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Set Partitions that Require a Maximum Number of Sorts Through the $aba-$avoiding Stack

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Abstract: Recently, Xia introduced a deterministic variation ϕ_σ of Defant and Kravitz's stack-sorting maps for set partitions and showed that any set partition p is sorted by $\phi_{aba}^{N(p)}$, where $N(p)$ is the number of distinct letters in p. Xia then asked which set partitions p are not sorted by $\phi_{aba}^{N(p)-1}$. In this note, we prove that the minimal length of a set partition p that is not sorted by $\phi_{aba}^{N(p)-1}$ is $2N(p)$. Then we show that there is only one set partition of length $2N(p)$ and $\binom{N(p)+1}{2} + 2\binom{N(p)}{2}$ set partitions of length $2N(p)+1$ that are not sorted by $\phi_{aba}^{N(p)-1}$.

Keywords: Pattern avoidance; Stack sort; Set partitions
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1. Introduction

In 1973, Knuth [6] introduced a non-deterministic stack-sorting machine that at each step, either pushes the leftmost remaining entry of the input permutation into the stack or pops the topmost entry of the stack. In 1990, West [8] modified Knuth's stack-sorting machine to make it deterministic. In West's deterministic stack-sorting map s, the input permutation is sent through a stack in a right-greedy manner, while insisting that the stack is increasing from top to bottom (see for example, Figure 1). Put differently, the stack in West's stack-sorting map s must avoid subsequences that are order-isomorphic to 21. It is well-known that $s^{n-1}(\pi) = \text{id}$ for any $\pi \in S_n$.

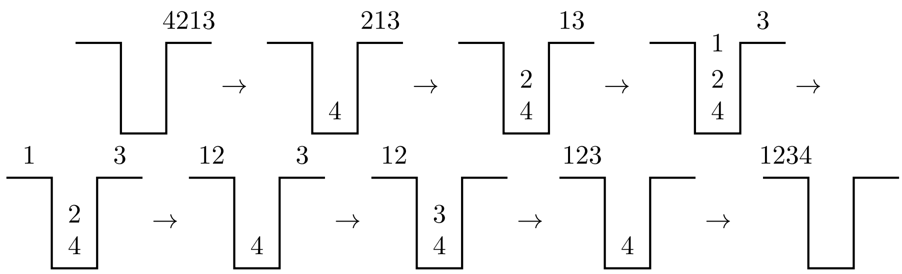

Figure 1: West's stack-sorting map s on $\pi = 4213$

West's stack-sorting map [8] has been extended since. In 2002, Atkinson, Murphy, and Ruškuc [1] introduced a stack-sorting map that processes the input permutation in a left-greedy manner instead of in a right-greedy manner as in West's stack-sorting map [8]. In 2014, Smith [7] extended West's stack-sorting map so that the stack decreases from top to bottom as opposed to increase as in West's stack-sorting map [8]. In 2020, Cerbai, Claesson, and Ferrari [3] extended West's stack-sorting map s to $s \circ s_\sigma$, where the map s_σ sends the input permutation through a stack in a right greedy manner, while maintaining that the stack avoids subsequences that are order-isomorphic to some permutation σ (Note that $s_{21} = s$). In the following year, Berlow [2] generalized s_σ to s_T, in which the stack must simultaneously avoid subsequences that are order isomorphic to any of the permutations in the set T, while Defant and Zheng [5] generalized s_σ to $s_{\overline{\sigma}}$, in which the stack must avoid substrings that are order isomorphic to σ at all times.

More recently, in 2024, Defant and Kravitz [4] generalized Knuth's non-deterministic stack-sorting-machine [6] to *set partitions*, which are sequences of (possibly repeated) letters from some infinite alphabet A. In the same year, Xia [9] introduced a deterministic variation ϕ_σ of Defant and Kravitz's stack-sorting map for set partitions [4] as West did [8] of Knuth's stack-sorting machine [6]. A set partition is said to be *sorted* if all occurrences of the same letter appear consecutively in the set partition, and two set partitions $p = p_1p_2\cdots p_n$ and $q = q_1q_2\cdots q_n$ are *equivalent* if there exists some bijection $f : A \to A$ such that $q = f(p_1)f(p_2)\cdots f(p_n)$. In Xia's deterministic stack-sorting map ϕ_σ for set partitions, the input set partition is sent through a stack in a right-greedy manner, while insisting that the stack avoids subsequences that are equivalent to the set partition σ (see for example, Figure 2).

