only this one will lead to a solution that is analytic at $x = 0$. Taking $v = v_0$ in (2) gives

$$A(x;0) = \frac{v_0}{q} = \frac{1 - x - \sqrt{1-2x+(1-4pq)x^2}}{2pqx}.$$

Solving for $A(x;v)$ in (2) now implies

$$A(x;v) = \frac{x(v - v_0)}{v - (pv+1)vx - qx},$$

and taking $v = 1$ gives

$$A(x;1) = \frac{x(1-v_0)}{1-(p+q+1)x} = \frac{(2p+1)x - 1 + \sqrt{1-2x+(1-4pq)x^2}}{2p(1-(p+q+1)x)},$$

which completes the proof. □

By (3), we have

$$f_{12}(x;q) = A(x;1,q,1) = \frac{(2q+1)x - 1 + \sqrt{1-2x+(1-4q)x^2}}{2q(1-(q+2)x)} \tag{5}$$

and

$$f_{21}(x;q) = A(x;1,1,q) = \frac{3x - 1 + \sqrt{1-2x+(1-4q)x^2}}{2(1-(q+2)x)}. \tag{6}$$

Since the sum of the number of ascents, descents, and levels is $n-1$ in any integral sequence of length n, we also get

$$f_{11}(x;q) = (1/q)A(qx;1,1/q,1/q) = \frac{(q+2)x - 1 + \sqrt{1-2qx+(q^2-4)x^2}}{2(1-(q+2)x)}. \tag{7}$$

Note that each of the expressions in (5)–(7) reduces to $L(x)$ when $q = 1$, as expected.

Given a subword pattern ρ, let $\mathrm{tot}_n(\rho)$ and $\mathrm{tot}'_n(\rho)$ denote the total number of occurrences of ρ in all the members of $\mathcal{U}_n$ and $\mathcal{V}_n$, respectively. We have the following explicit formulas for 12, 21, and 11.

Corollary 2.1. *If $n \geq 1$, then*

$$tot_n(12) = 3^{n-1} - L_n + \sum_{i=1}^{n-1}(L_i - G_{i-1})3^{n-i-1},$$

$$tot_n(21) = \sum_{i=1}^{n-1}(L_i - G_{i-1})3^{n-i-1},$$

and

$$tot_n(11) = (1/2)(3^{n-1} - G_{n-1}) + \sum_{i=1}^{n-1}(L_i - G_{i-1})3^{n-i-1}.$$

Proof. Note that all three formulas are seen to hold for $n = 1$, so we may assume $n \geq 2$. By (5), we have

$$\begin{aligned}\frac{\partial}{\partial q} f_{12}(x;q)\mid_{q=1} &= \frac{\partial}{\partial q}\left(\frac{(2q+1)x - 1 + \sqrt{1-2x+(1-4q)x^2}}{2q(1-(q+2)x)}\right)_{q=1} \\ &= L(x)\left(\frac{x}{1-3x} - 1\right) + \frac{x}{1-3x}\left(1 - \frac{x}{\sqrt{1-2x-3x^2}}\right),\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{tot}_n(12) = [x^n]\frac{\partial}{\partial q} f_{12}(x;q)\mid_{q=1} &= \sum_{i=1}^{n-1} L_i 3^{n-i-1} - L_n + 3^{n-1} - \sum_{i=1}^{n-1} G_{i-1}3^{n-i-1} \\ &= 3^{n-1} - L_n + \sum_{i=1}^{n-1}(L_i - G_{i-1})3^{n-i-1}.\end{aligned}$$

By (6), we have

$$\begin{aligned}[x^n]\frac{\partial}{\partial q} f_{21}(x;q)\mid_{q=1} &= [x^n]\left(\frac{x}{1-3x}L(x) - \frac{x^2}{(1-3x)\sqrt{1-2x-3x^2}}\right) \\ &= \sum_{i=1}^{n-1}(L_i - G_{i-1})3^{n-i-1},\end{aligned}$$

which implies the second formula. Finally, by (7), we have

$$\begin{aligned}[x^n]\frac{\partial}{\partial q} f_{11}(x;q)\mid_{q=1} &= [x^n]\left(\frac{x}{1-3x}L(x) + \frac{x}{2(1-3x)}\left(1 - \frac{1-x}{\sqrt{1-2x-3x^2}}\right)\right) \\ &= \sum_{i=1}^{n-1} L_i 3^{n-i-1} + \frac{1}{2}\left(3^{n-1} - \sum_{i=1}^{n} G_{i-1}3^{n-i} + \sum_{i=1}^{n-1} G_{i-1}3^{n-i-1}\right) \\ &= \sum_{i=1}^{n-1} L_i 3^{n-i-1} + \frac{1}{2}\left(3^{n-1} - 2\sum_{i=1}^{n-1} G_{i-1}3^{n-i-1} - G_{n-1}\right),\end{aligned}$$

which leads to the third formula and completes the proof. □

The generating function $A(x;0)$ is symmetric in p and q, which may be realized directly by considering the reversal operation. By (4), we have

$$g_{12}(x;q) = g_{21}(x;q) = A(x;0,1,q) = \frac{1 - x - \sqrt{1-2x-(1-4q)x^2}}{2qx}$$

and

$$g_{11}(x;q) = (1/q)A(qx;0,1/q,1/q) = \frac{1 - qx - \sqrt{1-2qx+(q^2-4)x}}{2x}.$$

Taking $q = 1$ in both of these formulas gives $M(x)$, as expected.

Differentiating with respect to q, setting $q = 1$, and extracting the coefficient of x^n yields the following formulas for the totals of the corresponding statistics on $\mathcal{V}_n$.

Corollary 2.2. *If $n \geq 1$, then $tot'_n(12) = tot'_n(21) = G_{n-1} - M_{n-1}$ and $tot'_n(11) = (1/2)(G_n - G_{n-1})$.*

It should be noted that $\text{tot}_n(11) = A132894(n-1)$ for $n \geq 1$, $\text{tot}'_n(11) = A005717(n-1)$ for $n \geq 2$, and $\text{tot}'_n(12) = A014531(n-2)$ for $n \geq 3$, where $A\#\#\#\#\#\#(m)$ denotes the OEIS sequence parameterized as in the indicated entry.

Given $\lambda \in \mathcal{U}_n$, consider marking the second letter of some level within λ, or equivalently some h step within the lattice path $\jmath(\lambda)$. Since there are $n-1$ possible positions for the marked step, with all other steps of $\jmath(\lambda)$ constituting an arbitrary member of $\mathcal{L}_{n-1}$, it follows that there are $(n-1)L_{n-1}$ h steps altogether in $\mathcal{L}_n$, and hence levels in $\mathcal{U}_n$. Likewise, there are $(n-1)M_{n-2}$ total levels in $\mathcal{V}_n$. Equating these expressions with those from Corollaries 2.1 and 2.2, and replacing n with $n+1$, yields the following pair of identities.

Corollary 2.3. *If $n \geq 1$, then*

$$nL_n = (1/2)(3^n - G_n) + \sum_{i=1}^{n}(L_i - G_{i-1})3^{n-i}$$

and

$$nM_{n-1} = (1/2)(G_{n+1} - G_n).$$

Using Corollaries 2.1 and 2.3, one may obtain explicit formulas without summations for the totals above on $\mathcal{U}_n$ for $n \geq 1$ as follows: $\text{tot}_n(11) = (n-1)L_{n-1}$, $\text{tot}_n(21) = (n-1)L_{n-1} - (1/2)(3^{n-1} - G_{n-1})$, and $\text{tot}_n(12) = (n-1)L_{n-1} + (1/2)(3^{n-1} + G_{n-1}) - L_n$.

