ecajournal.haifa.ac.il
Enumerative Combinatorics and Applications
ECA **5:3** (2025) Article #S2R17

Hankel Determinants of Convoluted Catalan Numbers and Nonintersecting Lattice Paths: A Bijective Proof of Cigler's Conjecture

Markus Fulmek

Faculty of Mathematics, University of Vienna
Email: Markus.Fulmek@univie.ac.at

Received: January 20, 2025, **Accepted**: April 8, 2025, **Published**: April 18, 2025

ABSTRACT: In recent preprints, Cigler considered certain Hankel determinants of convoluted Catalan numbers and conjectured identities for these determinants. In this note, we shall give a bijective proof of Cigler's Conjecture by interpreting determinants as generating functions of nonintersecting lattice paths: this proof employs the reflection principle, the Lindström–Gessel–Viennot–method, and a certain construction involving reflections and overlays of nonintersecting lattice paths. Shortly after the bijective proof was presented here, Cigler provided a shorter proof based on earlier results.

Keywords: Convoluted Catalan numbers; Hankel determinants; Lindström–Gessel–Viennot involution
2020 Mathematics Subject Classification: 05A15; 05A19

1. Introduction

In recent preprints [3, 4], Cigler considered Hankel determinants

$$D_{K,M}(N) \stackrel{\text{def}}{=} \det\left(C_{K,i+j+M}\right)_{i,j=0}^{N-1}, \tag{1}$$

for $M \in \mathbb{Z}$ and $K, N \in \mathbb{N}$, $K \geq 1$, where the entries are convolution powers of Catalan numbers for $p \in \mathbb{N}$

$$C_{K,p} \stackrel{\text{def}}{=} \binom{2p+K-1}{p} - \binom{2p+K-1}{p-1} = \frac{K}{p+K}\binom{2p+K-1}{p} = \frac{K}{2p+K}\binom{2p+K}{p},$$

and $C_{K,p} = 0$ for $p \in \mathbb{Z}$ with $p < 0$.

Cigler presented the following conjecture [4, Conjecture 1, equations (7) and (8)] regarding these determinants, which we formulate as a theorem since we will give a (bijective) proof for it. This is justified even more, since shortly after this proof was presented, Cigler [6] provided an elegant (and much shorter) proof, based on earlier results of Cigler [5, Proposition 2.5] and Andrews and Wimp [2].

Theorem 1.1 (Cigler's Conjecture). *Let $m, k \in \mathbb{N}$, with $m, k > 0$. Then for $K = 2k$ we have the* even identities

$$\begin{aligned} D_{2k,1-k-m}(0) &= 1 \textit{ (true by definition)},\\ D_{2k,1-k-m}(N) &= 0 \textit{ for } N = 1, 2, \ldots, m+k-1, \end{aligned} \tag{2}$$

and for all $n \in \mathbb{N}$

$$D_{2k,1-k-m}(n+m+k) = (-1)^{\binom{m+k}{2}} D_{2k,1-k+m}(n) \tag{3}$$

Moreover, for $K = 2k-1$ we have the odd identities

$$\begin{aligned} D_{2k-1,2-k-m}(0) &= 1 \textit{ (true by definition)},\\ D_{2k-1,2-k-m}(N) &= 0 \textit{ for } N = 1, 2, \ldots, m+k-2, \end{aligned} \tag{4}$$

and for all $n \in \mathbb{N}$

$$D_{2k-1,2-k-m}(n+m+k-1) = (-1)^{\binom{m+k-1}{2}} D_{2k-1,1-k+m}(n) \tag{5}$$

We shall prove this by interpreting the determinants as *generating functions* of tuples of *lattice paths*, for which we apply the well–known Lindström–Gessel–Viennot–method, i.e., we construct a sign–reversing involution for both sides of the identities: for the set of "survivors" of this involution on the left–hand side of the identities we shall construct a *second* sign–reversing involution, and we shall show that all lattice paths that "survive" also this second involution have a very special shape, from which one immediately sees a bijective correspondence with the "survivors" of the right–hand side of the identities. But while this final argument is immediately obvious from a picture, the road to get there is unfortunately long: we will therefore try to explain it through numerous pictures.

