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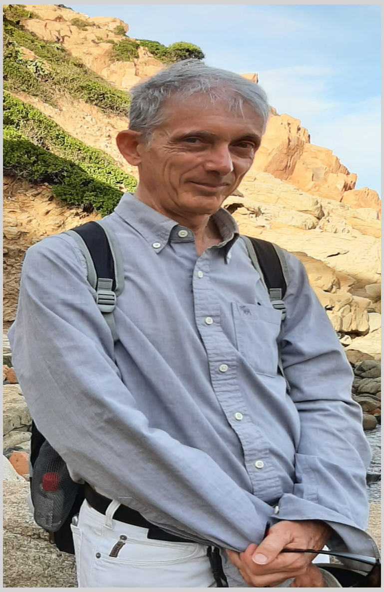
Interview with Francesco Brenti

Toufik Mansour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Francesco Brenti completed his Ph.D. at the Massachusetts Institute of Technology (MIT) in 1988 under the supervision of Richard Stanley. During his career, he worked at MIT (1984-1988), the University of Michigan (1988-1991), the Mittag-Leffler Institute (1991-1992), the University of Perugia (1992-1997), and the University of Rome "Tor Vergata" (1997-present). Since 2013, he has been a full professor at the University of Rome "Tor Vergata." His visiting positions include the Hebrew University of Jerusalem; the University of Marne-la-Vallée, Noisy-le-Grand; the Institute for Advanced Study, Princeton; the Mathematical Sciences Research Institute, Berkeley; the Royal Institute of Technology, Stockholm; the Mathematical Research Centre, Aarhus; Bar-Ilan University, Ramat Gan; MIT; the Mittag-Leffler Institute, Djursholm. His research interests are in algebraic combinatorics, Kazhdan-Lusztig theory, the theory of symmetric functions, and total positivity. He is a member of the editorial boards of the Séminaire Lotharingien de Combinatoire (since 1997), the European Journal of Combinatorics (since 2008), Algebraic Combinatorics (since 2020), and Bollettino dell'Unione Matematica Italiana (since 2020).

Mansour: Professor Brenti, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Brenti: In my view, combinatorics is the study of configurations of finite sets.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Brenti: I think that this is a wonderful and natural development that has put combinatorics firmly in the mathematical landscape.

Mansour: What have been some of the main goals of your research?

Brenti: I think that my research had two main goals: (1) prove that certain sequences, particularly arising from combinatorics, but not only, are unimodal and (2) understand the Kazhdan-Lusztig polynomials from a combinatorial point of view.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Brenti: When I was 12 years old I was almost flunked in mathematics, at that same age my father brought home a book (which I still have) entitled "Caso e probabilità" ("Chance and Probability"). I read it and was fascinated by the fact that pure reasoning could enable you to "predict" the future (for example, that if you throw two dice then you will get 6 or 7 more often than other totals). From that point on, I never stopped reading mathematics books and later articles.

Toufik Mansour is a professor of mathematics at the University of Haifa, Israel. His email address is tmansour@univ.haifa.ac.il

Mansour: Were there specific problems that made you first interested in combinatorics?

Brenti: Yes, when I was about to finish my undergraduate degree, Professor Adriano Barlotti told me about some open problems in finite geometries, and I was particularly fascinated by Graeco-Latin squares, and by the existence of a finite projective plane of order 10.

Mansour: What was the reason you chose MIT for your Ph.D. and your advisor Richard Stanley?

Brenti: I had always been fascinated by the United States, and when I asked Professor Barlotti about the possibilities of going there for a Ph.D. he suggested Caltech, MIT, and the University of Waterloo. I applied to all of these places, got accepted by all three, and chose MIT because I thought that it was the most prestigious school, and because Gian-Carlo Rota was there. When I got to MIT I started working, during my first summer there, with Rota. At the time he was very involved with invariant theory and umbral calculus. It was notationally very heavy stuff, and I did not like it very much (I remember him telling me once that "You have to work on ugly stuff in order to make it beautiful" and a few weeks later saying to him "Ugly it is, from that point of view we are going well"). At the same time, I had been exposed to enumerative and algebraic combinatorics through Richard Stanley's "Combinatorial Theory" graduate course and research seminars. I had also started reading in my spare time, for my own curiosity and pleasure, Stanley's two papers on plane partitions. After a while, I decided that this was crazy, and switched to Stanley.

Mansour: What would guide you in your research? A general theoretical question or a specific problem?

Brenti: I have almost always been guided in my research by a specific problem.

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Brenti: Oh sure! Sometimes even too much!

Mansour: What three results do you consider the most influential in combinatorics during the last thirty years?