Taking $q = 0$ in (7) gives

$$[x^n]f_{11}(x;0) = [x^n]\left(-\frac{1}{2} + \frac{1+2x}{2\sqrt{1-4x^2}}\right) = \begin{cases} \binom{2m}{m}, & \text{if } n = 2m+1; \\ \binom{2m-1}{m}, & \text{if } n = 2m. \end{cases}$$

To realize this last formula directly, note first that members of $\mathcal{U}_{2m+1}$ containing no levels are synonymous with lattice paths of length $2m$ starting from the origin and staying above the x-axis. It is well-known that such lattice paths number $\binom{2m}{m}$, see, e.g., [25, A000984], which implies the odd case of the formula. Members of $\mathcal{U}_{2m}$ are synonymous with first-quadrant lattice paths wherein there are $2m-1$ steps. Consider appending either a u or d to such a path, and note that all first-quadrant paths of length $2m$ arise uniquely in this manner. Thus, there are $\frac{1}{2}\binom{2m}{m} = \binom{2m-1}{m}$ paths of length $2m-1$, which implies the even case of the formula.

We now provide bijective arguments for the tot'_n expressions in Corollary 2.2 and seek comparable proofs of the formulas in Corollary 2.1.

Combinatorial proof of Corollary 2.2:

To find $\text{tot}'_n(21)$, we count equivalently the d steps within all the members of $\mathcal{M}_{n-1}$. Upon replacing n by $n+1$, we show that the number of d steps altogether in $\mathcal{M}_n$ is given by $G_n - M_n$. To do so, we count marked members of $\mathcal{M}_n$ wherein some d step is marked. Let $\lambda \in \mathcal{M}_n$ be decomposed as $\lambda = \lambda'\underline{d}\lambda''$, where the marked step is underlined and λ'' is possibly empty. Let $\nu(\lambda) = \widetilde{\text{rev}}(\lambda')u\widetilde{\text{rev}}(\lambda'')$, where $\widetilde{\text{rev}}(\rho)$ for a lattice path ρ consisting of u, d, and h steps is obtained by reading ρ backwards and replacing each u with d and d with u, leaving all h steps unchanged. One may verify $\nu(\lambda) \in \mathcal{G}_n - \mathcal{M}_n$ for all λ. Further, it is seen that ν is onto and may be reversed by considering the position of the rightmost point of minimum height within a member of $\mathcal{G}_n - \mathcal{M}_n$. Since ν is a bijection, it follows that the total number of d steps in members of $\mathcal{M}_n$ is given by $G_n - M_n$, as desired.

To establish the second formula in Corollary 2.2, it suffices to show that there are in total $(1/2)(G_{n+1} - G_n)$ h steps in all the members of $\mathcal{M}_n$. Suppose some h within $\rho \in \mathcal{M}_n$ is marked, which we decompose as $\rho = \rho'\underline{h}\rho''$, where the marked h is underlined and ρ' or ρ'' may be empty. Let $\mu(\rho) = \widetilde{\text{rev}}(\rho')d\widetilde{\text{rev}}(\rho'')u$. Then μ is onto the subset of $\mathcal{G}_{n+1}$ whose members end in u and is reversible, upon considering the leftmost point of minimum height. By subtraction, there are $G_{n+1} - G_n$ members of $\mathcal{G}_{n+1}$ ending in u or d, and hence $(1/2)(G_{n+1} - G_n)$ that end in u, by symmetry. Since μ is a bijection with such members of $\mathcal{G}_{n+1}$, the formula for $\text{tot}'_n(11)$ is established. □

Figures 1 and 2 below illustrate the bijections ν and μ applied respectively to the lattice paths λ and ρ in $\mathcal{M}_{12}$. The marked steps of λ and ρ are indicated in red, with the corresponding steps in $\nu(\lambda)$ and $\mu(\rho)$ in green.

We now demonstrate how the well-known two-term Motzkin recurrence given by $(n+2)M_n = (2n+1)M_{n-1} + 3(n-1)M_{n-2}$ for $n \geq 2$ can be derived from the preceding bijective arguments.

Combinatorial derivation of Motzkin recurrence:

We establish the recurrence, rewritten as

$$4(n-1)M_{n-2} = (n+2)M_n - nM_{n-1} - ((n+1)M_{n-1} - (n-1)M_{n-2}), \qquad n \geq 2. \tag{8}$$

From the bijective proof given above for the $\mathrm{tot}'_n(11)$ formula, we have $2(n-1)M_{n-2} = G_n - G_{n-1}$. Hence, to establish (8), it suffices to show

$$(n+2)M_n - nM_{n-1} = 2G_n. \tag{9}$$

For (9), note first that there are nM_n letters within all the members of $\mathcal{V}_{n+1}$, ignoring the initial 1 in each word. Classifying each such letter as the second letter in an ascent, descent, or level, we also have that there are $2(G_n - M_n) + nM_{n-1}$ letters which do not start words in all the members of $\mathcal{V}_{n+1}$, by the combinatorial argument given for Corollary 2.2. Equating expressions yields (9) and completes the proof of (8), as desired. For a different combinatorial proof of (9), see [12]. □

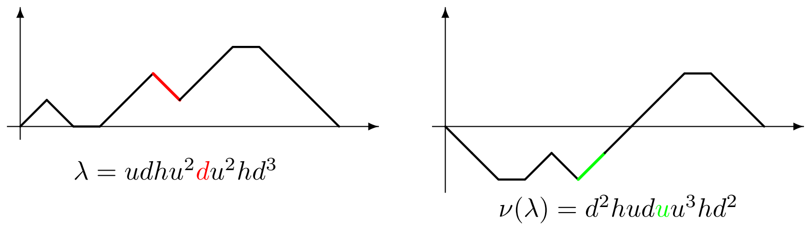


Figure 1: Lattice paths $\lambda \in \mathcal{M}_{12}$ and $\nu(\lambda) \in \mathcal{G}_{12} - \mathcal{M}_{12}$.

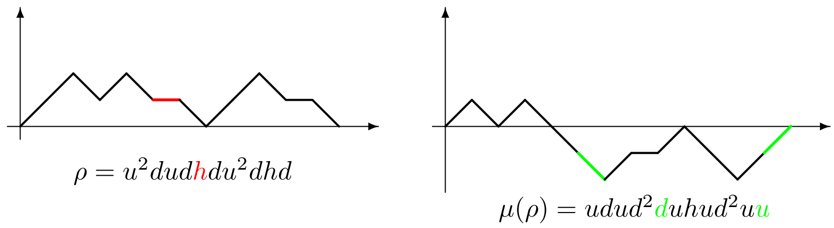


Figure 2: Lattice paths $\rho \in \mathcal{M}_{12}$ and $\mu(\rho) \in \mathcal{G}_{13}$.

3. Distribution of 123 and 321 subwords

Let $b_{n,i,j} = b_{n,i,j}(p,q)$ denote the joint distribution of the statistics on $\mathcal{U}_{n,i,j}$ recording the number of occurrences of the subwords 123 and 321 (marked by p and q, respectively). Let $b_{n,j}$ for $n \geq 1$ and $1 \leq j \leq n$ denote the same distribution on $\mathcal{U}_{n,j}$. Define $b_n(v) = \sum_{j=1}^{n} b_{n,j}v^{j-1}$ for $n \geq 1$. To aid in finding the generating function for the sequence of polynomials $b_n(v)$, we define the auxiliary sequences

$$b'_n(v) = \sum_{j=2}^{n} b_{n,j-1,j}v^{j-1} \quad \text{and} \quad b^*_n(v) = \sum_{j=1}^{n-2} b_{n,j+1,j}v^{j-1}.$$

The $b_n(v)$, $b'_n(v)$, and $b^*_n(v)$ satisfy the following system of recurrences.

