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Canon Permutations and Generalized Descents of Standard Young Tableaux

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Abstract: Canon permutations are permutations of the multiset consisting of k copies of each integer between 1 and n, with the property that the subsequences obtained by taking the jth copy of each entry, for each fixed j, are all the same. When $k = 2$, canon permutations are sometimes called nonnesting permutations, as they are in bijection with labeled nonnesting matchings. The polynomial that enumerates nonnesting permutations by the number of descents factors as a product of an Eulerian polynomial and a Narayana polynomial. In this paper, we extend this result to arbitrary k, and we relate the enumeration of canon permutations by the number of descents to the enumeration of standard Young tableaux of rectangular shape with respect to generalized descent statistics. Our proof is bijective, and it also settles a conjecture of Sulanke about the distribution of certain lattice path statistics.

Keywords: Canon permutation; Descent; Narayana number; Nonnesting; Standard Young tableau
2020 Mathematics Subject Classification: 05A05; 05A15; 05A19

1. Introduction

1.1 Canon permutations

Let $\mathcal{S}_n$ be the set of permutations of $[n] = \{1, 2, \dots, n\}$. Let $\mathcal{M}_n^k = \{1^k, 2^k, \dots, n^k\}$ be the multiset consisting of k copies of each number in $[n]$. Given a permutation π of $\mathcal{M}_n^k$, one can consider the subsequence obtained by taking the first (i.e. leftmost) copy of each number in $[n]$, and more generally, the subsequence obtained by taking the jth copy from the left of each number in $[n]$, for any given $j \in [k]$. If the subsequences obtained in this way (which must be permutations in $\mathcal{S}_n$) are the same for every j, then π is called a *canon permutation*. We denote by $\mathcal{C}_n^k$ the set of canon permutations of $\mathcal{M}_n^k$, and by $\mathcal{C}_n^{k,\sigma}$ the subset of those where the jth copies of each entry form the subsequence $\sigma \in \mathcal{S}_n$, sometimes called the *underlying permutation*. For example, $351335212514424 \in \mathcal{C}_5^3$, since the subsequences of first, second, and third copies of each entry, respectively, are all equal to $\sigma = 35124$.

The term *canon permutation* was coined in [5] in reference to the musical form where different voices play the same melody, starting at different times.

When $k = 2$, canon permutations are sometimes called *nonnesting permutations* since they can be interpreted as nonnesting matchings of $[2n]$ where each of the n arcs is labeled with a distinct element in $[n]$. Recall that a perfect matching of $[2n]$ is nonnesting if it does not contain a pair of arcs (i_1, i_3) and (i_2, i_4) where $i_1 < i_2 < i_3 < i_4$. The corresponding nonnesting permutation is obtained by simply reading, for each i from 1 to $2n$, the label of the arc containing i; see Figure 1 for an example. Indeed, the nonnesting condition on the matching is equivalent to the fact that the left endpoints of the arcs occur in the same order as their right endpoints.

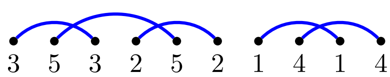

Figure 1: The nonnesting permutation $3532521414 \in \mathcal{C}_5^2$ viewed as a labeled nonnesting matching.

Nonnesting permutations have recently been considered in [5] as a variation of *noncrossing permutations* (also called *quasi-Stirling permutations*) [1, 4, 7], which in turn generalize the well-known *Stirling permutations* introduced by Gessel and Stanley [8]. Separately, nonnesting permutations naturally index the regions of the Catalan hyperplane arrangement, as shown by Bernardi in [3], where they are called *annotated* 1*-sketches*.

In most of the above papers, an important role is played by the number of descents of the permutations. We say that i is a *descent* of $\pi = \pi_1\pi_2\dots\pi_m$ (a word over the positive integers) if $\pi_i > \pi_{i+1}$, and that it is a *plateau* if $\pi_i = \pi_{i+1}$. Denote the *descent set* and the *plateau set* of π by

$$\mathrm{Des}(\pi) = \{i \in [m-1] : \pi_i > \pi_{i+1}\},$$
$$\mathrm{Plat}(\pi) = \{i \in [m-1] : \pi_i = \pi_{i+1}\},$$

respectively, its number of descents by $\mathrm{des}(\pi) = |\mathrm{Des}(\pi)|$, and its number of plateaus by $\mathrm{plat}(\pi) = |\mathrm{Plat}(\pi)|$.

The distribution of the number of descents on $\mathcal{S}_n$ is given by the Eulerian polynomial

$$A_n(t) = \sum_{\sigma\in\mathcal{S}_n} t^{\mathrm{des}(\sigma)},$$

whose exponential generating function can be expressed as

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{t-1}{t-e^{(t-1)z}};$$

see [10, Prop. 1.4.5].

In this paper, we are interested in the polynomials

$$C_n^k(t,u) = \sum_{\pi\in\mathcal{C}_n^k} t^{\mathrm{des}(\pi)}u^{\mathrm{plat}(\pi)}, \tag{1}$$

which give the distribution of the number of descents and plateaus on canon permutations. Separating canon permutations according to their underlying permutation, we can write $C_n^k(t,u) = \sum_{\sigma\in\mathcal{S}_n} C_n^{k,\sigma}(t,u)$, where

$$C_n^{k,\sigma}(t,u) = \sum_{\pi\in\mathcal{C}_n^{k,\sigma}} t^{\mathrm{des}(\pi)}u^{\mathrm{plat}(\pi)}.$$

The main result in [5] is a formula for these polynomials in the case $k = 2$. Before we can state it, we need to define another family of polynomials. Let $\mathcal{D}_n$ be the set of Dyck paths of semilength n, that is, lattice paths from $(0,0)$ to $(2n,0)$ with steps $\mathtt{u} = (1,1)$ and $\mathtt{d} = (1,-1)$ that do not go below the x-axis. A *peak* in a Dyck path is an occurrence of two adjacent steps $\mathtt{ud}$; it is called *low peak* if these steps touch the x-axis, and a *high peak* otherwise. Let $\widehat{N}_n(t,u)$ be the polynomial for paths in $\mathcal{D}_n$ where t and u mark the number of high peaks and the number of low peaks, respectively. The generating function for these polynomials, called *Narayana polynomials*, is

$$\sum_{n\geq 0} \widehat{N}_n(t,u)z^n = \frac{2}{1+(1+t-2u)z+\sqrt{1-2(1+t)z+(1-t)^2z^2}}. \tag{2}$$

It is shown in [5] that, when $k = 2$, the polynomials in (1) factor as follows.