In order to keep things as brief as possible, from now on we shall refer to

- the *determinants* to the left and to the right in identities (2), (3), (4) or (5),
- and/or their parameters K, M, N, which depend on k, m, n, and on the *parity* p (even or odd) of the identity
$$K = K(k, m, n, p), M = M(k, m, n, p), N = N(k, m, n, p)$$
- and/or the lattice path interpretation for these determinants and their parameters (yet to be explained)

simply as

- *right–hand side* (which we abbreviate as **rhs**)
- and *left–hand side* (which we abbreviate as **lhs**),

respectively. Moreover, we shall refer to relations that are "geometrically obvious" in the illustrations we provide (like "above", "between", or "to the left") in simple terms (instead of stating everything with coordinates or equations).

In section 2, we briefly recall basic concepts, and in section 3 we present the bijective proof of Cigler's Conjecture.

2. Preliminary considerations

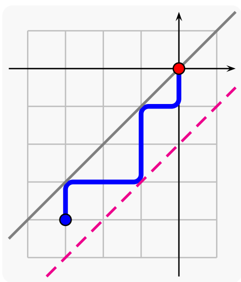
We recall basic facts and make some elementary observations.

2.1 Binomial coefficients and lattice paths

If we restrict the binomial coefficient $\binom{2m+k-1}{m}$ to the case $m, 2m+k-1 \in \mathbb{N}$, then we may interpret it as the number of *lattice paths* in the integer lattice $\mathbb{Z}^2$ consisting of

- horizontal steps to the right, i.e., from (x, y) to $(x+1, y)$
- or vertical steps upwards, i.e., from (x, y) to $(x, y+1)$,

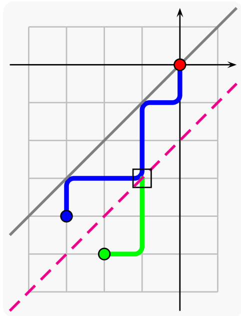
which start at $(a-m, a-m-k+1) \in \mathbb{Z}^2$ and end at $(a, a) \in \mathbb{Z}^2$ for some $a \in \mathbb{Z}$, see the following picture illustrating such path for $a = 0$, $m = 3$ and $k = 2$:



If we want to count only paths that do never touch the *forbidden line* $y = x - k$ (this is the dashed line in the above picture), then we must subtract

- from the number $\binom{2m+k-1}{m}$ of *all* lattice paths
- the number of all lattice paths which *do* touch the *forbidden line*,

and by the well–known *reflection principle* [1] we have that the latter equals $\binom{2m+k-1}{m-1}$, see the following picture where the initial segment of the path up to the first point of intersection (indicated by a small square in the picture) with the forbidden line is *reflected* on the forbidden line:



2.2 Determinants interpreted as generating functions

So, we may interpret the entry

$$C_{K,i+j+M} = \binom{2\,(i+j+M)+K-1}{i+j+M} - \binom{2\,(i+j+M)+K-1}{i+j+M-1}$$

of the determinant (1) as the *number* of lattice paths which

- start at *initial* point $A_i = (-i-M, -i-M-K+1)$
- and end at *terminal* point $B_j = (j, j)$
- and do not touch the *forbidden line* $y = x - K$,

as long as $2\,(i+j+M)+K-1 \geq 0$ (otherwise, the number of such lattice paths clearly is zero, while the binomial coefficients might be nonzero*).

By assigning constant weight 1 to each single lattice path, we may interpret the *number* $C_{K,i+j+M}$ as the *generating function* of the set of all such lattice paths.