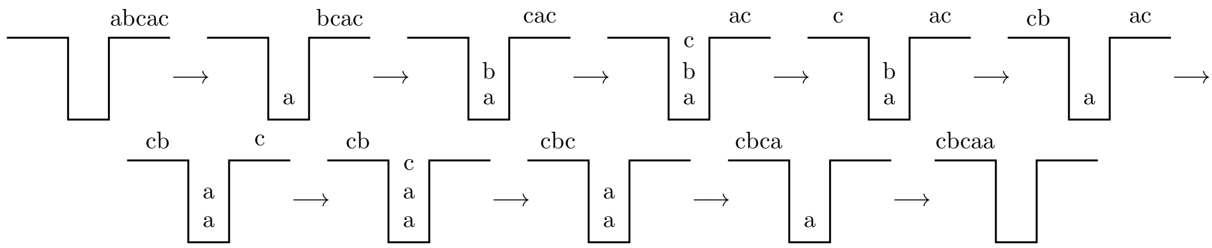

Figure 2: Xia's stack-sorting map ϕ_{aba} on $p = abcac$

In addition to introducing ϕ_σ, Xia [9, Proposition 5.2] showed that ϕ_{aba} is the only ϕ_σ that eventually sorts all set partitions. Then Xia [9, Theorem 3.1] showed that any set partition p is sorted after applying $\phi_{aba}^{N(p)}$, where $N(p)$ is the number of distinct letters in p, and demonstrated the sharpness of her bound by proving that $p = (a_1a_2\cdots a_{N(p)})^2$ is not sorted after applying $\phi_{aba}^{N(p)-1}$ for any $N(p) \geq 3$. Finally, Xia [9, Question 6.1] asked which set partitions p are not sorted after applying $\phi_{aba}^{N(p)-1}$. We first answer Xia's question with the restriction that $|p| \leq 2N(p)$.

Theorem 1.1. *If set partition p satisfies $|p| \leq 2N(p)$ for some $N(p) \geq 3$ and is not sorted after applying $\phi_{aba}^{N(p)-1}$, then p is equivalent to $(a_1a_2\cdots a_{N(p)})^2$.*

Theorem 1.1 proves that for any fixed $N(p) \geq 3$, Xia's example in [9, Theorem 3.1] is, up to equivalence, the only shortest set partition p that is not sorted after applying $\phi_{aba}^{N(p)-1}$. In Theorem 1.2, we enumerate the set partitions of length $2N(p)+1$ that are not sorted after applying $\phi_{aba}^{N(p)-1}$.

Theorem 1.2. *For a fixed $N(p) \geq 3$, the number of inequivalent set partitions p that satisfy $|p| = 2N(p)+1$ and are not sorted after applying $\phi_{aba}^{N(p)-1}$ is $\binom{N(p)+1}{2} + 2\binom{N(p)}{2}$.*

The rest of this note is organized as follows. In Section 2, we establish the preliminaries. In Section 3, we prove Theorems 1.1 and 1.2.

2. Preliminaries

Let A be an infinite alphabet. In this note, we use $a_1, a_2, a_3, \ldots$ or the standard Latin alphabet $a, b, c, \ldots$ to refer to the letters of A. Unless otherwise specified, $a_1, a_2, a_3, \ldots$ are distinct letters of A.

First, for a (possibly empty) set partition p, let $|p|$ be its length, and let $p^m = \underbrace{pp\cdots p}_{m\text{ times}}$. In addition, for a (possibly empty) set partition $p = p_1p_2\cdots p_{|p|}$, let $p_{[i:j]} = p_ip_{i+1}\cdots p_j$. Next, let the *reverse* of a set partition p be $\mathrm{r}(p) = p_{|p|}p_{|p|-1}\cdots p_1$. For example, if $p = abcac$, then $\mathrm{r}(p) = cacba$.