Theorem 1.1 ([5])**.** *For $n \geq 1$,*

$$C_n^2(t,u) = A_n(t)\,\widehat{N}_n(t,u).$$

More specifically, for all $\sigma \in \mathcal{S}_n$,

$$C_n^{2,\sigma}(t,u) = t^{\mathrm{des}(\sigma)}\widehat{N}_n(t,u).$$

The main goal of this paper is to generalize this result to the polynomials $C_n^k(t,u)$ for arbitrary k. As we will see, in the general version, the role of Dyck paths will be played by standard Young tableaux of rectangular shape.

2. Generalized descents of standard Young tableaux

2.1 Generalized Narayana polynomials

For $n,k \geq 1$, let $\mathrm{SYT}(k^n)$ denote the set of standard Young tableaux of rectangular shape consisting of n rows and k columns. Such tableaux have kn cells, filled with the numbers $1,2,\dots,kn$ so that numbers in rows increase from left to right, and numbers in columns increase from top to bottom. The *descent set* of $T \in \mathrm{SYT}(k^n)$, denoted by $\mathrm{Des}(T)$, is the set of $i \in [kn-1]$ such that i appears in a row higher than $i+1$ in T. The number of descents of T is $\mathrm{des}(T) = |\mathrm{Des}(T)|$. Denoting by $R_T(i)$ the index of the row in T where i appears, where rows are indexed from 1 to n starting from the top row, we can write

$$\mathrm{Des}(T) = \{i \in [kn-1] : R_T(i) < R_T(i+1)\}.$$

Similarly, define the ascent and plateau sets of T by

$$\mathrm{Asc}(T) = \{i \in [kn-1] : R_T(i) > R_T(i+1)\},$$
$$\mathrm{Plat}(T) = \{i \in [kn-1] : R_T(i) = R_T(i+1)\},$$

respectively, and let $\mathrm{asc}(T) = |\mathrm{Asc}(T)|$ and $\mathrm{plat}(T) = |\mathrm{Plat}(T)|$.

The distribution of the number of ascents and descents on standard Young tableaux of rectangular shape is given by the *generalized Narayana numbers*, which were studied by Sulanke [11] in connection to higher-dimensional lattice paths, and are defined as

$$N(n, k, h) = |\{T \in \mathrm{SYT}(k^n) : \mathrm{asc}(T) = h\}|.$$

It is also shown in [11] that

$$N(n, k, h) = |\{T \in \mathrm{SYT}(k^n) : \mathrm{des}(T) = h + n - 1\}|. \tag{3}$$

Sulanke obtains the following formula for the generalized Narayana numbers, which is implicit in MacMahon's work on plane partitions [9].

Proposition 2.1 ([11, Prop. 1])**.** *For $0 \le h \le (k-1)(n-1)$,*

$$N(n, k, h) = \sum_{j=0}^{h} (-1)^{h-j} \binom{kn+1}{h-j} \prod_{i=0}^{n-1} \binom{k+i+j}{k} \binom{k+i}{k}^{-1}.$$

We define the *generalized Narayana polynomials* as

$$N_{n,k}(t, u) = \sum_{T \in \mathrm{SYT}(k^n)} t^{\mathrm{asc}(T)} u^{\mathrm{plat}(T)}. \tag{4}$$

When disregarding the number of plateaus, their coefficients are the generalized Narayana numbers, that is,

$$N_{n,k}(t, 1) = \sum_{h=0}^{(k-1)(n-1)} N(n, k, h)\, t^h.$$

Example 2.1. *We have*

$$\begin{aligned} N_{4,3}(t, u) &= u^8 + \left(5u^4 + 8u^5 + 9u^6\right) t + \left(1 + 6u + 21u^2 + 36u^3 + 41u^4 + 8u^5\right) t^2 \\ &\quad + \left(16 + 54u + 79u^2 + 36u^3 + 5u^4\right) t^3 + \left(38 + 54u + 21u^2\right) t^4 + (16 + 6u)\, t^5 + t^6, \\ N_{4,3}(t, 1) &= 1 + 22\,t + 113\,t^2 + 190\,t^3 + 113\,t^4 + 22\,t^5 + t^6. \end{aligned}$$

For $k = 2$, there is a straightforward bijection between $\mathrm{SYT}(2^n)$ and $\mathcal{D}_n$, obtained by letting the ith step of the Dyck path be $\mathtt{u}$ if i is in the first column of the tableau, and $\mathtt{d}$ otherwise, for each $i \in [2n]$. Through this bijection, ascents of the tableau become high peaks of the Dyck path, and plateaus become low peaks. It follows that $N_{n,2}(t, u) = \widehat{N}_n(t, u)$, the polynomials from (2).

2.2 Generalized descents

Next, we generalize the definitions of descents and ascents in standard Young tableaux of rectangular shape.

Definition 2.1. *For any permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$, define the σ-*descent set *of $T \in \mathrm{SYT}(k^n)$ as*

$$\mathrm{Des}_\sigma(T) = \{i \in [kn-1] : \sigma_{R_T(i)} > \sigma_{R_T(i+1)}\},$$

and the number of σ-descents by $\mathrm{des}_\sigma(T) = |\mathrm{Des}_\sigma(T)|$.

For example, if $\sigma = 35142 \in \mathcal{S}_5$ and $T \in \mathrm{SYT}(3^5)$ is the tableau on the left of Figure 2, we have $\mathrm{Des}_\sigma(T) = \{2, 4, 7, 9, 10, 11, 14\}$ and $\mathrm{des}_\sigma(T) = 7$. Definition 2.1 generalizes the usual descent and ascent sets of standard Young tableaux, since

$$\mathrm{Des}(T) = \mathrm{Des}_{n\dots21}(T) \quad \text{and} \quad \mathrm{Asc}(T) = \mathrm{Des}_{12\dots n}(T) \tag{5}$$

for all $T \in \mathrm{SYT}(k^n)$.