Denoting the permutation group on $\{0, 1, \dots, N-1\}$ by $\mathfrak{S}_N$, the determinant $D_{K,M}(N)$

$$\det\left(C_{K,i+j+M}\right)_{i,j=0}^{N-1} = \sum_{\pi \in \mathfrak{S}_N} sgn\,(\pi) \cdot \prod_{i=0}^{N-1} C_{K,i+\pi(i)+M}$$

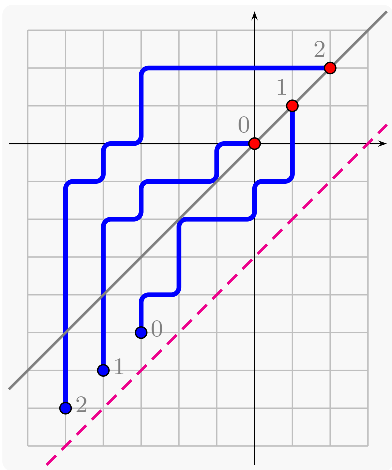
may be viewed as the *generating function* of the set $\mathcal{P}$ of all N–tuples t of lattice paths

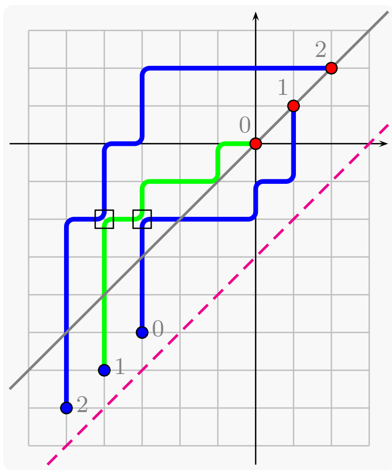
- from initial points A_i
- to terminal points $B_{\pi(i)}$,

for some $\pi(t) = \pi \in \mathfrak{S}_N$ which we call the *permutation of* t, where the *weight* of such N–tuple t equals $sgn\,(\pi(t))$:

$$\det\left(C_{K,i+j+M}\right)_{i,j=0}^{N-1} = \sum_{t \in \mathcal{P}} sgn\,(\pi(t)). \tag{6}$$

The following pictures illustrate this idea by two triples of lattice paths for $K = M = N = 3$ and $\pi = (1, 0, 2)$ (in one–line–notation):





*For instance, if $M = -1$, $K = 1$ and $i = j = 0$, then $\binom{2(i+j+M)+K-1}{i+j} = \binom{-2}{0} = 1$.

Note that in the right picture (where different colours for the paths are used just to make their course more visible), there are two points of intersections of lattice paths, indicated in the picture by small squares: if some N–tuple of lattice paths *has* such points of intersections, then it is called *intersecting*, otherwise it is called *nonintersecting* (so the left picture above shows a *nonintersecting* triple of lattice paths).

In the following considerations, we will also encounter initial and terminal points A'_i, B'_j in *other* positions, and we will sloppily refer to any such configuration of initial and terminal points as *situation*. If we *know* in some situation that initial point A'_i is (or must be) connected with a terminal point $B'_{\pi(i)}$ by a lattice path, we call π the permutation *corresponding to this situation*. (So $\pi = (1, 0, 2)$ is the permutation corresponding to the situation shown in the above pictures.)

2.3 The Lindström–Gessel–Viennot method

The well–known Lindström–Gessel–Viennot method [7,8] constructs an *involution* φ on the set $\mathcal{P}$ of all N–tuples of lattice paths, i.e.,

$$\varphi\colon\ \mathcal{P} \to \mathcal{P} \text{ is bijective and } \varphi^{-1} = \varphi,$$

which is *sign–reversing*, i.e.,

$$\varphi(t) \neq t \implies sgn(\varphi(t)) = -\,sgn(t)\,.$$

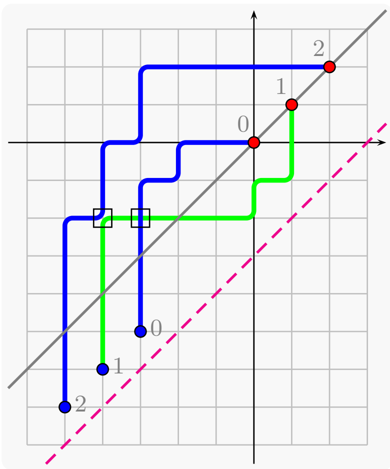
Clearly, such *sign–reversing involution* φ results in a *cancellation* of summands in (6), the only "survivors" of which are the fixed points of φ (i.e., N–tuples t of lattice paths with $\varphi(t) = t$).