Next, for $p = p_1p_2\cdots p_{|p|}$ and $a \in A$, say that $a \in p$ if there exists some i such that $p_i = a$. Furthermore, as in Xia [9], let $I(p, B)$ be the set of i such that $p_i \in B$ for a set of letters $B \subseteq A$. If $|B| = 1$, then we omit the brackets around the set B. For example, if $p = a_1a_2a_2a_3a_1a_1$, then $I(p, a_1) = \{1, 5, 6\}$, and $I(p, \{a_1, a_3\}) = \{1, 4, 5, 6\}$. Let the i^{th} smallest number in the set $I(p, B)$ be $I^i(p, B)$.

Next, for any p, let $\text{mcount}(p) = \max_{a\in A} |I(p, a)|$. For example, $\text{mcount}(p) = 2$ for $p = a_1a_2a_3a_1a_3$. Now, for $\{a_j, a_k\} \subseteq p$ such that $|I(p, a_j)| \geq 2$ and $|I(p, a_k)| \geq 2$, say that a_j and a_k are *crossing* in p if

$$\min(I(p, a_j)) < \min(I(p, a_k)) < \max(I(p, a_j)) < \max(I(p, a_k)).$$

For example, a_1 and a_2 are crossing in $p = a_1a_2a_1a_3a_2a_3$, but a_1 and a_3 are not.

Also, as defined by Xia [9], say that a letter in p is *clumped* in p if all instances of the letter appear consecutively in p. Let $C(p)$ be the number of clumped letters in p, and let $\text{nc}(p)$ be the leftmost letter in p that is not clumped in p. For example, in $p = a_1a_1a_1a_2a_3a_4a_2a_4$, the letters a_1 and a_3 are clumped, so $C(p) = 2$ and $\text{nc}(p) = a_2$. Note that p is sorted if and only if $C(p) = N(p)$. Now, every set partition p can be uniquely written as $p = a_1^{\ell_1}a_2^{\ell_2}\cdots a_m^{\ell_m}$ for some possibly repeating set of letters $a_1, a_2, \ldots, a_m$ such that $a_i \neq a_{i+1}$ for all $1 \leq i \leq m-1$ and $\ell_i > 0$ for all $1 \leq i \leq m$. Then let the *truncation* of a set partition p be $\text{trunc}(p) = a_1a_2\cdots a_m$. For example, if $p = a_1a_1a_1a_2a_2a_1a_1a_3$, then $\text{trunc}(p) = a_1a_2a_1a_3$. We end this section by citing a lemma and a corollary in Xia [9].

Lemma 2.1 (Xia [9, Lemma 3.1])**.** *Let $p = p_1^{\ell_1}s_1p_1^{\ell_2}s_2\cdots p_1^{\ell_m}s_mp_1^{\ell_{m+1}}$ for $\ell_1, \ell_2, \ldots, \ell_m > 0$ and $\ell_{m+1} \geq 0$ such that p_1 is the first letter of p and s_i are nonempty set partitions such that $p_1 \notin s_i$ for all $1 \leq i \leq m$. Then*

$$\phi_{aba}(p) = \phi_{aba}(s_1)\phi_{aba}(s_2)\cdots\phi_{aba}(s_m)p_1^{\ell_1+\ell_2+\cdots+\ell_{m+1}}.$$

Now, it follows as a corollary of Lemma 2.1 that if p is not sorted, then $C(\phi_{aba}(p)) > C(p)$, because $\text{nc}(p)$ is not clumped in p but is clumped in $\phi_{aba}(p)$.

Corollary 2.1 (Xia [9, Proof of Theorem 3.1])**.** *If p is not sorted, then $C(\phi_{aba}(p)) > C(p)$.*

The following corollary follows immediately from Corollary 2.1.