Standard Young tableaux of rectangular shape are related to canon permutations via the following bijection; see Figure 2 for an example.

$$
\begin{array}{c|c|c|c|}
\sigma & \multicolumn{3}{c}{} \\
\cline{2-4}
3 & 1 & 3 & 6 \\ \cline{2-4}
5 & 2 & 4 & 9 \\ \cline{2-4}
1 & 5 & 8 & 12 \\ \cline{2-4}
4 & 7 & 10 & 14 \\ \cline{2-4}
2 & 11 & 13 & 15 \\ \cline{2-4}
\end{array}
\quad \mapsto \quad 353513415421242 = \pi
$$

Figure 2: The bijection from Lemma 2.1, with $\sigma = 35142$. We have $\mathrm{Des}_\sigma(T) = \mathrm{Des}(\pi) = \{2, 4, 7, 9, 10, 11, 14\}$.

Lemma 2.1. *For every $\sigma \in \mathcal{S}_n$, the map*

$$
\begin{array}{ccc}
\mathrm{SYT}(k^n) & \longrightarrow & \mathcal{C}_n^{k,\sigma} \\
T & \mapsto & \pi,
\end{array}
$$

where $\pi = \sigma_{R_T(1)}\sigma_{R_T(2)} \dots \sigma_{R_T(kn)}$, is a bijection. Additionally,

$$\mathrm{Des}_\sigma(T) = \mathrm{Des}(\pi) \quad \textit{and} \quad \mathrm{Plat}(T) = \mathrm{Plat}(\pi),$$

which imply that

$$\sum_{T \in \mathrm{SYT}(k^n)} t^{\mathrm{des}_\sigma(T)} u^{\mathrm{plat}(T)} = C_n^{k,\sigma}(t,u).$$

Proof. Let us show that, for fixed $\sigma \in \mathcal{S}_n$, the map $T \mapsto \pi$ is a bijection between $\mathrm{SYT}(k^n)$ and $\mathcal{C}_n^{k,\sigma}$. By construction, entry i is in row r and column j of T if and only if π_i is the jth copy (from the left) of σ_r in π. Thus, the entries in column j of T are the positions in π of the jth copies of each number, which, when read from left to right, form the permutation σ. This proves that $\pi \in \mathcal{C}_n^{k,\sigma}$. Conversely, given $\pi \in \mathcal{C}_n^{k,\sigma}$, the above observation uniquely determines T.

Finally, note that $i \in \mathrm{Des}(\pi)$ if and only if $\pi_i > \pi_{i+1}$, that is, $\sigma_{R_T(i)} > \sigma_{R_T(i+1)}$, which is equivalent to $i \in \mathrm{Des}_\sigma(T)$. Similarly, $i \in \mathrm{Plat}(\pi)$ if and only if $\pi_i = \pi_{i+1}$, that is, $\sigma_{R_T(i)} = \sigma_{R_T(i+1)}$, which is equivalent to $R_T(i) = R_T(i+1)$, and also to $i \in \mathrm{Plat}(T)$. □

By letting $\sigma \in \mathcal{S}_n$ vary, Lemma 2.1 gives a bijection

$$
\begin{array}{ccc}
\mathcal{S}_n \times \mathrm{SYT}(k^n) & \to & \mathcal{C}_n^k \\
(\sigma, T) & \mapsto & \pi.
\end{array}
$$

3. Main result

Our main result is the following generalization of Theorem 1.1 to arbitrary k.

Theorem 3.1. *For $n, k \geq 1$,*

$$C_n^k(t,u) = A_n(t)\, N_{n,k}(t,u).$$

More specifically, for all $\sigma \in \mathcal{S}_n$,

$$C_n^{k,\sigma}(t,u) = t^{\mathrm{des}(\sigma)} N_{n,k}(t,u). \tag{6}$$

Example 3.1. *For $n = k = 3$, we have the factorization*

$$C_3^3(t,u) = \left(1 + 4\,t + t^2\right)\left(u^6 + \left(4u^3 + 6u^4\right)t + \left(1 + 6u + 9u^2 + 4u^3\right)t^2 + (4 + 6u)\,t^3 + t^4\right).$$

The second statement in Theorem 3.1 immediately implies the first by summing over all $\sigma \in \mathcal{S}_n$. Using Lemma 2.1 and (4), we can rewrite (6) as

$$\sum_{T \in \mathrm{SYT}(k^n)} t^{\mathrm{des}_\sigma(T)} u^{\mathrm{plat}(T)} = t^{\mathrm{des}(\sigma)} \sum_{T \in \mathrm{SYT}(k^n)} t^{\mathrm{asc}(T)} u^{\mathrm{plat}(T)}.$$

By (5), we have $\mathrm{asc}(T) = \mathrm{des}_{12\dots n}(T)$. Thus, in order to prove Theorem 3.1, it suffices to construct a bijection $\phi_\sigma : \mathrm{SYT}(k^n) \to \mathrm{SYT}(k^n)$ such that, for all $T \in \mathrm{SYT}(k^n)$,

$$\mathrm{des}_\sigma(T) = \mathrm{des}_{12\dots n}(\phi_\sigma(T)) + \mathrm{des}(\sigma) \quad \text{and} \quad \mathrm{plat}(T) = \mathrm{plat}(\phi_\sigma(T)). \tag{7}$$

We will do this in Section 4. We end this section by comparing our bijection ϕ_σ to others in the literature.

The structure of ϕ_σ, described as a composition of smaller bijections, is similar to the construction given in [5] to prove Theorem 1.1. However, some of the steps of the bijection in [5] are specific to the geometry of Dyck paths, and they do not clearly extend to higher dimensions. In fact, our bijection ϕ_σ on $\mathrm{SYT}(k^n)$, in the case $k = 2$, is different—and arguably neater—than the analogous bijection from [5]; see Remark 4.1 for a specific example where the bijections differ.

In the special case that $\sigma = n \dots 21$, using that $\mathrm{des}_{n\dots21}(T) = \mathrm{des}(T)$ and $\mathrm{des}(n \dots 21) = n-1$, our bijection $\phi_{n\dots21}$ has the property that

$$\mathrm{des}(T) = \mathrm{asc}(\phi_{n\dots21}(T)) + n - 1.$$

A similar bijection with this property was given by Sulanke in [11, Prop. 2], in order to prove (3).