Consider the *lexicographic order on* $\mathbb{Z}^2$:

$$(r, s) \succ (u, v) \overset{\text{def}}{\iff} r > u \text{ or } (r = u \text{ and } s > v)\,.$$

The Lindström–Gessel–Viennot–involution φ is defined as follows: if t is a *nonintersecting* N–tuple of lattice paths, then $\varphi(t) = t$. Otherwise choose the *maximal* point of intersection P *in lexicographic order* (in the right picture above, this is the point with coordinates $(-3, -2)$): Assuming that (precisely) the two paths ending in B_{j_1} and B_{j_2} meet in P, we obtain another (again intersecting) N–tuple t' of lattice paths with the *opposite* sign by *exchanging* the *terminal* path segments from P up to B_{j_1} and B_{j_2}, respectively; and we set $\varphi(t) = t'$.

For instance, after applying this involution to the situation of the right picture above, we obtain the following picture for $\pi' = (0, 1, 2)$ (in one–line–notation, again):



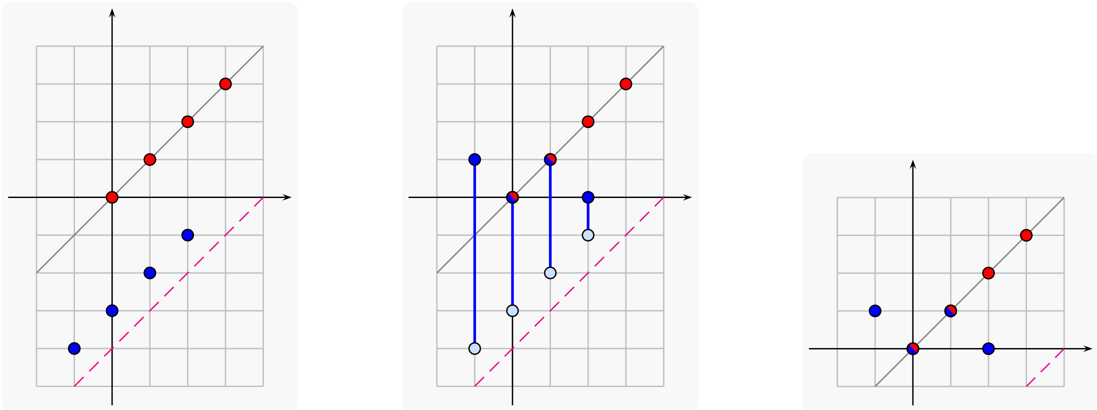
Clearly, the fixed points of this involution are precisely the *nonintersecting* N–tuples of lattice paths. For the rest of this paper, we shall only consider such N–tuples of nonintersecting lattice paths, which we call *survivors* (since they "survive" the cancellation implied by the Lindström–Gessel–Viennot–involution): note that the generating of these survivors equals the determinant $D_{K,M}(N)$ in (6), where the *sign* $sgn(o)$ of a survivor o connecting initial points A_i to terminal points $B_{\pi(i)}$ is defined as the sign $sgn(\pi)$ of the permutation π.

In the typical application of the Lindström–Gessel–Viennot method, there is only one permutation π (typically: the identity) for which a tuple of nonintersecting paths is possible: note that this is *not necessarily* the case in the situation we are considering here, see again the above pictures.