Lemma 3.1. *If $n \geq 2$, then*

$$b_n(v) = b'_n(v) + b^*_n(v) + b_{n-1}(v), \tag{10}$$

$$b'_n(v) = (p-1)vb'_{n-1}(v) + vb_{n-1}(v), \tag{11}$$

and

$$b^*_n(v) = \frac{q-1}{v}(b^*_{n-1}(v) - b^*_{n-1}(0)) + \frac{1}{v}(b_{n-1}(v) - b_{n-1}(0)), \tag{12}$$

with $b_1(v) = 1$ and $b'_1(v) = b^*_1(v) = 0$.

Proof. We may assume $n \geq 3$ since (10)–(12) are seen to hold for $n = 2$, as $b_2(v) = v + 1$, $b'_2(v) = v$, and $b^*_2(v) = 0$. Let $b_{n,i,j} = 0$ if $(i,j) \notin [n-1] \times [n]$ and $b_{n,j} = 0$ if $j \notin [n]$. We then have $b_{n,j,j} = b_{n-1,j}$ for $1 \leq j \leq n-1$,

$$b_{n,j-1,j} = (p-1)b_{n-1,j-2,j-1} + b_{n-1,j-1}, \qquad 2 \leq j \leq n,$$

and

$$b_{n,j+1,j} = (q-1)b_{n-1,j+2,j+1} + b_{n-1,j+1}, \qquad 1 \leq j \leq n-2.$$

If $2 \le j \le n-1$, then

$$b_{n,j} = b_{n,j-1,j} + b_{n-1,j} + b_{n,j+1,j},$$

which is also seen to hold for $j = 1$ and $j = n$. Multiplying both sides of the last equality by v^{j-1}, and summing over $1 \le j \le n$, then yields

$$\begin{aligned} b_n(v) &= \sum_{j=2}^{n} b_{n,j-1,j} v^{j-1} + \sum_{j=1}^{n-1} b_{n-1,j} v^{j-1} + \sum_{j=1}^{n-2} b_{n,j+1,j} v^{j-1} \\ &= b_n'(v) + b_{n-1}(v) + b_n^*(v), \end{aligned}$$

which gives (10). Also, for $n \ge 3$, we have

$$b_n'(v) = (p-1)\sum_{j=3}^{n} b_{n-1,j-2,j-1} v^{j-1} + \sum_{j=2}^{n} b_{n-1,j-1} v^{j-1} = (p-1)vb_{n-1}'(v) + vb_{n-1}(v)$$

and

$$\begin{aligned} b_n^*(v) &= (q-1)\sum_{j=1}^{n-4} b_{n-1,j+2,j+1} v^{j-1} + \sum_{j=1}^{n-2} b_{n-1,j+1} v^{j-1} \\ &= \frac{q-1}{v}(b_{n-1}^*(v) - b_{n-1}^*(0)) + \frac{1}{v}(b_{n-1}(v) - b_{n-1}(0)), \end{aligned}$$

which gives (11) and (12). □

Define the joint distribution generating function $B(x; v) = B(x; v, p, q)$ by $B(x; v) = \sum_{n \ge 1} b_n(v)x^n$, and also $B'(x; v) = \sum_{n \ge 2} b_n'(v)x^n$ and $B^*(x; v) = \sum_{n \ge 3} b_n^*(v)x^n$. Multiplying both sides of (10)–(12) by x^n, and summing over all $n \ge 2$, leads to the following system of functional equations.

Lemma 3.2. *We have*

$$(1-x)B(x; v) = x + B'(x; v) + B^*(x; v), \tag{13}$$

$$B'(x; v) = \frac{vx}{1-(p-1)vx} B(x; v), \tag{14}$$

and

$$B^*(x; v) = \frac{(q-1)x}{v}(B^*(x; v) - B^*(x; 0)) + \frac{x}{v}(B(x; v) - B(x; 0)). \tag{15}$$

By solving the preceding system of functional equations, we obtain an explicit formula for the generating function of the joint distribution.

Theorem 3.1. *The generating functions for the joint distributions of the* 123 *and* 321 *subwords on* $\mathcal{U}_n$ *and* $\mathcal{V}_n$ *for* $n \ge 1$ *are given respectively by*

$$B(x; 1) = \frac{x(1-(p-1)x)(1-v_0)}{1-(p+q+1)x + (pq+p+q-3)x^2 - (p-1)(q-1)x^3} \tag{16}$$

and

$$B(x; 0) = \frac{v_0}{q + (1-q)x}, \tag{17}$$

where $v_0 = v_0(x; p, q)$ *is given by*

$$\begin{aligned} v_0 &= \frac{1-x+(pq-1)x^2 - (p-1)(q-1)x^3}{2x(p+(1-p)x)} \\ &\quad - \frac{\sqrt{(1-x+(pq-1)x^2-(p-1)(q-1)x^3)^2 - 4x^2(p+(1-p)x)(q+(1-q)x)}}{2x(p+(1-p)x)}. \end{aligned}$$

Proof. By (13) and (14), we have

$$(1-x)B = x + \frac{vx}{1-(p-1)vx}B + B^*,$$

and hence

$$B = \frac{x+B^*}{1-x-\frac{vx}{1-(p-1)vx}} = \frac{(1-(p-1)vx)(x+B^*)}{1-x-pvx+(p-1)vx^2}, \tag{18}$$

where the arguments for $B(x;v)$ and $B^*(x;v)$ have been suppressed. By (15), we have

$$B^* = \frac{(q-1)x}{v}(B^* - B^*(0)) + \frac{x}{v}\left(\frac{(1-(p-1)vx)(x+B^*)}{1-x-pvx+(p-1)vx^2} - \frac{x+B^*(0)}{1-x}\right),$$

and thus

$$\begin{aligned}&\left(1-\frac{(q-1)x}{v} - \frac{x(1-(p-1)vx)}{v(1-x-pvx+(p-1)vx^2)}\right)B^* \\ &\quad = \frac{x^2(1-(p-1)vx)}{v(1-x-pvx+(p-1)vx^2)} - \frac{x^2}{v(1-x)} + \frac{xB^*(0)}{v}\left(1-q-\frac{1}{1-x}\right), \end{aligned} \tag{19}$$

where $B^*(0)$ denotes evaluation of $B^*(x;v)$ at $v=0$.