In the same paper, in a remark at the end of [11, Section 3.1], Sulanke defines certain statistics asc_τ on higher-dimensional lattice paths, which are closely related to our statistics des_σ, and he conjectures that they have a (shifted) generalized Narayana distribution. Sulanke's conjecture can be shown to be equivalent to our (6) with the specialization $u = 1$.

4. The bijections

The goal of this section is to construct a bijection ϕ_σ, for any given $\sigma \in \mathcal{S}_n$, that satisfies (7), relating the statistics des_σ and $\mathrm{des}_{12\dots n}$ while preserving the number of plateaus. This will be achieved in two steps. In Section 4.1, we describe a sequence of bijections that relates the statistics des_σ and des_λ, where λ is a very specific permutation having the same descent set as σ. In Section 4.2, we describe a sequence of bijections that relates the statistics des_λ and $\mathrm{des}_{12\dots n}$. The bijection ϕ_σ will be obtained by composing the two sequences of bijections.

It will be convenient to identify $T \in \mathrm{SYT}(k^n)$ with its *Yamanouchi word* $w_T = R_T(1)R_T(2)\dots R_T(kn)$, which records the row where each entry is. This encoding is simply the map from Lemma 2.1 when σ is the identity permutation and so $w_T \in \mathcal{C}_n^{k,12\dots n}$. As an alternative characterization of the set $\mathcal{C}_n^{k,12\dots n}$, note that a permutation of $\mathcal{M}_n^k$ is the Yamanouchi word of some $T \in \mathrm{SYT}(k^n)$ if and only if, in every prefix, the number of copies of j is greater than or equal to the number of copies of $j+1$, for all $j \in [k-1]$. We will call this the *prefix condition*.

We say that two statistics stat_1 and stat_2 on a set X are *equidistributed* if there is a bijection $f : X \to X$ such that, for all $x \in X$, we have $\mathrm{stat}_1(x) = \mathrm{stat}_2(f(x))$. We allow stat_1 and stat_2 to be tuples of statistics, which can be integer-valued or set-valued.

4.1 Switching non-adjacent entries of σ

Let $r, s \in [n]$ be such that $|r - s| > 1$. We start by defining two bijections $f_{rs}, F_{rs} : \mathrm{SYT}(k^n) \to \mathrm{SYT}(k^n)$ that affect only the entries in rows r and s of the tableau. Given $T \in \mathrm{SYT}(k^n)$, we will describe $f_{rs}(T)$ and $F_{rs}(T)$ in terms of their Yamanouchi words, which will be obtained from w_T by applying certain permutations to each of the maximal consecutive blocks B having entries in $\{r, s\}$. Since rows r and s are not adjacent, permuting the entries within each such block B does not violate the prefix condition, and so the resulting word is guaranteed to encode a tableau in $\mathrm{SYT}(k^n)$.

Definition 4.1. *To construct the Yamanouchi word of $f_{rs}(T)$, we change each block B to B' as follows.*

(1) If B starts and ends with the same letter, let $B' = B$.

(2) Otherwise, if $B = r^{a_1}s^{b_1}\dots r^{a_j}s^{b_j}$ for some $j \geq 1$ and $a_i, b_i \geq 1$ for all i, let $B' = s^{b_1}r^{a_1}\dots s^{b_j}r^{a_j}$; conversely, if $B = s^{b_1}r^{a_1}\dots s^{b_j}r^{a_j}$, let $B' = r^{a_1}s^{b_1}\dots r^{a_j}s^{b_j}$.

For example, if $B = srrsssrr$, we get $B' = rrsrrsss$.

Definition 4.2. *To construct the Yamanouchi word of $F_{rs}(T)$, we change each block B to B'' as follows.*

(1) Break B into subblocks by splitting at each s followed by r. For this, we view B as a cyclic word so that its last entry is immediately followed by its first entry. In particular, unless B ends with an s and starts with an r, one of the subblocks straddles from the end to the beginning of B.

(2) For each subblock, write it as $r^a s^b$ for some $a, b \geq 1$, and change it to $s^b r^a$, while keeping the splitting points between subblocks. Let B'' be the resulting block.

For example, if $B = srrsssrr$, we split it into two sublocks as $B = s/rrsss/rr$ (with the block rrs stradling across the endpoints), and then change it into $B'' = r/sssrr/sr$. See Figure 3 for a complete example of f_{rs} and F_{rs} applied to a tableau.

It is clear from the definitions that $f_{rs}^{-1} = f_{sr} = f_{rs}$, so f_{rs} is an involution, and that $F_{rs}^{-1} = F_{sr}$. Next, we look at the effect of these bijections on descents and plateaus.

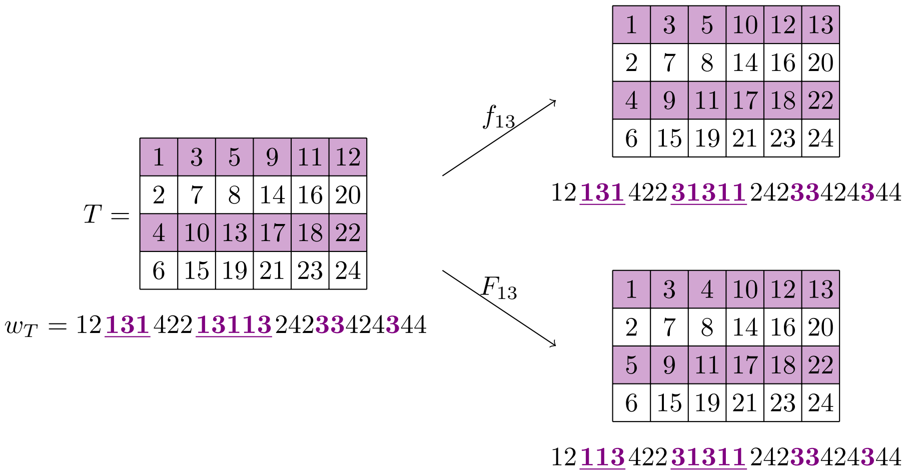

Figure 3: The bijections f_{rs} and F_{rs} for $r = 1$ and $s = 3$. The blocks B, B', B'' containing two different letters are underlined. Letting $\sigma = 2431$ and $\tau = 3421$, we have $\mathrm{des}_\sigma(T) = 10 = \mathrm{des}_\tau(f_{13}(T))$, $\mathrm{plat}(T) = 4 = \mathrm{plat}(f_{13}(T))$, and $\mathrm{Des}_\sigma(T) = \{2, 4, 5, 8, 10, 14, 16, 18, 20, 22\} = \mathrm{Des}_\tau(F_{13}(T))$.