2.4 Enforced segments for survivors

Observe that the nonintersecting lattice paths starting in A_i, $i = 0, 1, \ldots, N-1$, *must not* start with a horizontal step to the right (since they must not touch the *forbidden line*): so each lattice path starts with an *enforced* *initial segment* of vertical upward steps (which might be empty in special cases), and since the lattice paths are *nonintersecting*, the length of the *enforced initial segment* starting in A_{i+1} exceeds the length of the *enforced initial segment* starting in A_i by *two*, *except* in cases where this enforced initial segment runs into one of the endpoints B_j, where it is “stopped” immediately (so if some initial point *coincides* with some terminal point, the enforced initial segment starting there has length 0). Following these enforced initial segments from their (original) initial points leads to (new) *enforced initial points*: so instead of considering the *original* initial points, we may always consider these *enforced* initial points.

The meaning of this will become clear by looking at the following pictures (where $K = 4$, $M = -2$ and $N = 4$). Here, the *enforced initial segments* are shown as blue vertical lines in the middle picture, and the *enforced initial points* are either blue or *bicoloured* (blue *and* red; if the enforced initial point *coincides* with a terminal point):



Note that in the situation shown in the above pictures, for any nonintersecting 4–tuple the lattice path

- starting in A_0 or A_3 *must* end in B_2 *or* in B_3,
- starting in A_1 *must* end in B_1,
- starting in A_2 *must* end in B_0,

and the corresponding *enforced sub–permutation*

$$\pi\colon\ 1 \mapsto 1,\ 2 \mapsto 0$$

is of length 2 is *descending* and thus contributes $(-1)^{\binom{2}{2}} = -1$ to the sign of the overall permutation (which depends on whether A_0 is connected to B_2 or B_3; for both possibilities there are nonintersecting 4–tuples of lattice paths).

2.5 Positions of enforced initial points

As a preparation for the proof, we examine the *enforced initial points* of the nonintersecting lattice paths corresponding to the **lhs** and **rhs** of

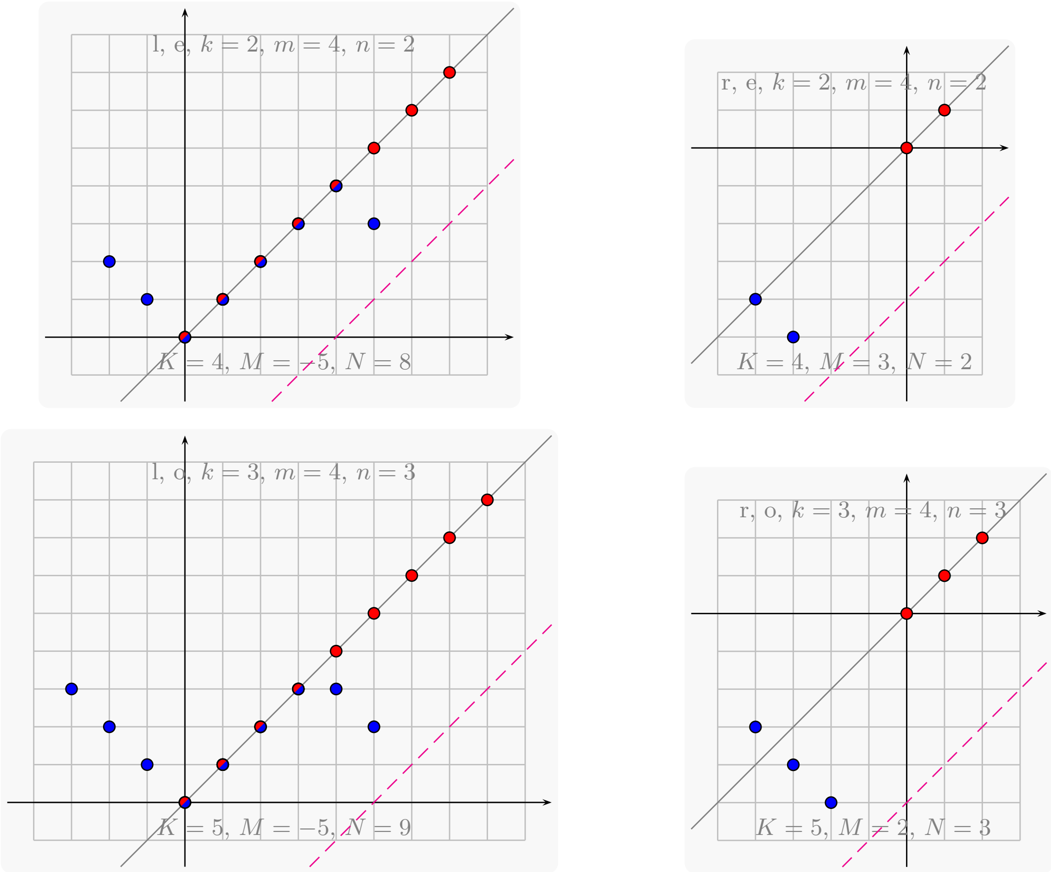
- the even identities (2) and (3),
- and the odd identities (4) and (5).