Brenti: The proof of the log-concavity of the chromatic polynomial by Huh[1], the strong perfect graph theorem by Chudnovsky[2], Robertson, Seymour, and Thomas[3], and the proof of the generalized lower bound conjecture by Karu[4].

Mansour: What are the top three open questions in your list?

Brenti: (1) The combinatorial invariance conjecture for Kazhdan-Lusztig polynomials[5]; (2) Finding a combinatorial interpretation for Kazhdan-Lusztig polynomials[6]; and (3) Finding a combinatorial interpretation for the coefficients obtained by expanding the plethysm of two Schur functions into the basis of Schur functions[7].

Mansour: What kind of mathematics would you like to see in the next ten-to-twenty years as the continuation of your work?

Brenti: Whether as a continuation of my work or not, I would like to see the solutions of the three conjectures/open problems just mentioned.

Mansour: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

Brenti: I feel that there are definitely mainstream areas in mathematics (i.e., areas in which many mathematicians work). I also think that the importance of a topic is proportional to the applications (i.e., consequences) that it has.

Mansour: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Brenti: It seems to me that it is natural to

[1] J. Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, J. Amer. Math. Soc. 25 (2012), 907–927.

[2] See, Interview with Maria Chudnovsky, ECA 1:2 (2021) Interview S3I4.

[3] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The strong perfect graph theorem*, Ann. of Math. 164 (2006), 51–229.

[4] K. Karu, *Hard Lefschetz theorem for nonrational polytopes*, Invent. Math., 157(2004), 419–447.

[5] F. Brenti, *Kazhdan-Lusztig polynomials: history problems, and combinatorial invariance*, Sém. Lothar. Combin. 49 (2002/04), Article B49b.

[6] See `https://www.samuelfhopkins.com/OPAC/files/proceedings/brenti.pdf`.

[7] R. Stanley, *Enumerative combinatorics*, Volume 2, Cambridge Stud. Adv. Math. 62, Cambridge University Press, Cambridge, see Theorem A2.7 and the comments following it.

call "applied" mathematics that has applications outside of mathematics, so I feel that the distinction is reasonable. I also think that "pure" mathematics can unexpectedly become applied, and conversely. I would consider myself a "pure" mathematician. I see the relationship between "pure" and "applied" mathematics as mutually beneficial.

Mansour: You have supervised many Ph.D. students for their thesis. What do you think about working with graduate students and passing knowledge to them? Do you follow your students after their graduation?

Brenti: It is wonderful to pass on to a young person who wishes to engage in research the things that you have learned. Sometimes you may be disappointed by the decisions that they take... but it is their life. I like to follow my students after their Ph.D., although this is slightly more difficult with students that go out of academia.

Mansour: What advice would you give to young people thinking about pursuing a research career in mathematics?

Brenti: My advice is to pursue a research career in mathematics if research is what you really want to do, your core motivation, your passion. You should not pursue a research career in mathematics for the perceived "prestige" of being a university professor or a scientist.

Mansour: Would you tell us about your interests besides mathematics?

Brenti: I am passionate about tennis and karate (particularly Shotokan karate, I got my black belt in 1978). I also enjoy art and architecture.

Mansour: Rome is one of the oldest imperial cities and metropolises. It is one of the most popular tourist destinations because of its history, art, architecture, beauty, and perhaps pasta and pizza! Is it also an inspiring city for a mathematician?

Brenti: There is a fair amount of mathematical activity going on in Rome: three public universities; the Italian Institute of Higher Mathematics (INdAM); and the Accademia dei Lincei. There is also a rather large number of visitors. The place that I find most inspiring for doing mathematics in Rome is the library of the Department of Mathematics (Istituto Guido Castelnuovo) of the University of Rome "La Sapienza". Other ancient libraries in Rome are also very inspiring.

Mansour: You have given talks at numerous conferences, workshops, and seminars. What do you think about the importance of such activities for the researchers? During the pandemic, virtual meetings became popular. Should we continue having online meetings even after the pandemic? What do you think about the advantages and disadvantages of face-to-face and online meetings?

Brenti: I believe that attending a couple of conferences/workshops each year is very important for a researcher to keep abreast of the field of research. I feel that in-person meetings are scientifically more valuable because of the many occasions for chance encounters that they offer. On the other hand, online meetings can be attended also by mathematicians that do not have access to funds, and they are much better for the environment. I think that we should continue having online meetings even after the pandemic.

Mansour: Few women indeed pursue a career in mathematics. There has been discussion about how to get more women interested in math. You have served on several committees that promote women's participation in mathematics, including the Joint Committee on Women in Mathematical Sciences (JCW) and the World Meeting for Women in Mathematics in 2018. What should be done in the next ten years to get more women interested in mathematics?