Lemma 4.1. *Let $\sigma \in \mathcal{S}_n$, let $r, s \in [n]$ be such that $|r - s| > 1$, and suppose that $\sigma_r = \ell$ and $\sigma_s = \ell + 1$. Let $\tau \in \mathcal{S}_n$ be obtained from σ by switching these two entries, that is, $\tau_r = \ell + 1$, $\tau_s = \ell$, and $\tau_i = \sigma_i$ for $i \notin \{r, s\}$. Then the bijections $f_{rs}, F_{rs} : \mathrm{SYT}(k^n) \to \mathrm{SYT}(k^n)$ satisfy that, for all $T \in \mathrm{SYT}(k^n)$,*

$$\mathrm{des}_\sigma(T) = \mathrm{des}_\tau(f_{rs}(T)), \quad \mathrm{plat}(T) = \mathrm{plat}(f_{rs}(T)),$$

$$\mathrm{Des}_\sigma(T) = \mathrm{Des}_\tau(F_{rs}(T)).$$

Proof. Let us first consider the effect of f_{rs} on the generalized descents and plateaus of the tableau. Clearly, $\mathrm{plat}(T) = \mathrm{plat}(f_{rs}(T))$, since plateaus correspond to pairs of adjacent equal entries in the Yamanouchi word, and the number of these is not affected by f_{rs}.

As for descents, recall that $i \in \mathrm{Des}_\sigma(T)$ if $\sigma_{R_T(i)} > \sigma_{R_T(i+1)}$. Since the values ℓ and $\ell+1$ are consecutive, the only elements where $\mathrm{Des}_\sigma(T)$ and $\mathrm{Des}_\tau(f_{rs}(T))$ may differ must come from positions inside the blocks B and B' in Definition 4.1: adjacent pairs sr in B contribute to $\mathrm{Des}_\sigma(T)$, whereas adjacent pairs rs in B' contribute to $\mathrm{Des}_\tau(f_{rs}(T))$. In case (1), the block $B = B'$ has the same number of adjacent pairs sr as adjacent pairs rs. In case (2), the number of adjacent pairs sr in B is equal to the number of adjacent pairs rs in B'. It follows that $\mathrm{des}_\sigma(T) = \mathrm{des}_\tau(f_{rs}(T))$.

Let us now consider the effect of F_{rs} on the generalized descents. The only elements where $\mathrm{Des}_\sigma(T)$ and $\mathrm{Des}_\tau(F_{rs}(T))$ may differ must come from positions inside the blocks B and B'' in Definition 4.2: adjacent pairs sr in B contribute to $\mathrm{Des}_\sigma(T)$, whereas adjacent pairs rs in B'' contribute to $\mathrm{Des}_\sigma(F_{rs}(T))$. The locations of these adjacent pairs are precisely the splitting points for the subblocks (not including a potential splitting point at the very end of the block), so they are preserved. We conclude that $\mathrm{Des}_\sigma(T) = \mathrm{Des}_\tau(F_{rs}(T))$. □

When $k = 2$, the blocks B can have size at most two, and the bijections f_{rs} and F_{rs} coincide, sending Des_σ to Des_τ while simultaneously preserving plat. However, for $k \geq 3$, the pairs of statistics $(\mathrm{Des}_\sigma, \mathrm{plat})$ and $(\mathrm{Des}_\tau, \mathrm{plat})$ are not equidistributed on $\mathrm{SYT}(k^n)$ in general. For example, letting $\sigma = 132$ and $\tau = 231$, the tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & 8 \\ \hline 5 & 7 & 9 \\ \hline \end{array} \in \mathrm{SYT}(3^3)$$

has $\mathrm{Des}_\sigma(T) = \{2, 4, 5, 8\}$ and $\mathrm{plat}(T) = 0$, but there is no $\hat{T} \in \mathrm{SYT}(3^3)$ with $\mathrm{Des}_\tau(\hat{T}) = \{2, 4, 5, 8\}$ and $\mathrm{plat}(\hat{T}) = 0$.

Recall that a permutation is *reverse-layered* if it can be decomposed as a sequence of monotone increasing blocks where the entries in each block are larger than the entries in the next block. Each block in this decomposition is called a layer. For example, $789|6|345|12 \in \mathcal{S}_9$ is reverse-layered, with layers of size $3, 1, 3, 2$, separated by vertical bars. For any subset $S \subseteq [n-1]$, there is a unique reverse-layered permutation $\lambda \in \mathcal{S}_n$ with $\mathrm{Des}(\lambda) = S$. If the elements of S are $i_1 < i_2 < \cdots < i_d$, then

$$\lambda = (n{-}i_1{+}1)(n{-}i_1{+}2)\ldots n|(n{-}i_2{+}1)(n{-}i_2{+}2)\ldots(n{-}i_1)|\ldots|12\ldots(n{-}i_d).$$

Given any two permutations in $\mathcal{S}_n$ with the same descent set, we can transform one into the other by repeatedly switching non-adjacent entries with consecutive values. Indeed, suppose that $\sigma \in \mathcal{S}_n$ is not reverse-layered and that $\lambda \in \mathcal{S}_n$ is the unique reverse-layered permutation such that $\mathrm{Des}(\sigma) = \mathrm{Des}(\lambda)$. Note that λ has the maximum number of inversions among all the permutations with the same descent set. If $\sigma \neq \lambda$, we can find a pair of non-adjacent entries in σ with consecutive values that do not create an inversion, that is, indices r, s such that $s - r > 1$ and $\sigma_s = \sigma_r + 1$. Transposing the entries in positions r, s increases the number of inversions of the permutation, while preserving the descent set. For concreteness, we consider the following specific choice of r, s, as described in [5]: let m be the largest entry that is not in the same position in σ as in λ, let r be such that $\lambda_r = m$, and let s be such that $\sigma_s = \sigma_r + 1$. By repeatedly applying such transpositions, one eventually reaches the permutation λ. For example, if $\sigma = 14235$, then $\lambda = 45123$, and the sequence of transpositions is

$$14235 \xrightarrow{2,5} 15234 \xrightarrow{1,3} 25134 \xrightarrow{1,4} 35124 \xrightarrow{1,5} 45123.$$