Let $v_0 = v_0(x;p,q)$ satisfy

$$1 - \frac{(q-1)x}{v} - \frac{x(1-(p-1)vx)}{v(1-x-pvx+(p-1)vx^2)} = 0,$$

and thus v_0 is as given above. Taking $v = v_0$ in (19) implies

$$\begin{aligned}\frac{1}{v_0}B^*(0)\left(1-q-\frac{1}{1-x}\right) &= \frac{x}{(1-x)v_0} - \frac{x(1-(p-1)xv_0)}{(1-x-pxv_0+(p-1)x^2v_0)v_0} \\ &= \frac{x}{(1-x)v_0} + \frac{(q-1)x}{v_0} - 1,\end{aligned}$$

where the latter inequality follows from the equation for v_0. Hence,

$$B^*(0) = -x + \frac{1-x}{1-(1-q)(1-x)}v_0. \tag{20}$$

By (19) and (20), we have

$$\begin{aligned}\alpha(x;v,p,q)B^* &= \frac{x^2(1-(p-1)vx)}{1-x-pvx+(p-1)vx^2} - \frac{x^2}{1-x} + \frac{xB^*(0)}{1-x}((1-q)(1-x)-1) \\ &= \frac{x^2(1-(p-1)vx)}{1-x-pvx+(p-1)vx^2} - \frac{x^2}{1-x} + x\left(\frac{x(1-(1-q)(1-x))}{1-x} - v_0\right) \\ &= \frac{x^2(1-(p-1)vx)}{1-x-pvx+(p-1)vx^2} + (q-1)x^2 - xv_0 \\ &= \frac{x^2(q+(1-q)x) - (pq-1)vx^3 + (p-1)(q-1)vx^4}{1-x-pvx+(p-1)vx^2} - xv_0,\end{aligned}$$

where $\alpha(x;v,p,q)$ is given by

$$\frac{-x(q+(1-q)x) + (1-x+(pq-1)x^2-(p-1)(q-1)x^3)v - x(p+(1-p)x)v^2}{1-x-pvx+(p-1)vx^2}.$$

Adding $x\alpha(x;v,p,q)$ to both sides of the last equation gives

$$\begin{aligned}&\alpha(x;v,p,q)(B^*+x) \\ &= \frac{x^2(q+(1-q)x) - (pq-1)vx^3 + (p-1)(q-1)vx^4}{1-x-pvx+(p-1)vx^2} - xv_0 \\ &\quad + \frac{-x^2(q+(1-q)x) + x(1-x+(pq-1)x^2-(p-1)(q-1)x^3)v - x^2(p+(1-p)x)v^2}{1-x-pvx+(p-1)vx^2} \\ &= \frac{x(1-x)v - x^2(p+(1-p)x)v^2}{1-x-pvx+(p-1)vx^2} - xv_0 \\ &= \frac{xv(1-x-x(p+(1-p)x)v)}{1-x-x(p+(1-p)x)v} - xv_0 = x(v-v_0),\end{aligned}$$

and hence by (18),

$$B = \frac{x(1-(p-1)vx)(v_0-v)}{x(q+(1-q)x) - (1-x+(pq-1)x^2-(p-1)(q-1)x^3)v + x(p+(1-p)x)v^2}. \tag{21}$$

Formulas (16) and (17) now follow respectively from taking $v=1$ and $v=0$ in (21). □

Letting $p = q = 1$ in (16) and (17) recovers the formulas for $L(x)$ and $M(x)$, respectively. From (16), we obtain

$$\begin{aligned} f_{123}(x;q) &= B(x;1,q,1) \\ &= \frac{(1+(1-q)x)(3(1-q)x^2+(1+2q)x-1+\sqrt{(1-x-(1-q)x^2)^2-4x^2(q+(1-q)x)})}{2(q+(1-q)x)(1-(2+q)x-2(1-q)x^2)} \end{aligned}$$

and

$$f_{321}(x;q) = B(x;1,1,q) = \frac{(1-q)x^2+3x-1+\sqrt{(1-x-(1-q)x^2)^2-4x^2(q+(1-q)x)}}{2(1-(2+q)x-2(1-q)x^2)}.$$

Differentiation with respect to q of the formula for $f_{123}(x;q)$ leads to

$$\text{tot}_n(123) = L_{n-1} - L_n + 3^{n-2} + (1/2)\sum_{i=1}^{n-2}(2L_i - G_i + G_{i-1})3^{n-i-2}, \qquad n \geq 2.$$

This expression may be simplified using the fact $2L_i = G_i + G_{i-1}$ for $i \geq 1$, with a similar proof applying to the formula for $\text{tot}_n(321)$, which yields the following result.

Corollary 3.1. *If $n \geq 2$, then*

$$tot_n(123) = L_{n-1} - L_n + 3^{n-2} + \sum_{i=0}^{n-3} G_i 3^{n-i-3}$$

and

$$tot_n(321) = tot_n(123) + L_n - 2 \cdot 3^{n-2}.$$

Setting p or q equal to unity in (17) gives

$$g_{123}(x;q) = g_{321}(x;q) = B(x;0,1,q) = \frac{1-x-(1-q)x^2-\sqrt{(1-x-(1-q)x^2)^2-4x^2(q+(1-q)x)}}{2x(q+(1-q)x)},$$

which leads to the following result.

Corollary 3.2. *If $n \geq 2$, then $tot'_n(123) = tot'_n(321) = M_{n-2} - M_{n-1} + (1/2)(G_{n-1} - G_{n-2})$.*

Note that $\text{tot}'_n(123)$ coincides with the sequence $A014532(n-4)$ for $n \geq 5$. We conclude this section by providing a bijective proof of the prior corollary.

Combinatorial proof of Corollary 3.2:

Note first that 123 and 321 are equally distributed on $\mathcal{V}_n$, by symmetry, and hence their total number of occurrences is the same. We may assume $n \geq 5$, as the formula is seen to hold for $n = 2, 3, 4$ (it is zero in each case). Replacing n by $n+1$, we count occurrences of 321 in $\mathcal{V}_{n+1}$, or equivalently, occurrences of d^2 in $\mathcal{M}_n$ where $n \geq 4$. Suppose an occurrence of d^2 in $\pi \in \mathcal{M}_n$ is marked, which we decompose as $\pi = \pi' \underline{d^2} \pi''$, where the marked d^2 is underlined. Let $\gamma(\pi) = \widetilde{\text{rev}}(\pi') d \widetilde{\text{rev}}(\pi'')$. Let $\mathcal{T}$ denote the set of lattice paths from $(0,0)$ to $(n-1,-3)$ using u, d, and h steps. One may verify that $\gamma(\pi) \in \mathcal{T}$ for all π, and is onto $\mathcal{T}$ and reversible, upon considering the position of the leftmost point of minimum height. Thus, to complete the proof, it suffices to show

$$|\mathcal{T}| = (1/2)(G_n - G_{n-1}) - (M_n - M_{n-1}). \tag{22}$$

To do so, note first that there are $(1/2)(G_n - G_{n-1})$ members of $\mathcal{G}_n$ starting with u, and of these, $M_n - M_{n-1}$ belong to $\mathcal{M}_n$. Thus, by subtraction, the right-hand side of (22) enumerates members of $\mathcal{G}_n$ starting with u and dipping below the x-axis at least once, the subset of $\mathcal{G}_n$ of which we denote by $\mathcal{S}$. Let $\rho \in \mathcal{S}$ be decomposed as $\rho = u\rho'\rho''$, where ρ' ends at the leftmost d step of ρ having final height -1. Define $\delta(\rho) = \rho' \widetilde{\text{rev}}(\rho'')$ and note $\delta(\rho) \in \mathcal{T}$ for all ρ. Further, the mapping δ is seen to be onto $\mathcal{T}$ and reversible, upon considering the position of the leftmost d step terminating at height -2. Thus, δ provides a bijection between the sets $\mathcal{S}$ and $\mathcal{T}$, which implies (22) and completes the proof. □

Figures 3 and 4 illustrate the bijections γ and δ applied respectively to $\pi \in \mathcal{M}_{12}$ and $\rho \in \mathcal{S}$. The marked d^2 of π and the section ρ' of ρ are indicated in red, with the corresponding steps in $\gamma(\pi)$ and $\delta(\rho)$ in green.