As examples we consider

- the even identity for $k = 2$, $m = 4$ and $n = 2$,
- and the odd identity for $k = 3$, $m = 4$ and $n = 3$;

see the following pictures. As can be seen in the pictures, there are

- enforced initial points which are not terminal points (coloured blue),
- terminal points which are not enforced initial points (coloured red),
- and enforced initial points which are *also* terminal points (with colours blue *and* red, we call such points *two–faced*).



In order to show that there are

- *always* two–faced points in the **lhs**,
- but *never* two–faced points in the **rhs**,

(as the above pictures suggest), note that the *first enforced initial points* lie on the line

$$(y = -x - 2M - K + 2)$$

with slope -1 through the point $(-M, -M - K + 2)$. This line intersects the diagonal $(y = x)$ (which contains the terminal points) in the point

$$S = \left(-M + 1 - \frac{K}{2}, -M + 1 - \frac{K}{2}\right),$$

so terminal points which are *also* initial points only exist if the following inequality holds:

$$-M + 1 - \frac{K}{2} \geq 0 \iff \frac{K}{2} \leq -M + 1. \tag{7}$$

If (7) holds, then the number of two–faced points is $\left\lfloor -M + 1 - \frac{K}{2} \right\rfloor + 1$.

2.5.1 Even identities

In the even case, the parameters for the **lhs** are

$$K = 2k, M = 1 - k - m, N = n + m + k,$$

so the point of intersection is $S = (m, m)$, and inequality (7) reads

$$k \leq m + k,$$

which is true for $m \geq 0$ (and there are $m + 1$ two–faced points), and the parameters for the **rhs** are

$$K = 2k, M = 1 - k + m, N = n$$

so the point of intersection is $S = (-m, -m)$, and (7) reads

$$k \leq k - m,$$

which is false for $m \geq 1$ (and there are no two–faced points).

2.5.2 Odd identities

In the odd case, the parameters for the **lhs** are

$$K = 2k - 1, M = 2 - k - m, N = n + m + k - 1,$$

so the point of intersection is $S = \left(m - \frac{1}{2}, m - \frac{1}{2}\right)$, and (7) reads

$$k - \frac{1}{2} \leq m + k - 1$$

which is true for $m \geq 1$ (and there are m two–faced points), and the parameters for the **rhs** are

$$K = 2k - 1, M = 1 - k + m, N = n,$$

so so the point of intersection is $S = \left(-m + \frac{1}{2}, -m + \frac{1}{2}\right)$, and (7) reads

$$k - \frac{1}{2} \leq -m + k,$$

which is false for $m \geq 1$ (and there are no two–faced points).