Denote by $f_\sigma : \mathrm{SYT}(k^n) \to \mathrm{SYT}(k^n)$ the composition of the bijections f_{rs} corresponding to the above sequence of transpositions that transform σ into λ, and define F_σ similarly. In the above example, $f_{14235} = f_{15} \circ f_{14} \circ f_{13} \circ f_{25}$. Lemma 4.1 implies that, for all $T \in \mathrm{SYT}(k^n)$,

$$\mathrm{des}_\sigma(T) = \mathrm{des}_\lambda(f_\sigma(T)), \quad \mathrm{plat}(T) = \mathrm{plat}(f_\sigma(T)), \tag{8}$$

$$\mathrm{Des}_\sigma(T) = \mathrm{Des}_\lambda(F_\sigma(T)).$$

The compositions $f_\tau^{-1} \circ f_\sigma$ and $F_\tau^{-1} \circ F_\sigma$ give a bijective proof of the following.

Proposition 4.1. *If $\sigma, \tau \in \mathcal{S}_n$ are such that $\mathrm{Des}(\sigma) = \mathrm{Des}(\tau)$, then*

(i) the pairs of statistics $(\mathrm{des}_\sigma, \mathrm{plat})$ and $(\mathrm{des}_\tau, \mathrm{plat})$ are equidistributed on $\mathrm{SYT}(k^n)$,

(ii) the statistics Des_σ and Des_τ are equidistributed on $\mathrm{SYT}(k^n)$.

4.2 Removing descents

The previous section allows us to focus on the statistics des_λ where $\lambda \in \mathcal{S}_n$ is a reverse-layered permutation. Recall that, for each $S \subseteq [n-1]$, there is one such permutation having S as its descent set. In this section, we analyze how these statistics change when removing elements from S one at a time, with the goal of relating des_λ with $\mathrm{des}_{12\dots n}$.

Let $0 \le \ell < m < n$. Next, we define a bijection $g_{\ell m} : \mathrm{SYT}(k^n) \to \mathrm{SYT}(k^n)$, which will be used to analyze the removal of the largest element from S. Given $T \in \mathrm{SYT}(k^n)$, we will describe $g_{\ell m}(T)$ in terms of its Yamanouchi word, similarly to the bijections in Section 4.1. Each entry in w_T must belong to one of the sets $X = \{1, \dots, \ell\}$ (which is empty if $\ell = 0$), $Y = \{\ell+1, \dots, m\}$, or $Z = \{m+1, \dots, n\}$. The three subsequences of w_T obtained by restricting to each one of these sets are not changed by $g_{\ell m}$; rather, $g_{\ell m}$ only affects the interleaving among these subsequences. Thus, to describe $g_{\ell m}$, it will be enough to explain how it changes the simplified Yamanouchi word $\overline{w_T}$, defined as the word over $\{x, y, z\}$ whose ith entry records which of the sets X, Y, Z the ith entry of w_T belongs to.

Definition 4.3. *To construct the simplified Yamanouchi word of $g_{\ell m}(T)$, we consider each maximal block B of $\overline{w_T}$ having entries in $\{x, z\}$, and we change it to B' as follows.*

(1) If B starts and ends with the same letter, let $B' = B$.

(2) Otherwise, if $B = x^{a_1} z^{b_1} \dots x^{a_j} z^{b_j}$ for some $j \ge 1$ and $a_i, b_i \ge 1$ for all i, let $B' = z^{b_1} x^{a_1} \dots z^{b_j} x^{a_j}$; conversely, if $B = z^{b_1} x^{a_1} \dots z^{b_j} x^{a_j}$, let $B' = x^{a_1} z^{b_1} \dots x^{a_j} z^{b_j}$.

See Figure 4 for a complete example of $g_{\ell m}$ applied to a tableau. By definition, $g_{\ell m}$ is an involution. Note that, in the special case $\ell = 0$, the word $\overline{w_T}$ has no xs, and so g_{0m} is simply the identity map. Next, we look at the effect of $g_{\ell m}$ on descents and plateaus.

Lemma 4.2. *Let $S \subseteq [n-1]$ be nonempty, let $m = \max S$, and let $S' = S \setminus \{m\}$. Let $\ell = \max S'$ if S' is nonempty, otherwise let $\ell = 0$. Let $\lambda, \lambda' \in \mathcal{S}_n$ be the reverse-layered permutations with descent sets S and S', respectively. Then the bijection $g_{\ell m} : \mathrm{SYT}(k^n) \to \mathrm{SYT}(k^n)$ satisfies that, for all $T \in \mathrm{SYT}(k^n)$,*

$$\mathrm{des}_\lambda(T) = \mathrm{des}_{\lambda'}(g_{\ell m}(T)) + 1, \qquad \mathrm{plat}(T) = \mathrm{plat}(g_{\ell m}(T)).$$

Proof. It is clear that the number of plateaus is preserved by $g_{\ell m}$ since the number of pairs of equal adjacent letters in the Yamanouchi words is unchanged.