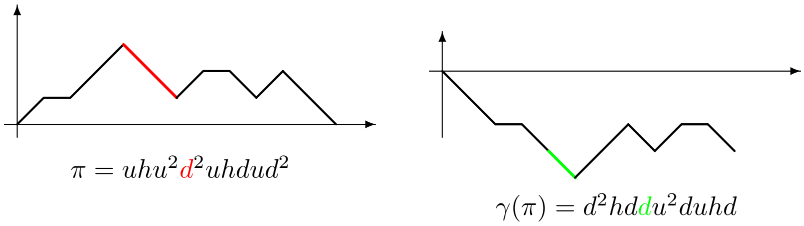


Figure 3: Lattice paths $\pi \in \mathcal{M}_{12}$ and $\gamma(\pi) \in \mathcal{T}$.

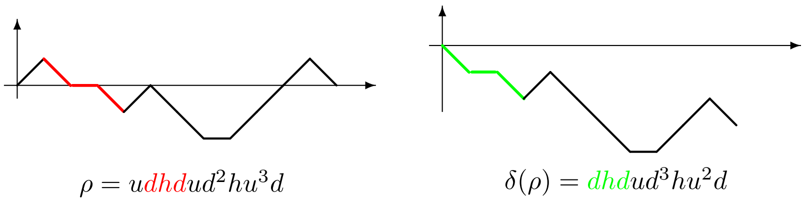


Figure 4: Lattice paths $\rho \in \mathcal{S}$ and $\delta(\rho) \in \mathcal{T}$ where $n = 12$.

4. Subwords with repeated letters

4.1 The patterns 112 and 212

Let $d_{n,i,j} = d_{n,i,j}(p,q)$ denote the joint distribution of 112 and 212 on $\mathcal{U}_{n,i,j}$, and $d_{n,j}$ the comparable distribution on $\mathcal{U}_{n,j}$. Assume $d_{n,i,j}$ or $d_{n,j}$ to be zero whenever the corresponding subset of $\mathcal{U}_n$ is empty. Define $d_n(v) = \sum_{j=1}^{n} d_{n,j}v^{j-1}$ for $n \geq 1$ and the auxiliary sequence $d'_n(v) = \sum_{j=2}^{n} d_{n,j-1,j}v^{j-1}$ for $n \geq 2$.

The sequences $d_n(v)$ and $d'_n(v)$ satisfy the following system of intertwined recurrences.

Lemma 4.1. *We have*

$$d_n(v) = d'_n(v) + d_{n-1}(v) + \frac{1}{v}(d_{n-1}(v) - d_{n-1}(0)), \qquad n \geq 2, \tag{23}$$

and

$$d'_n(v) = vd'_{n-1}(v) + pvd_{n-2}(v) + q(d_{n-2}(v) - d_{n-2}(0)), \qquad n \geq 3, \tag{24}$$

with $d_1(v) = 1$ and $d'_2(v) = v$.

Proof. The initial conditions and the $n = 2$ case of (23) are readily verified, so we may assume $n \geq 3$. Considering the penultimate letter within a member of $\mathcal{U}_{n,j}$ implies

$$d_{n,j} = d_{n,j-1,j} + d_{n,j,j} + d_{n,j+1,j} = d_{n,j-1,j} + d_{n-1,j} + d_{n-1,j+1}, \qquad 1 \leq j \leq n.$$

This yields

$$\begin{aligned} d_n(v) &= \sum_{j=2}^{n} d_{n,j-1,j}v^{j-1} + \sum_{j=1}^{n-1} d_{n-1,j}v^{j-1} + \sum_{j=1}^{n-2} d_{n-1,j+1}v^{j-1} \\ &= d'_n(v) + d_{n-1}(v) + \frac{1}{v}(d_{n-1}(v) - d_{n-1}(0)). \end{aligned}$$

Further, considering the antepenultimate letter within a member of $\mathcal{U}_{n,j-1,j}$, we have

$$\begin{aligned} d'_n(v) &= \sum_{j=2}^{n} d_{n,j-1,j}v^{j-1} \\ &= \sum_{j=3}^{n} d_{n-1,j-2,j-1}v^{j-1} + p\sum_{j=2}^{n-1} d_{n-2,j-1}v^{j-1} + q\sum_{j=2}^{n-2} d_{n-2,j}v^{j-1} \\ &= vd'_{n-1}(v) + pvd_{n-2}(v) + q(d_{n-2}(v) - d_{n-2}(0)), \end{aligned}$$

which completes the proof. □

Define the generating functions $D(x;v)$ and $D'(x;v)$ by

$$D(x;v) = \sum_{n\geq 1} d_n(v)x^n \quad \text{and} \quad D'(x;v) = \sum_{n\geq 2} d'_n(v)x^n.$$

From (23) and (24), one obtains

$$\begin{aligned}\left(1 - x - \frac{x}{v}\right) D(x;v) &= x + D'(x;v) - \frac{x}{v}D(x;0),\\ D'(x;v) &= \frac{x^2}{1-vx}\left(v + (pv+q)D(x;v) - qD(x;0)\right).\end{aligned}$$

Substituting the second equation into the first, and rearranging, leads to the following functional equation.

Lemma 4.2. *We have*

$$\left(1 - x - \frac{x}{v} - \frac{(pv+q)x^2}{1-vx}\right) D(x;v) = \frac{x}{1-vx} - \left(\frac{x}{v} + \frac{qx^2}{1-vx}\right) D(x;0). \tag{25}$$

Solving (25) yields the following result.

Theorem 4.1. *The generating functions for the joint distributions of the* 112 *and* 212 *subwords on* $\mathcal{U}_n$ *and* $\mathcal{V}_n$ *for* $n \geq 1$ *are given respectively by*

$$\begin{aligned}D(x;1) &= \frac{-(q-1)^2x^3 + 2(p-q)x^2 + (q+2)x - 1}{2(q+(p-q)x)(1-3x+(2-p-q)x^2)}\\ &\quad + \frac{(1+(q-1)x)\sqrt{(1-x-(q-1)x^2)^2 - 4x^2(1+(p-1)x)}}{2(q+(p-q)x)(1-3x+(2-p-q)x^2)}\end{aligned} \tag{26}$$

and

$$D(x;0) = \frac{1 - x + (q-1)x^2 - \sqrt{(1-x-(q-1)x^2)^2 - 4x^2(1+(p-1)x)}}{2x(q+(p-q)x)}. \tag{27}$$

Proof. Let $v_0 = v_0(x;p,q)$ satisfy $1 - x - \frac{x}{v} - \frac{(pv+q)x^2}{1-vx} = 0$, and hence it is given by

$$v_0 = \frac{1 - x - (q-1)x^2 - \sqrt{(1-x-(q-1)x^2)^2 - 4x^2(1+(p-1)x)}}{2x(1+(p-1)x)}.$$

Letting $v = v_0$ in (25), and solving for $D(x;0)$, yields

$$D(x;0) = \frac{v_0}{1+(q-1)xv_0}.$$

Note

$$\frac{1}{v_0} = \frac{1 - x - (q-1)x^2 + \sqrt{(1-x-(q-1)x^2)^2 - 4x^2(1+(p-1)x)}}{2x},$$

and thus

$$\begin{aligned}D(x;0) &= \frac{1}{\frac{1}{v_0} + (q-1)x}\\ &= \frac{1 - x + (q-1)x^2 - \sqrt{(1-x-(q-1)x^2)^2 - 4x^2(1+(p-1)x)}}{2x(q+(p-q)x)},\end{aligned}$$

which gives (27).