Corollary 2.2. *If p is not sorted by $\phi_{aba}^{N(p)-1}$, then $C(\phi_{aba}^i(p)) = i$ for all $0 \leq i \leq N(p)$.*

3. Proofs of the Main Results

To prove Theorem 1.1, we first note that the following proposition follows directly from the definition of truncation.

Proposition 3.1. *For any p, it holds that $\text{trunc}(\phi_{aba}(p)) = \text{trunc}(\phi_{aba}(\text{trunc}(p)))$.*

We now prove Theorem 1.1 through Lemma 2.1, Corollary 2.2, and Proposition 3.1.

Proof of Theorem 1.1. By Xia [9, Theorem 3.1], any set partition that is equivalent to $(a_1a_2\cdots a_{N(p)})^2$ is not sorted after applying $\phi_{aba}^{N(p)-1}$ for $N(p) \geq 3$. It thus suffices to show that if p satisfies $|p| \leq 2N(p)$ and is not sorted after applying $\phi_{aba}^{N(p)-1}$, then it is equivalent to $(a_1a_2\cdots a_{N(p)})^2$, towards which, we induct on $N(p)$.

The statement clearly holds for $N(p) = 3$. Now, suppose that $N(p) \geq 4$ and that if some set partition q satisfies $|q| \leq 2N(q) - 2$ and is not sorted after applying $\phi_{aba}^{N(q)-2}$, then it is equivalent to $(a_1a_2\cdots a_{N(q)-1})^2$. First, by Corollary 2.2, $C(\phi_{aba}^0(p)) = C(p) = 0$, so every $a \in p$ must satisfy $|I(p, a)| \geq 2$. But because $|p| \leq 2N(p)$, it must be that $|I(p, a)| = 2$ for all $a \in p$.

Now, let $p = p_1s_1p_1s_2$ for some set partitions s_1 and s_2. Then because $C(\phi_{aba}(p)) = 1$, each $a(\neq p_1) \in p$ satisfies $a \in s_1$ and $a \in s_2$; otherwise, by Lemma 2.1, at least one of $\text{nc}(s_1)$ or $\text{nc}(s_2)$ are clumped in $\phi_{aba}(p)$ in addition to p_1, which negates Corollary 2.2 for $i = 1$.

Next, because all $a(\neq p_1) \in p$ satisfy $a \in s_1$ and $a \in s_2$, if p is not equivalent to $(a_1a_2\cdots a_{N(p)})^2$, then some a_j and a_k must not be crossing in p. Furthermore, by Lemma 2.1, the same a_j and a_k must not be crossing in $\phi_{aba}(p)$ as well. Now, by Proposition 3.1, the set partition $q = \phi_{aba}(p)_{[1:2N(p)-2]}$ satisfies $|q| = 2N(p) - 2 = 2N(q)$, and $\phi_{aba}^{N(q)-1}(q)$ must not be sorted; otherwise, $\phi_{aba}^{N(p)-1}(p)$ will be sorted. Thus, by the induction hypothesis, a_j and a_k must be crossing in $q = \phi_{aba}(p)_{[1:2N(p)-2]}$. Therefore, p must be equivalent to $(a_1a_2\cdots a_{N(p)})^2$. □

Next, we prove auxiliary lemmas that lead up to Theorem 1.2. First, for a set partition p such that $|p| = 2N(p) + 1$ and p is not sorted after applying $\phi_{aba}^{N(p)-1}$, we prove that $|I(p, a)| = 2$ for all but one $a \in p$ and $|I(p, a_*)| = 3$ for exactly one $a_* \in p$.