The reverse-layered permutations λ and λ' only differ in the relative order of the entries in positions Y and Z: in λ, entries in positions Y are larger than those in positions Z, whereas the opposite is true in λ'. The

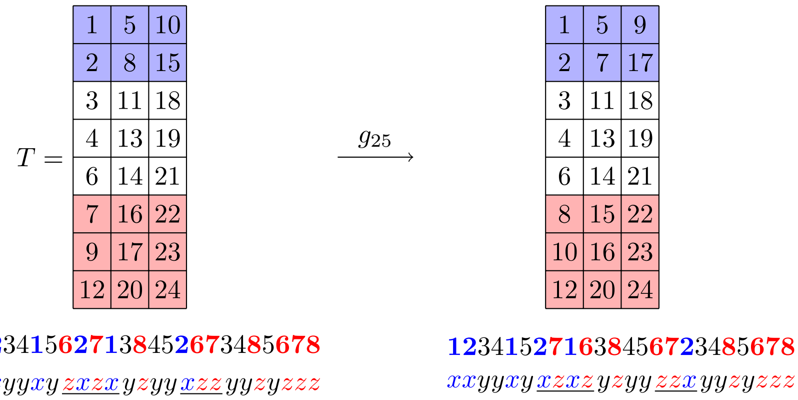

Figure 4: The bijection $g_{\ell m}$ for $\ell = 2$ and $m = 5$. The blocks B and B' in case (2) of Definition 4.3 are underlined. Letting $S = \{2, 5\}$, $\lambda = 78456123$ and $\lambda' = 78123456$, we have $\mathrm{des}_\lambda(T) = 9 = \mathrm{des}_{\lambda'}(g_{25}(T)) + 1$ and $\mathrm{plat}(T) = \mathrm{plat}(g_{25}(T)) = 0$.

relative order of entries in positions within the same set X, Y or Z does not change, and neither does the fact that entries in positions X are larger than those in positions Y and Z. It follows that the sets $\mathrm{Des}_\lambda(T)$ and $\mathrm{Des}_{\lambda'}(T)$ only differ in those i such that $R_T(i) \in Y$ and $R_T(i+1) \in Z$, or viceversa. In terms of the simplified Yamanouchi word $\overline{w_T}$, an adjacent pair yz contributes to $\mathrm{Des}_\lambda(T)$ but not to $\mathrm{Des}_{\lambda'}(T)$, whereas an adjacent pair zy contributes to $\mathrm{Des}_{\lambda'}(T)$ but not to $\mathrm{Des}_\lambda(T)$.

Thus, the only elements where $\mathrm{Des}_\lambda(T)$ and $\mathrm{Des}_{\lambda'}(g_{\ell m}(T))$ may differ must come from positions inside or on the boundary of the blocks B and B' in Definition 4.3. Note that the word $\overline{w_T}$ must start with a nonempty sequence of xs before the first y appears, and it must end with a nonempty sequence of zs following the last y. In particular, all the blocks B that contain both xs and zs are preceded and followed by a y.

In case (1), we have $B = B'$, so any discrepancies must come from adjacent pairs yz and zy on the boundaries of this block. These can only occur when the block starts and ends in z. In this case, the adjacent pair yz on the left boundary only contributes to $\mathrm{des}_\lambda(T)$, whereas the adjacent pair zy on the right boundary only contributes to $\mathrm{des}_{\lambda'}(g_{\ell m}(T))$, so the total contributions are the same. There is, however, one important exception: the block of zs at the end of $\overline{w_T}$ is not followed by a y, so the contribution of this block to $\mathrm{des}_{\lambda'}(g_{\ell m}(T))$ is one less than its contribution to $\mathrm{des}_\lambda(T)$.

In case (2), the contributions of $B = x^{a_1} z^{b_1} \dots x^{a_j} z^{b_j}$ to $\mathrm{des}_\lambda(T)$ are the j adjacent pairs xz, whereas the contributions of $B' = z^{b_1} x^{a_1} \dots z^{b_j} x^{a_j}$ to $\mathrm{des}_{\lambda'}(g_{\ell m}(T))$ are the $j-1$ pairs xz and the pair xy on the right boundary, for a total of j in both instances. Similarly, the contributions of $B = z^{b_1} x^{a_1} \dots z^{b_j} x^{a_j}$ to $\mathrm{des}_\lambda(T)$ are the $j-1$ pairs xz, the pair yz on the left boundary, and the pair xy on the right boundary, whereas the contributions of $B' = x^{a_1} z^{b_1} \dots x^{a_j} z^{b_j}$ to $\mathrm{des}_{\lambda'}(g_{\ell m}(T))$ are the j pairs xz and the pair zy on the right boundary, for a total of $j+1$ in both instances.

Adding the contributions of all the blocks, it follows that $\mathrm{des}_\lambda(T) = \mathrm{des}_{\lambda'}(g_{\ell m}(T)) + 1$. □

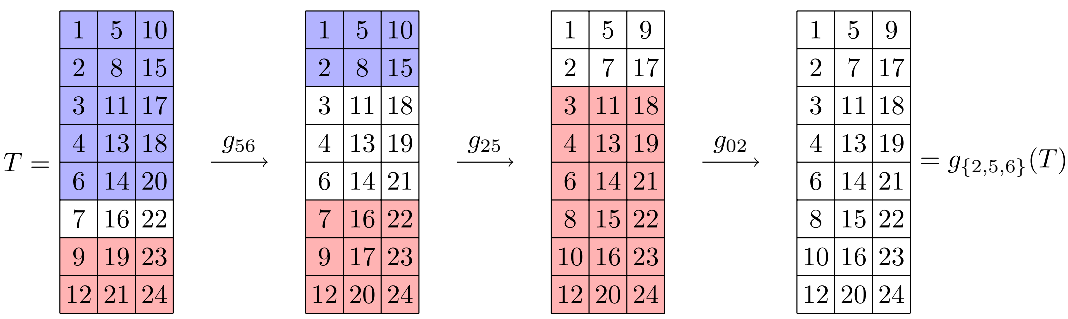

Figure 5: The bijection $g_S = g_{02} \circ g_{25} \circ g_{56}$ for $S = \{2, 5, 6\}$. Letting $\lambda = 78456312$, we have $\mathrm{des}_\lambda(T) = 10 = \mathrm{des}_{12\dots8}(g_S(T)) + 3$ and $\mathrm{plat}(T) = \mathrm{plat}(g_S(T)) = 0$.

By repeatedly applying Lemma 4.2, we can construct a bijection that relates the statistics des_λ and $\mathrm{des}_{12\dots n}$. Specifically, if the elements of $S \subseteq [n-1]$ are $i_1 < i_2 < \dots < i_d$, let

$$g_S = g_{0i_1} \circ g_{i_1 i_2} \circ \dots \circ g_{i_{d-1} i_d}.$$

Then, if $\lambda \in \mathcal{S}_n$ is the reverse-layered permutation with descent set S, Lemma 4.2 implies that

$$\operatorname{des}_\lambda(T) = \operatorname{des}_{12\dots n}(g_S(T)) + d \quad \text{and} \quad \operatorname{plat}(T) = \operatorname{plat}(g_S(T)) \tag{9}$$

for all $T \in \operatorname{SYT}(k^n)$. See Figure 5 for an example of this bijection.