2.5.3 Even and odd identities

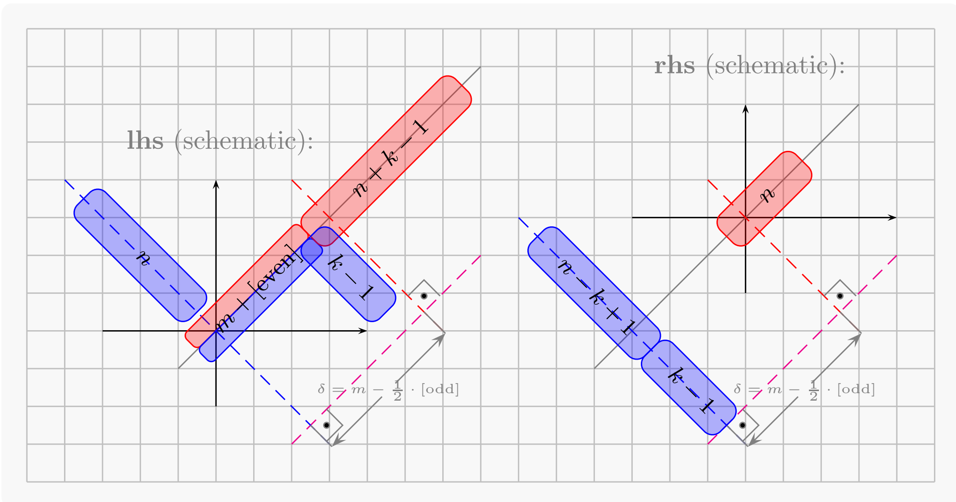
It is easy to see that for the **lhs** *and* the **rhs**, the number of initial points *strictly below* the diagonal $(y = x)$ is always $\min(k - 1, N)$, and that for the **lhs** and $n \geq 0$, the number of initial points

- *strictly above* the diagonal $(y = x)$ is always n,
- *on* the diagonal (these are the *two–faced* points) is
  - $m + 1$ for the even case
  - and m for the odd case,

  which we express uniformly as $(m + [\text{even}])$, using Iverson's bracket.

(*Iverson's bracket* ["some assertion"] is defined as 1 if "some assertion" is true, otherwise as 0: we shall employ this useful notation also in the following.)

The following pictures give a schematic illustration of the situation (here, the two–faced points for the **lhs** are indicated by a box coloured blue *and* red, and all other initial and terminal points are indicated by blue and red boxes, respectively):



Note that the distance δ between the orthogonal projections

- of the first initial point (for the **lhs**: *above* the diagonal $(y = x)$)
- and of the first terminal point (for the **lhs**: which is *not two–faced*)

on the forbidden line is the *same* for the **lhs** and the **rhs** (namely $\delta = m + \frac{1}{2} \cdot [\text{odd}]$, see the above pictures).

2.5.4 A first consequence

These simple observations already prove identities (2) and (4): if for the **lhs**

- $N < m + k$ in the even case
- or $N < m + k - 1$ in the odd case,

then the terminal point $(0, 0)$ cannot be reached by *any* initial point, and therefore the generating function of the corresponding nonintersecting lattice paths is zero. So it only remains to prove identities (3) and (5)

2.6 A warmup–exercise: Cigler's Conjecture for $k = 1$

The simple observations we made so far already give a simple bijective proof (essentially by looking at suitable pictures) for the special case $k = 1$ of Theorem 1.1 (the *odd identities* for $k = 1$ are, in fact, equivalent to Cigler's earlier result [4, equation (5)], see also [3, Theorem 1, equation (6)]):

Proof of Cigler's Conjecture for special case $k = 1$. We start with the *odd identities* with parameters $K = 2k - 1 = 1$ and

- $M = 1 - m$ and $N = n + m$ for the **lhs**,
- $M = m$ and $N = n$ for the **rhs**.

Observe that in this situation the initial points A_i and the terminal points B_j are *collinear*.

Recall that (4) was already proved in section 2.5.4.