Brenti: I have never participated in the conferences/committees that you mention. I think that striving for gender balance whenever possible (in conferences, academic recruiting, etc.) and having funding and conferences reserved for women should help solve this problem.

Mansour: You coauthored the book on *Combinatorics of Coxeter Groups* with Anders Björner[8], which became highly influential in the field. What is a Coxeter group, and how is it related to enumerative combinatorics? Would you tell us about some important open questions related to Coxeter Groups?

Brenti: A Coxeter group is a group generated by involutions in which the only relations are given by the orders of pairwise products

[8] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, 231, Springer, New York, 2005.

of generators. The connections to enumerative combinatorics come up in a number of ways. As with any group given by generators and relations, there is a notion of length, so there is an associated length-generating function and various q-analogues of it that can naturally be defined, and there is a Cayley graph. Also, Coxeter groups possess a natural partial order structure on their elements and this partial order (Bruhat order) is a rich source of enumerative questions. A Coxeter group also has associated an important family of polynomials with integer coefficients in one variable, indexed by pairs of elements in the group, known as the Kazhdan-Lusztig polynomials of the group. It has been proved, using techniques from category theory, that these coefficients are always nonnegative and a natural and long-standing open problem is that of finding a combinatorial interpretation for them. Another related open problem is the combinatorial invariance conjecture which asserts that the Kazhdan-Lusztig polynomial associated with a pair of elements only depends on the interval determined by them under Bruhat order as an abstract poset. Given that the polynomials are defined by a complicated algebraic recursion, and that in many cases (e.g., for Weyl groups) they have important geometric and representation-theoretic interpretations the conjecture would say that, in fact, they are combinatorial objects that can be associated to any poset with suitable properties.

Mansour: *Unimodality*[9] is an interesting topic. Would you tell us about it and why it is important?

Brenti: A sequence of real numbers is unimodal if it first weakly increases and then weakly decreases, so it never has a "valley". Related concepts are log-concavity (the square of each number in the sequence is greater than or equal to the product of the two adjacent numbers), and the generating polynomial having only real roots. The study of these concepts has led to amazing discoveries including multivariate analogs of the last two of them (Lorentzian and stable, respectively) and the realization that often the first two are shadows of algebraic structures that closely resemble various cohomology theories of complex algebraic varieties. Aside from this, these properties have applications to probability theory (for example, if a sequence is log-concave then an estimate of its maximum can be obtained just from knowledge of the first few terms).

Mansour: You have a paper titled *Combinatorics and total positivity*[10]. What is total positivity? How do we use them in enumeration problems?

Brenti: A matrix is totally positive (more appropriately called totally nonnegative) if all its minors are nonnegative. Many matrices arising in enumerative combinatorics are totally positive, and total positivity is closely related to polynomials with only real roots and therefore also to log-concavity and unimodality. If a matrix arising from combinatorics is totally positive then of course one would like to have a combinatorial interpretation of the various minors. In some cases, this can be done using non-intersecting lattice paths but for many other matrices, this is a challenging open problem. So, totally positive matrices have not really been *used* in enumerative combinatorics, but are a rich source of combinatorial problems.

Mansour: You have papers[11] related to the hyperoctahedral group. How do the group structures change a combinatorial question?

Brenti: When passing from the symmetric group to the hyperoctahedral one (i.e., the group of signed permutations) questions usually become more complicated, but not more difficult. So typically the proof of a certain result is twice as long as one of the corresponding statements for the symmetric group, but the basic ideas are the same.

Mansour: In recent joint work with Roberto Conti, Gleb Nenashev, *Permutative automorphisms of the Cuntz algebras: Quadratic cycles, an involution, and a box product*[12], you obtained some interesting results on stable permutations from a combinatorial point of view.

[9]P. Brändén, *Unimodality, log-concavity, real-rootedness and beyond*, Handbook of enumerative combinatorics, CRC Press, Boca Raton, FL, 2015, 437–483.

[10]F. Brenti, *Combinatorics and total positivity*, J. Combin. Theory Ser. A 71:2 (1995), 175–218.

[11]F. Brenti, *Parabolic Kazhdan-Lusztig polynomials for Hermitian symmetric pairs*, Trans. Amer. Math. Soc. 361:4 (2009), 1703–1729.

[12]F. Brenti, R. Conti, and G. Nenashev, *Permutative automorphisms of the Cuntz algebras: quadratic cycles, an involution and a box product*, Adv. in Appl. Math. 143 (2023), Paper 102447.