Finally, for any $\sigma \in \mathcal{S}_n$, let $S = \operatorname{Des}(\sigma)$ and define $\phi_\sigma = g_S \circ f_\sigma$. Combining (8) and (9), we see that ϕ_σ satisfies the property stated in (7). See Figure 6 for a complete example of this bijection.

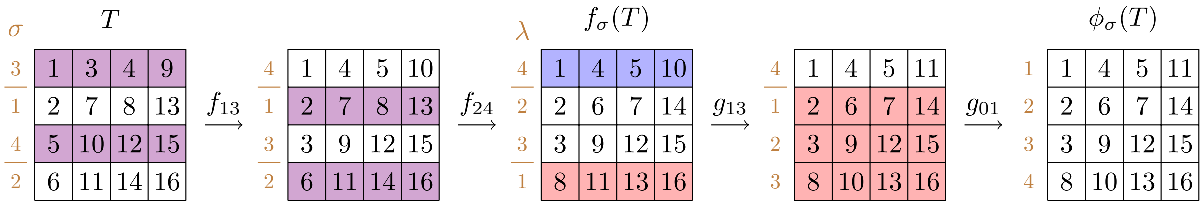

Figure 6: The bijection $\phi_\sigma = g_{\operatorname{Des}(\sigma)} \circ f_\sigma$ for $\sigma = 3142$, which has $\operatorname{Des}(\sigma) = \{1,3\}$. In this example, $\operatorname{des}_\sigma(T) = 6 = \operatorname{des}_{1234}(\phi_\sigma(T)) + 2$ and $\operatorname{plat}(T) = \operatorname{plat}(\phi_\sigma(T)) = 2$.

Remark 4.1. *In the case $k = 2$, our bijection ϕ_σ does not always agree with the analogous bijection described in [5] in terms of Dyck paths. To illustrate this, let us compare the bijection $g_{\ell m} : \operatorname{SYT}(2^n) \to \operatorname{SYT}(2^n)$ from Definition 4.3 with the analogous bijection $\hat{g}$ on Dyck paths described in [5, Lem. 3.6], using the example on the top of [5, Fig. 6], where $n = 11$ and $S = \{2,4,7\}$. Translating this example into tableaux in $\operatorname{SYT}(2^{11})$ via the usual correspondence, we obtain the pair $T \mapsto \hat{g}(T)$ shown on the left-hand side of Figure 7. On the other hand, the right-hand side of this figure shows the tableau $g_{47}(T)$ produced by our map from Definition 4.3.*

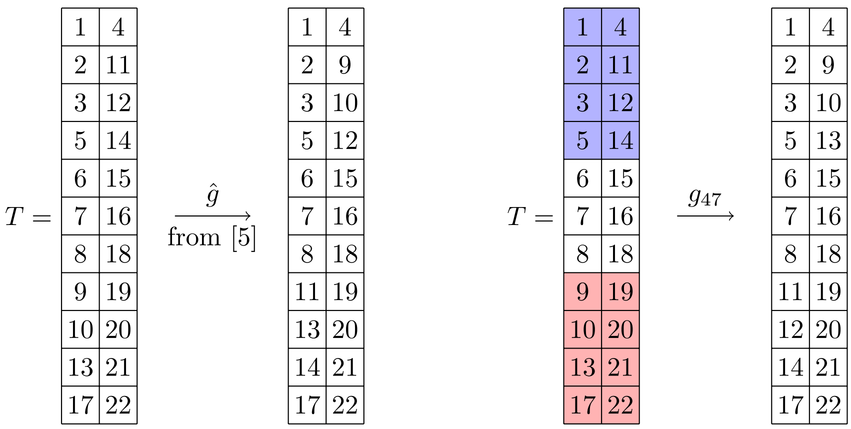

Figure 7: The bijection from [5, Lem. 3.6] does not agree with our bijection from Definition 4.3 for $k = 2$.

5. Symmetries and upcoming work

A consequence of Theorem 3.1 is that the distribution of the number of descents on canon permutations is symmetric. For $k = 2$, this fact was already noted and proved bijectively in [5].

Corollary 5.1. *For every $0 \le h \le k(n-1)$,*

$$|\{\pi \in \mathcal{C}_n^k : \operatorname{des}(\pi) = h\}| = |\{\pi \in \mathcal{C}_n^k : \operatorname{des}(\pi) = k(n-1) - h\}|.$$

Proof. It is well known that the Eulerian polynomials are palindromic, that is, $A_n(t) = t^{n-1} A_n(1/t)$. This is because $\operatorname{des}(\sigma) = n - 1 - \operatorname{des}(\sigma_n \sigma_{n-1} \dots \sigma_1)$ for all $\sigma = \sigma_1 \dots \sigma_n \in \mathcal{S}_n$.

A much less obvious fact is that the generalized Narayana polynomials are palindromic as well. Indeed, it is shown by Sulanke [11, Cor. 1] that

$$N(n,k,h) = N(n,k,(k-1)(n-1) - h) \tag{10}$$

for all $0 \le h \le (k-1)(n-1)$, or equivalently, $N_{n,k}(t,1) = t^{(k-1)(n-1)} N_{n,k}(1/t,1)$ (see Example 2.1).

Using these symmetries together with Theorem 3.1,

$$C_n^k(t,1) = A_n(t)\, N_{n,k}(t,1) = t^{k(n-1)} A_n(1/t) N_{n,k}(1/t,1) = t^{k(n-1)} C_n^k(1/t,1).$$

Taking the coefficient of t^h on both sides, we obtain the stated equality. □

The above proof of Corollary 5.1 is not bijective, since it relies on the symmetry of the generalized Narayana numbers, stated in (10), for which no bijective proofs appear in the literature. A bijective proof of this symmetry will be given in upcoming work [6], from where a bijective proof of Corollary 5.1 will follow.

Finally, we remark that a recent preprint of Beck and Deligeorgaki [2] studies canon permutations from the viewpoint of Stanley's theory of (P, w)-partitions, and gives a different proof of our Theorem 3.1.

Acknowledgements

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