Lemma 3.1. *If p satisfies $|p| = 2N(p) + 1$ and is not sorted after applying $\phi_{aba}^{N(p)-1}$, then there exists exactly one $a_* \in p$ such that $|I(p, a_*)| = 3$, and for any other $a \in p$, it holds that $|I(p, a)| = 2$.*

Proof. By Corollary 2.2, $C(p) = 0$. Thus, $|I(p, a)| \geq 2$ for all $a \in p$ and because $|p| = 2N(p) + 1$, all but one $a \in p$ must satisfy $|I(p, a)| = 2$ and one $a_* \in p$ must satisfy $|I(p, a_*)| = 3$. □

Next, we show that if a set partition p satisfies the statement of Theorem 1.2 and in addition $|I(p, p_1)| = 2$, then a_* as in the statement of Lemma 3.1 appears exactly twice in $p_{[1:I^2(p,p_1)-1]}$ or $p_{[I^2(p,p_1)+1:|p|]}$ and any other $a \in p$ that satisfies $a \notin \{p_1, a_*\}$ appears exactly once in both $p_{[1:I^2(p,p_1)-1]}$ and $p_{[I^2(p,p_1)+1:|p|]}$.

Lemma 3.2. *If p satisfies $|p| = 2N(p) + 1$, is not sorted after applying $\phi_{aba}^{N(p)-1}$, and satisfies $|I(p, p_1)| = 2$, then either $\mathrm{mcount}(p_{[1:I^2(p,p_1)-1]}) = 2$ or $\mathrm{mcount}(p_{[I^2(p,p_1)+1:|p|]}) = 2$.*

Proof. Let $p = p_1 s_1 p_1 s_2$ for (possibly empty) set partitions s_1 and s_2, and let $a_* \in p$ be as defined in the statement of Lemma 3.1. Note that $a_* \neq p_1$, because $|I(p, p_1)| = 2$. Now, for any $s \in \{s_1, s_2\}$, if $a \in s$ satisfies $|I(p, a)| = |I(s, a)|$, then p_1 and $\mathrm{nc}(s)$ are clumped in $\phi_{aba}(p)$ by Lemma 2.1. But this negates Corollary 2.2 for $i = 1$. Thus, $|I(s, a)| < |I(p, a)|$ for $s \in \{s_1, s_2\}$ for all $a \in s$. Now, by Lemma 3.1, every $a \in s$ satisfies $|I(p, a)| \in \{2, 3\}$ for $s \in \{s_1, s_2\}$. Thus, either $\mathrm{mcount}(s_1) = \mathrm{mcount}(p_{[1:I^2(p,p_1)-1]}) = 2$ or $\mathrm{mcount}(s_2) = \mathrm{mcount}(p_{[I^2(p,p_1)+1:|p|]}) = 2$. □

Next, we count the number of inequivalent set partitions p that satisfy the conditions of Theorem 1.2 and contain 2 occurrences of p_1 and a letter that appears twice to the right of the rightmost p_1.

Lemma 3.3. *The number of inequivalent p that satisfy $|p| = 2N(p) + 1$, are not sorted after applying $\phi_{aba}^{N(p)-1}$, and satisfy $|I(p, p_1)| = \mathrm{mcount}(p_{[I^2(p,p_1)+1:|p|]}) = 2$ is $\binom{N(p)}{2}$.*

Proof. Let p be a set partition that satisfies the lemma statement. Let a_* be defined as in the statement of Lemma 3.1, and let $p = p_1 s_1 p_1 s_2 a_* s_3 a_* s_4$ for (possibly empty) set partitions s_1, s_2, s_3, and s_4. In addition, let $S = \{s_1, s_2, s_3, s_4\}$. Now, by Lemma 3.2, $a_* \in s_1$. Furthermore, for any $s \in S$, if some $a \in s$ satisfies $|I(s, a)| = 2$, then $\mathrm{nc}(s)$ is clumped in $\phi_{aba}(p)$ by Lemma 2.1. But this negates Corollary 2.2 for $i = 1$. Therefore, all $a \in s$ for each $s \in S$ must satisfy $|I(s, a)| = 1$.