Taking $v = 1$ in (25), we have

$$\left(1 - 2x - \frac{(p+q)x^2}{1-x}\right) D(x;1) = \frac{x}{1-x} - \left(x + \frac{qx^2}{1-x}\right) D(x;0),$$

and hence

$$(1 - 3x + (2-p-q)x^2)D(x;1) = x - (x + (q-1)x^2)D(x;0)$$

$$= x - \frac{(1+(q-1)x)(1-x+(q-1)x^2 - \sqrt{(1-x-(q-1)x^2)^2 - 4x^2(1+(p-1)x)})}{2(q+(p-q)x)}$$
$$= \frac{-(q-1)^2x^3 + 2(p-q)x^2 + (q+2)x - 1}{2(q+(p-q)x)}$$
$$+ \frac{(1+(q-1)x)\sqrt{(1-x-(q-1)x^2)^2 - 4x^2(1+(p-1)x)}}{2(q+(p-q)x)},$$

which implies (26). □

Taking p or q to be unity in (26) and (27) gives the generating functions of the corresponding univariate distributions on $\mathcal{U}_n$ or $\mathcal{V}_n$ for 112 and 212. Differentiation of these formulas with respect to q, and setting $q = 1$, gives the following expressions for the totals on $\mathcal{U}_n$ and $\mathcal{V}_n$.

Corollary 4.1. *If $n \geq 2$, then*

$$tot_n(112) = 3^{n-2} - L_{n-1} + \sum_{i=1}^{n-2}(L_i - G_{i-1})3^{n-i-2}$$

and

$$tot_n(212) = L_{n-1} - L_n + (1/2)(3^{n-2} + G_{n-1}) + \sum_{i=1}^{n-2}(L_i - G_{i-1})3^{n-i-2}.$$

Corollary 4.2. *If $n \geq 2$, then $tot'_n(112) = G_{n-2} - M_{n-2}$ and $tot'_n(212) = M_{n-2} - M_{n-1} + (1/2)(G_{n-1} - G_{n-2})$.*

Comparing the last two results with Corollaries 2.1 and 2.2, we have $\mathrm{tot}_n(112) = \mathrm{tot}_{n-1}(12)$ and $\mathrm{tot}'_n(112) = \mathrm{tot}'_{n-1}(12)$ for all $n \geq 2$. This can be explained directly by distinguishing some ascent within a member of $\mathcal{U}_{n-1}$ or $\mathcal{V}_{n-1}$ and inserting an extra copy of the smaller letter in the ascent directly prior to it. A simpler formula for $\mathrm{tot}_n(212)$ can be realized via the apparently new identity

$$L_{n-1} - L_n + (1/2)(3^{n-2} + G_{n-1}) + \sum_{i=1}^{n-2}(L_i - G_{i-1})3^{n-i-2} = (n-2)L_{n-2} + 2L_{n-1} - L_n, \qquad n \geq 2,$$

which may be shown by computing the generating function of each side.

It is possible to give a combinatorial explanation of this latter expression for $\mathrm{tot}_n(212)$ as well as the one above for $\mathrm{tot}'_n(212)$.

Combinatorial proof of $\mathrm{tot}_n(212)$ and $\mathrm{tot}'_n(212)$ formulas:

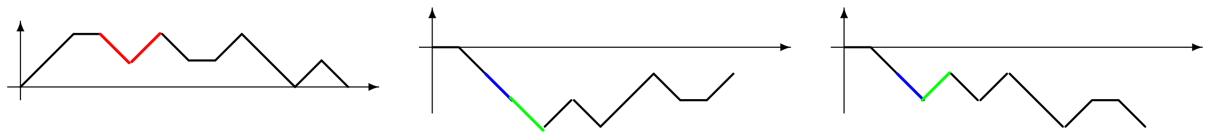
Both formulas are seen to hold for $n = 2, 3, 4$, so we may assume $n \geq 5$. First note that the number of 212's in $\mathcal{U}_n$ equals the number of non-1 letters in $\mathcal{U}_{n-2}$, upon inserting $a-1, a$ directly after some letter $a > 1$ within an arbitrary member of $\mathcal{U}_{n-2}$ and marking the resulting occurrence of 212. By a *return* within a member of $\mathcal{L}_m$, we mean an h or d step terminating on the x-axis. Equivalently, we find the number of steps within members of $\mathcal{L}_{n-2}$ not corresponding to returns, which is given by $(n-3)L_{n-2} - (\#\text{ returns in } \mathcal{L}_{n-2})$. To establish the formula for $\mathrm{tot}_n(212)$, it then suffices to show that the number of returns in $\mathcal{L}_m$ is given by $a_m = L_{m+2} - 2L_{m+1} - L_m$ for all $m \geq 2$.

To do so, let $\mathcal{P}$ denote the subset of $\mathcal{L}_{m+2}$ consisting of those lattice paths π expressible as $\pi = u\pi' d\pi''$, where π' is a non-empty Motzkin path. Let $\alpha(\pi) = \pi'\pi''$, where we mark the return within $\alpha(\pi)$ corresponding to the final step of π'. Then α is a bijection from $\mathcal{P}$ to the set of marked members of $\mathcal{L}_m$ wherein some return is marked. Thus, to complete the proof of the formula for $\mathrm{tot}_n(212)$, one can show $|\mathcal{P}| = a_m$ for all $m \geq 2$. To do so, first note that there are L_{m+1} members of $\mathcal{L}_{m+2}$ starting with h and the same number that start u but do not return to the x-axis. Also, there are L_m additional members of $\mathcal{L}_{m+2}$ which start ud. Combining the preceding cases gives $2L_{m+1} + L_m$ members of $\mathcal{L}_{m+2} - \mathcal{P}$ altogether, which implies $|\mathcal{P}| = a_m$, as desired.

To establish the formula for $\mathrm{tot}'_n(212)$ (with n replaced by $n+1$), first note that the number of 212's in $\mathcal{V}_{n+1}$ equals the number of *valleys* (i.e., occurrences of du) within the members of $\mathcal{M}_n$. Suppose that a valley of some $\rho \in \mathcal{M}_n$ is marked and we decompose ρ as $\rho = \rho' \underline{du} \rho''$, where the marked valley is underlined. Let $\beta(\rho) = \widetilde{\mathrm{rev}}(\rho') d \widetilde{\mathrm{rev}}(\rho'')$. Note that β is onto the set $\mathcal{R}$ consisting of the lattice paths from $(0,0)$ to $(n-1,-1)$ using u, d, and h steps and having minimum height -2 or less, and may be reversed by considering the position of the leftmost step of minimum height. Thus, the total number of valleys in $\mathcal{M}_n$ is given by $|\mathcal{R}|$. To complete the proof, it suffices to show $|\mathcal{R}| = |\mathcal{T}|$, where $\mathcal{T}$ is the set of lattice paths from $(0,0)$ to $(n-1,-3)$ considered in the combinatorial proof of Corollary 3.2 above and shown there to have cardinality $M_{n-1} - M_n + (1/2)(G_n - G_{n-1})$. Let $\tau \in \mathcal{R}$ be decomposed as $\tau = \tau' d \tau''$, where the d denotes the leftmost step of τ terminating at height -2.

Let ref(τ'') be obtained from τ'' by replacing each u with d and d with u, leaving all h steps unchanged (i.e., reflecting the subpath τ'' in the line $y=-2$). Let $\theta(\tau)=\tau' d\,\mathrm{ref}(\tau'')$. Then θ is seen to provide a bijection between $\mathcal{R}$ and $\mathcal{T}$, which completes the proof. □

Figure 5 illustrates the bijections β and θ from the preceding proof applied in that order to $\rho\in\mathcal{M}_{12}$ wherein the marked du in ρ is indicated in red. The steps corresponding to the marked du in ρ within the lattice paths $\beta(\rho)$ and $\theta(\beta(\rho))$ are indicated in green, with the leftmost step ending at height -2 in blue.