Among the main results of the past is a characterization of the stable cycles of rank one in $S([n]^2)$, thereby proving a conjecture of yourself and Conti. Would you say more about this conjecture and the main ideas behind its proof?

Brenti: The conjecture gives necessary and sufficient conditions for a cycle u of the discrete $n \times n$ square so that if you act on the discrete $n \times n \times n$ cube by u on the first two coordinates and then on the last two coordinates, and conversely (first on the last ones and then on the first ones) you get the same result (i.e., the two actions commute). The proof proceeds by considering the functional digraphs of these two permutations of the discrete cube so these digraphs consist of copies of the functional digraph of u on each "vertical" slice of the cube, and on each "horizontal" slice. One then shows that, if the two actions commute, then these two digraphs are edge-disjoint. On the other hand, if the condition in the conjecture were false, then there would be a vertex of degree one in both functional digraphs, and from this one derives a contradiction.

Mansour: In many of your publications, you propose conjectures and sometimes prove them. Which work is more interesting: claiming conjectures or proving them?

Brenti: I think that proving them is more interesting because the proof tells you at least one reason "why" the conjecture is true.

Mansour: In your work, you have extensively used combinatorial reasoning to address important problems. How do enumerative techniques engage in your research?

Brenti: I am a combinatorialist and I am always looking for combinatorics. If I find combinatorial concepts in a good problem arising from other areas of mathematics (typically algebra or geometry) I am usually interested and try to see what combinatorial techniques can bring to that problem. The combinatorial concepts that I have found most often have been permutations, posets, and lattice paths.

Mansour: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have a "eureka moment"?

Brenti: One of my favorite results is the proof of the combinatorial invariance conjecture[5] for lower intervals of the symmetric group. I became interested in the combinatorial invariance conjecture in 1991-92 during the special year devoted to combinatorics at the Mittag-Leffler Institut. I do not remember who first told me about the conjecture (probably either Anders Björner or Mark Haiman). In the spring semester of 2002, I had no teaching and I had gotten this idea, that seemed to work, that special matchings enabled you to compute the Kazhdan-Lusztig polynomial of a lower interval in the symmetric group just from the poset structure. I could see no reason why such a result should be true and so concluded that the only way to see if it was indeed true was to classify all the special matchings of any lower interval in S_n. I usually work on examples, and I knew that working out a non-trivial one took me about a day. As I said, I had no teaching that semester, so I selected 32 particularly interesting lower intervals and decided that I would devote one month to working them out. At the end of the month, I understood how to classify them. I did not have a "eureka moment". In general, my proofs are more of a slow coming together of various pieces, a bit like assembling a puzzle.

Mansour: Is there a specific problem you have been working on for many years? What progress have you made?

Brenti: Yes, the combinatorial invariance conjecture[5] that I have already mentioned a couple of times. In my mathematical life, I have proved it (often in collaboration with other mathematicians) in various special cases, including for lower intervals. In the last 2 years, there has been further progress, particularly in the symmetric group case. Using tools from deep learning, Williamson and coauthors[13] have proposed an explicit procedure that seems to compute, just from the poset structure, the Kazhdan-Lusztig polynomial of any interval in S_n. After that paper appeared

[13] C. Blundell, L. Buesing, A. Davies, P. Veličković, and G. Williamson, *Towards combinatorial invariance for Kazhdan-Lusztig polynomials*, Represent. Theory 26 (2022), 1145–1191.

[14] F. Brenti and M. Marietti, *Fixed points and adjacent ascents for classical complex reflection groups*, Adv. in Appl. Math. 101 (2018), 168–183.

[15] G. T. Barkley and C. Gaetz, *Combinatorial invariance for elementary intervals*, arXiv:2303.15577.

two more explicit procedures have been proposed, one by Marietti and myself[14], and one by Barkley and Gaetz[15].

Mansour: In a very recent short article[16] published in the Newsletter of the European Mathematical Society, Professor Melvyn B. Nathanson, while elaborating on the ethical aspects of the question "Who Owns the Theorem?" concluded that "Mathematical truths exist, and mathematicians only discover them." On the other side, there are opinions that "mathematical truths are invented". As a third way, some people claim that it is both invented and discovered. What do you think about this old discussion? More precisely, do you believe that you *invent* or *discover* your theorems?

Brenti: I believe that we discover mathematics.

Mansour: Professor Francesco Brenti, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

Brenti: Thank you for the opportunity.

[16] M. B. Nathanson, *Who Owns the Theorem?* The best writing on Mathematics 2021, Princeton: Princeton University Press, 2022, 255–257.