Next, no $a \in s_2$ satisfies $a \in s_3$ or $a \in s_4$, because if so, $\mathrm{nc}(s_2 a_* s_3 a_* s_4)$ is clumped in $\phi_{aba}(p)$ and $C(\phi_{aba}(p)) > 1$, which negates Corollary 2.2 for $i = 1$. Thus, $\phi_{aba}(p) = \mathrm{r}(s_1)\mathrm{r}(s_3)\mathrm{r}(s_4){a_*}^2\mathrm{r}(s_2)p_1^2$ and so, $\mathrm{trunc}(\phi_{aba}(p)) = \mathrm{r}(s_1)\mathrm{r}(s_3)\mathrm{r}(s_4)a_*\mathrm{r}(s_2)p_1$, because p_1 is the only letter clumped in $\phi_{aba}(p)$ by Corollary 2.2 for $i = 1$.

Next, by Proposition 3.1, if p is not sorted by $\phi_{aba}^{N(p)-1}$, then $\mathrm{r}(s_1)\mathrm{r}(s_3)\mathrm{r}(s_4)a_*\mathrm{r}(s_2)$ must not be sorted by $\phi_{aba}^{N(p)-2}$. Thus, by Theorem 1.1, it must be that

$$\mathrm{r}(s_1)\mathrm{r}(s_3)\mathrm{r}(s_4)a_*\mathrm{r}(s_2) = (\phi_{aba}(p)_1 \phi_{aba}(p)_2 \cdots \phi_{aba}(p)_{N(p)-1})^2.$$

Now, because $a_* \in s_1$ by Lemma 3.2 and no $a \in s_2$ satisfies $a \in s_3$ or $a \in s_4$, it must be that $|r(s_1)| \geq N(p)-1$. But because each $a \in s_1$ satisfies $|I(s_1, a)| = 1$, it holds that $|r(s_1)| \leq N(p) - 1$. Thus, $|r(s_1)| = N(p) - 1$. As a result,

$$\mathrm{r}(s_3)\mathrm{r}(s_4)a_*\mathrm{r}(s_2) = \mathrm{r}(s_1) = \mathrm{r}(p_2 p_3 \cdots p_{N(p)}).$$

Therefore, each ordered triple of nonnegative integers $(|s_2|, |s_3|, |s_4|)$ such that $|s_2| + |s_3| + |s_4| = N(p) - 2$ corresponds to a unique set partition p that satisfies the lemma statement. Thus, $\binom{N(p)}{2}$ set partitions satisfy the lemma statement. □

Next, we count the number of inequivalent set partitions p that satisfy the conditions of Theorem 1.2 and contain 2 occurrences of p_1 and a letter that appears twice to the left of the rightmost p_1. The proof follows in the same way as in Lemma 3.3 and is thus omitted.

Lemma 3.4. *The number of inequivalent p that satisfy $|p| = 2N(p) + 1$, are not sorted after applying $\phi_{aba}^{N(p)-1}$, and satisfy $|I(p, p_1)| = \mathrm{mcount}(p_{[1:I^2(p,p_1)-1]}) = 2$ is $\binom{N(p)}{2}$.*

Next, we count the number of inequivalent set partitions p that are not sorted after applying $\phi_{aba}^{N(p)-1}$ and contain each letter in p other than p_1 exactly twice.

Lemma 3.5. *The number of inequivalent set partitions p that satisfy $|p| = 2(N(p) - 1) + |I(p, p_1)|$ and are not sorted after applying $\phi_{aba}^{N(p)-1}$ is given by*

$$\binom{2N(p) + |I(p, p_1)| - 3}{|I(p, p_1)| - 1} - |I(p, p_1)| \binom{N(p) + |I(p, p_1)| - 3}{|I(p, p_1)| - 1}.$$