$\rho = u^2hdudhud^2ud$ $\qquad$ $\beta(\rho)=hdddudu^2dhu$ $\qquad$ $\theta(\beta(\rho))=hddudud^2uhd$

Figure 5: Lattice paths $\rho\in\mathcal{M}_{12}$, $\beta(\rho)\in\mathcal{R}$, and $\theta(\beta(\rho))\in\mathcal{T}$.

4.2 The patterns 121 and 221

Let $r_{n,i,j}=r_{n,i,j}(p,q)$ denote the joint distribution of 121 and 221 on $\mathcal{U}_{n,i,j}$, and $r_{n,j}$ the comparable distribution on $\mathcal{U}_{n,j}$. Define $r_n(v)=\sum_{j=1}^{n} r_{n,j}v^{j-1}$ for $n\geq 1$ and $r_n^*(v)=\sum_{j=1}^{n-2} r_{n,j+1,j}v^{j-1}$ for $n\geq 3$. Let $R(x;v)=\sum_{n\geq 1} r_n(v)x^n$ and $R^*(x;v)=\sum_{n\geq 3} r_n^*(v)x^n$.

Proceeding similarly as in the prior section, one can establish the following results.

Lemma 4.3. *We have*

$$r_n(v)=r_n^*(v)+(1+v)r_{n-1}(v),\qquad n\geq 2,$$

and

$$r_n^*(v)=pr_{n-2}(v)+\frac{1}{v}(r_{n-1}^*(v)-r_{n-1}^*(0))+\frac{q}{v}(r_{n-2}(v)-r_{n-2}(0)),\qquad n\geq 3,$$

with $r_1(v)=1$ *and* $r_2^*(v)=0$.

Lemma 4.4. *We have*

$$(1-(1+v)x)R(x;v)=x+R^*(x;v)$$

and

$$\left(1-\frac{x}{v}\right)R^*(x;v)=x^2\left(p+\frac{q}{v}\right)R(x;v)-\frac{qx^2}{v}R(x;0)-\frac{x}{v}R^*(x;0).$$

Theorem 4.2. *The generating functions for the joint distributions of the* 121 *and* 221 *subwords on* $\mathcal{U}_n$ *and* $\mathcal{V}_n$ *for* $n\geq 1$ *are given respectively by*

$$R(x;1)=\frac{3x-1-(1-p)x^2+\sqrt{(1-x+(1-p)x^2)^2-4x^2(1-(1-q)x)}}{2(1-3x+(2-p-q)x^2)}$$

and

$$R(x;0)=\frac{1-x+(1-p)x^2-\sqrt{(1-x+(1-p)x^2)^2-4x^2(1-(1-q)x)}}{2x(1-(1-q)x)}.$$

Corollary 4.3. *If* $n\geq 2$, *then*

$$tot_n(121)=\sum_{i=0}^{n-3}G_i3^{n-i-3}$$

and

$$tot_n(221)=\sum_{i=1}^{n-2}(L_i-G_{i-1})3^{n-i-2}.$$

Corollary 4.4. *If* $n\geq 2$, *then* $tot'_n(121)=(1/2)(G_{n-1}-G_{n-2})$ *and* $tot'_n(221)=G_{n-2}-M_{n-2}$.

Comparing the last two results with Corollaries 2.1 and 2.2 shows $\text{tot}_n(221) = \text{tot}_{n-1}(21)$ and $\text{tot}'_n(221) = \text{tot}'_{n-1}(21)$ for all $n \geq 2$, which may be explained directly as before. Further, for $n \geq 3$, we have that the total number of 121's in $\mathcal{U}_n$ and $\mathcal{V}_n$ equals the number of letters in $\mathcal{U}_{n-2}$ and $\mathcal{V}_{n-2}$, respectively, upon inserting $a+1, a$ right after any letter a within an arbitrary member of either set and marking the resulting occurrence of 121. Since there are $(n-2)L_{n-2}$ letters altogether in the members of $\mathcal{U}_{n-2}$, equating this expression for $\text{tot}_n(121)$ with the one from Corollary 4.3, and replacing n by $n+2$, yields the following identity:

$$nL_n = \sum_{i=0}^{n-1} G_i 3^{n-i-1}, \qquad n \geq 1.$$

Likewise, there are $(n-2)M_{n-3}$ total letters in the members of $\mathcal{V}_{n-2}$, and equating this with the expression for $\text{tot}'_n(121)$ from Corollary 4.4 recovers the second identity in Corollary 2.3.

4.3 The patterns 111, 122, and 211

Let $s_{n,i,j} = s_{n,i,j}(p,q,r)$ denote the joint distribution of 111, 122, and 211 on $\mathcal{U}_{n,i,j}$ (marked by p, q, and r, respectively), and let $s_{n,j}$ denote the comparable distribution on $\mathcal{U}_{n,j}$. Define $s_n(v) = \sum_{j=1}^{n} s_{n,j} v^{j-1}$ for $n \geq 1$ and $s'_n(v) = \sum_{j=1}^{n-1} s_{n,j,j} v^{j-1}$ for $n \geq 2$. Let $S(x;v) = \sum_{n\geq 1} s_n(v)x^n$ and $S'(x;v) = \sum_{n\geq 2} s'_n(v)x^n$.

Proceeding as before, one can prove the following results.

Lemma 4.5. *We have*

$$s_n(v) = s'_n(v) + vs_{n-1}(v) + \frac{1}{v}(s_{n-1}(v) - s_{n-1}(0)), \qquad n \geq 2,$$

and

$$s'_n(v) = ps'_{n-1}(v) + qvs_{n-2}(v) + \frac{r}{v}(s_{n-2}(v) - s_{n-2}(0)), \qquad n \geq 3,$$

with $s_1(v) = s'_2(v) = 1$.

Lemma 4.6. *We have*

$$\left(1 - vx - \frac{x}{v}\right) S(x;v) = x + S'(x;v) - \frac{x}{v}S(x;0)$$

and

$$(1-px)S'(x;v) = x^2 + x^2\left(qv + \frac{r}{v}\right)S(x;v) - \frac{rx^2}{v}S(x;0).$$

Theorem 4.3. *The generating functions for the joint distributions of the* 111, 122, *and* 211 *subwords on* $\mathcal{U}_n$ *and* $\mathcal{V}_n$ *for* $n \geq 1$ *are given respectively by*

$$S(x;1) = \frac{(1+(1-p)x)(2(q-p)x^2 + (2+p)x - 1)}{2(1+(q-p)x)(1-(2+p)x+(2p-q-r)x^2)} + \frac{(1+(1-p)x)\sqrt{(1-px)^2 - 4x^2(1+(q-p)x)(1+(r-p)x)}}{2(1+(q-p)x)(1-(2+p)x+(2p-q-r)x^2)}$$

and

$$S(x;0) = \frac{(1+(1-p)x)(1-px-\sqrt{(1-px)^2 - 4x^2(1+(q-p)x)(1+(r-p)x)})}{2x(1+(q-p)x)(1+(r-p)x)}.$$

Corollary 4.5. *If* $n \geq 2$*, then*

$$tot_n(111) = tot_{n-1}(11), \;\; tot_n(122) = tot_{n-1}(12), \;\; \text{and } tot_n(211) = tot_{n-1}(21).$$

Corollary 4.6. *If* $n \geq 2$*, then*

$$tot'_n(111) = tot'_{n-1}(11) \text{ and } tot'_n(122) = tot'_n(211) = tot'_{n-1}(12).$$

The formulas in the preceding two corollaries follow from differentiation of the respective generating functions (and comparison with earlier results) or by direct reasoning. Differentiating $S(x;1)$ with respect to p, and setting $p = q = r = 1$, one gets

$$\frac{\partial}{\partial q} f_{111}(x;q) \mid_{q=1} = \frac{x(1-2x)}{1-3x}L(x) + \frac{x}{2(1-3x)}\left(1 - 2x - \frac{1-x-4x^2}{\sqrt{1-2x-3x^2}}\right).$$

Extracting the coefficient of x^n then gives

$$\mathrm{tot}_n(111) = L_{n-1} - 3^{n-2} + \sum_{i=1}^{n-2} L_i 3^{n-i-2} - (1/2)\sum_{i=2}^{n-1}(G_i - G_{i-1} - 4G_{i-2})3^{n-i-1}, \qquad n \geq 2.$$

Equating this expression with the obvious formula $\mathrm{tot}_n(111) = (n-2)L_{n-2}$ gives a further identity relating L_n and G_n.