Proof. Let p be a set partition that satisfies the lemma statement. By Corollary 2.2, $C(\phi^0_{aba}(p)) = C(p) = 0$. Thus, all $a(\neq p_1) \in p$ must satisfy $|I(p,a)| = 2$. Let $p = p_1 s_1 p_1 s_2 \cdots p_1 s_{|I(p,p_1)|}$ for (possibly empty) set partitions $s_1, s_2, \ldots, s_{|I(p,p_1)|}$. Also, let $S = \{s_1, s_2, \ldots, s_{|I(p,p_1)|}\}$. Now, if there exists some $s \in S$ and $a \in s$ such that $|I(s,a)| = 2$, then $\mathrm{nc}(s)$ is clumped in $\phi_{aba}(p)$. But if so, $C(\phi_{aba}(p)) > 1$, which negates Corollary 2.2 for $i = 1$. Thus, each $a \in s$ must satisfy $|I(s,a)| = 1$. In particular, $|s_i| \leq N(p)-1$ for all $1 \leq i \leq |I(p,p_1)|$. Now, by Lemma 2.1, $\phi_{aba}(p) = \mathrm{r}(s_1)\cdots \mathrm{r}(s_{|I(p,p_1)|})p_1^{|I(p,p_1)|}$. Thus, $\mathrm{trunc}(\phi_{aba}(p)) = \mathrm{r}(s_1)\cdots\mathrm{r}(s_{|I(p,p_1)|})p_1$, because p_1 is the only letter clumped in $\phi_{aba}(p)$ by Corollary 2.2 for $i = 1$.

Next, by Proposition 3.1, if p is not sorted by $\phi_{aba}^{N(p)-1}$, then $\mathrm{r}(s_1)\cdots\mathrm{r}(s_{|I(p,p_1)|})$ must not be sorted by $\phi_{aba}^{N(p)-2}$. Thus, by Theorem 1.1, it must be that

$$\mathrm{r}(s_1)\cdots\mathrm{r}(s_{|I(p,p_1)|}) = (\phi_{aba}(p)_1\phi_{aba}(p)_2\cdots\phi_{aba}(p)_{N(p)-1})^2.$$

Thus, each ordered $|I(p,p_1)|$-tuple of nonnegative integers $(|s_1|, |s_2|, \cdots, |s_{|I(p,p_1)|}|)$ such that $\sum_{i=1}^{|I(p,p_1)|}|s_i| = 2N(p)-2$ and $|s_i| \leq N(p)-1$ for all $1 \leq i \leq |I(p,p_1)|$ corresponds to a unique set partition p that satisfies the lemma statement. The number of $|I(p,p_1)|$-tuples of nonnegative integers $(|s_1|, |s_2|, \cdots, |s_{|I(p,p_1)|}|)$ such that $\sum_{i=1}^{|I(p,p_1)|}|s_i| = 2N(p)-2$ is $\binom{2N(p)+|I(p,p_1)|-3}{|I(p,p_1)|-1}$. However, of those, $|I(p,p_1)|\binom{N(p)+|I(p,p_1)|-3}{|I(p,p_1)|-1}$ tuples violate $|s_i| \leq N(p)-1$ for exactly one $1 \leq i \leq |I(p,p_1)|$; none violate $|s_i| \leq N(p)-1$ for more than one $1 \leq i \leq |I(p,p_1)|$. Thus, $\binom{2N(p)+|I(p,p_1)|-3}{|I(p,p_1)|-1} - |I(p,p_1)|\binom{N(p)+|I(p,p_1)|-3}{|I(p,p_1)|-1}$ set partitions satisfy the statement of the lemma. ☐

We end by using Lemmas 3.1, 3.2, 3.3, 3.4, and 3.5 to prove Theorem 1.2.

Proof of Theorem 1.2. Let a_* be as defined in Lemma 3.1. Lemma 3.2 shows that Lemmas 3.3 and 3.4 count all set partitions p that satisfy the statement of Theorem 1.2 and $a_* \neq p_1$. Therefore, by Lemmas 3.3 and 3.4, $2\binom{N(p)}{2}$ set partitions p satisfy the statement of Theorem 1.2 and $a_* \neq p_1$. Lastly, by Lemma 3.5, $\binom{2N(p)}{2} - 3\binom{N(p)}{2} = \binom{N(p)+1}{2}$ set partitions satisfy the statement of Theorem 1.2 and $a_* = p_1$. Thus, $\binom{N(p)+1}{2} + 2\binom{N(p)}{2}$ set partitions satisfy the statement of Theorem 1.2. ☐

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