As special cases of the main results of the prior sections, one obtains the formulas in Tables 1 and 2 below for the generating functions $f_\tau(x;q)$ and $g_\tau(x;q)$ of the corresponding univariate distributions.

τ	$f_\tau(x;q)$
11	$\frac{(q+2)x-1+\sqrt{1-2qx+(q^2-4)x^2}}{2(1-(q+2)x)}$
12	$\frac{(1+2q)x-1+\sqrt{1-2x+(1-4q)x^2}}{2q(1-(q+2)x)}$
21	$\frac{3x-1+\sqrt{1-2x+(1-4q)x^2}}{2(1-(2+q)x)}$
111	$\frac{2(1-q)x^2+(2+q)x-1+\sqrt{(1-qx)^2-4x^2(1+(1-q)x)^2}}{2(1-(2+q)x-2(1-q)x^2)}$
112	$\frac{2(q-1)x^2+3x-1+\sqrt{1-2x-3x^2-4(q-1)x^3}}{2(1+(q-1)x)(1-3x-(q-1)x^2)}$
121	$\frac{(q-1)x^2+3x-1+\sqrt{(1-x-(q-1)x^2)^2-4x^2}}{2(1-3x-(q-1)x^2)}$
122	same as 112
123	$\frac{(1+(1-q)x)(3(1-q)x^2+(1+2q)x-1+\sqrt{(1-x-(1-q)x^2)^2-4x^2(q+(1-q)x)})}{2(q+(1-q)x)(1-(2+q)x-2(1-q)x^2)}$
211	$\frac{3x-1+\sqrt{1-2x-3x^2+4(1-q)x^3}}{2(1-3x+(1-q)x^2)}$
212	$\frac{-(1-q)^2x^3+2(1-q)x^2+(2+q)x-1+(1-(1-q)x)\sqrt{(1-x+(1-q)x^2)^2-4x^2}}{2(q+(1-q)x)(1-3x+(1-q)x^2)}$
221	same as 211
321	$\frac{(1-q)x^2+3x-1+\sqrt{(1-x-(1-q)x^2)^2-4x^2(q+(1-q)x)}}{2(1-(2+q)x-2(1-q)x^2)}$

Table 1: The generating functions $f_\tau(x;q)$ for all subwords τ of length two or three.

τ	$g_\tau(x;q)$
11	$\frac{1-qx-\sqrt{1-2qx+(q^2-4)x}}{2x}$
12	$\frac{1-x-\sqrt{1-2x+(1-4q)x^2}}{2qx}$
21	same as 12
111	$\frac{1-qx-\sqrt{(1-qx)^2-4x^2(1+(1-q)x)^2}}{2x(1+(1-q)x)}$
112	$\frac{1-x-\sqrt{1-2x-3x^2-4(q-1)x^3}}{2x(1+(q-1)x)}$
121	$\frac{1-x+(1-q)x^2-\sqrt{(1-x+(1-q)x^2)^2-4x^2}}{2x}$
122	same as 112
123	$\frac{1-x-(1-q)x^2-\sqrt{(1-x-(1-q)x^2)^2-4x^2(q+(1-q)x)}}{2x(q+(1-q)x)}$
211	same as 112
212	same as 123
221	same as 112
321	same as 123

Table 2: The generating functions $g_\tau(x;q)$ for all subwords τ of length two or three.

5. Concluding remarks

From Table 1, one has the following pair of non-trivial equivalences on $\mathcal{U}_n$.

Theorem 5.1. *The* 211 *and* 221 *subword patterns have equal distributions on* $\mathcal{U}_n$ *for all* $n \geq 1$*, as do the patterns* 112 *and* 122*.*

Proof. To show the first equivalence bijectively, suppose $\pi = \pi_1 \cdots \pi_n \in \mathcal{U}_n$. By a (decreasing) *block* within π, we mean a subsequence B of consecutive letters $\pi_a \cdots \pi_b$ within π such that $\pi_a \geq \pi_{a+1} \geq \cdots \geq \pi_b$ with $\pi_a > \pi_b$ and that is not strictly contained within any other weakly decreasing string of letters. Note that B a block implies $a = 1$ or $a > 1$ with $\pi_{a-1} < \pi_a$ and either $b = n$ or $b < n$ with $\pi_{b+1} > \pi_b$. Suppose that the distinct letters of a block B of π are given by $x_1 > \cdots > x_r$ for some $r \geq 2$, with x_i for each i occurring exactly m_i times in B. That is, $B = x_1^{m_1} \cdots x_r^{m_r}$. We replace the sequence of letters comprising the block B with the sequence $B^* = x_1^{m_r} x_2^{m_2} \cdots x_{r-1}^{m_{r-1}} x_r^{m_1}$, and perform this operation on each block of π. Let π^* denote the resulting member of $\mathcal{U}_n$.

Let $\mathcal{U}_n^{(a,b)}$ denote the subset of $\mathcal{U}_n$ whose members contain a occurrences of the subword 211 and b occurrences of 221. Then the mapping $\pi \mapsto \pi^*$ is an involution on $\mathcal{U}_n$ which maps $\mathcal{U}_n^{(a,b)}$ to $\mathcal{U}_n^{(b,a)}$, and vice versa, for all a and b. To see this, suppose that an arbitrary block B within π contains exactly k occurrences of 211 and ℓ occurrences of 221. Then we have that k and ℓ equal respectively the number of indices $i \in [2, r]$ and $i \in [r-1]$ such that $m_i > 1$. Thus, the block B^* within π^* is seen to contain ℓ and k occurrences of 211 and 221, respectively. Since every occurrence of either pattern is contained in some block, it follows that $\pi \in \mathcal{U}_n^{(a,b)}$ for some a and b implies $\pi^* \in \mathcal{U}_n^{(b,a)}$, which completes the proof of the first equivalence. The second can be shown in a similar manner by considering increasing, instead of decreasing, blocks. □

Note that the pair of equivalences in Theorem 5.1 also holds for $\mathcal{V}_n$ since the mapping $\pi \mapsto \pi^*$ preserves final letters. All other equivalences of subword patterns on $\mathcal{V}_n$ may be explained using the reversal operation (except for the one between 123 and 212). In particular, the four patterns in Theorem 5.1 are equivalent on $\mathcal{V}_n$. Further, combining the reversal operation with the mapping $\pi \mapsto \pi^*$ accounts for the symmetry of q and r witnessed by the generating function formula for $S(x; 0)$ in Theorem 4.3. Finally, to realize the equivalence of 123 and 212 on $\mathcal{V}_n$, note that it is the same as the equivalence of the statistics recording the number of uu's and du's on $\mathcal{M}_{n-1}$. The latter equidistribution is true due to a previous bijection of Callan [9].

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