For $N = m$ (this is the special case $n = 0$ for the **lhs** of (5)), we have

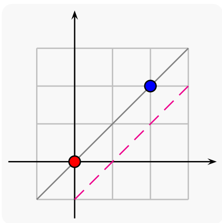
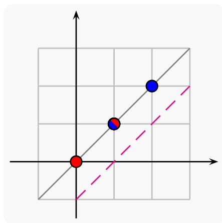
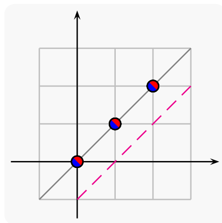
$$B_i = (i, i) = A_{m-1-i} \text{ for } i = 0, 1, \ldots, m - 1,$$

and there is *precisely one* N–tuple of nonintersecting lattice paths (where all paths have length 0), and the corresponding permutation

$$\pi\colon\ i \mapsto m - 1 - i \text{ for } i = 0, 1, \ldots, m - 1$$

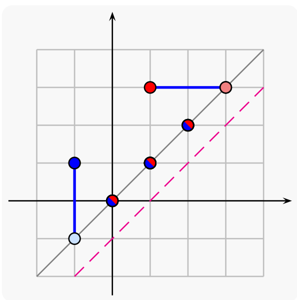
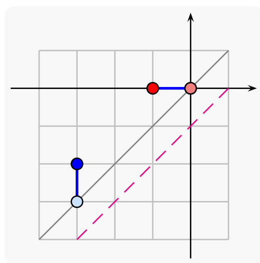
has sign $sgn(\pi) = (-1)^{\binom{m}{2}}$. Since the **rhs** of (5) equals 1 by convention for $N = n = 0$, equality holds in this special case.

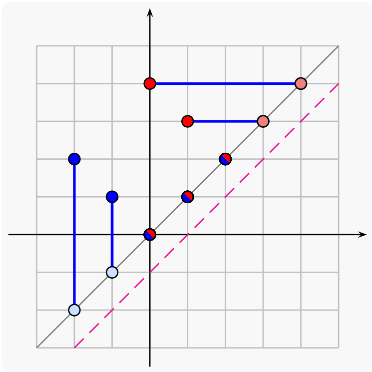
The following pictures illustrate these simple observations for $m = 3$ (i.e., $M = 2 - k - m = -2$) and $N = n = 1, 2, 3$ (again, two–faced points are coloured blue *and* red):

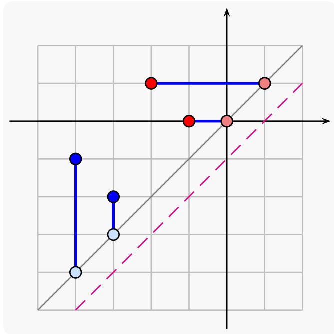




In order to show (5) for $n > 0$, observe that in this specific situation there are also "enforced *terminal* segments" of horizontal steps to the right for the lattice paths, since no terminal point B_j can be reached by a vertical upward step. The following pictures illustrate the situation for $m = 3$ and $n = 1, 2$, implying

- $M = 2 - k - m = -2$ and $N = 4, 5$ for the **lhs** of (5), shown in the left pictures below,
- $M = 1 - k + m = 3$ and $N = 1, 2$ for the **rhs** of (5), shown in the right pictures below:





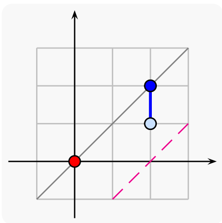
Now observe that the *enforced* initial and terminal points corresponding to the **lhs** of (5) are simply obtained by a *translation* of the *enforced* initial and terminal points corresponding to the **rhs** (recall the considerations and pictures in section 2.5.3), and that the m *enforced two–faced points*

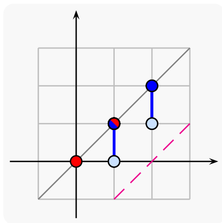
- correspond to the *same* translation of the *forbidden line*
- and contribute a factor $(-1)^{\binom{m}{2}}$ to the sign of the corresponding permutation (which has fixed points $m+1, m+2, \ldots, N-1$).

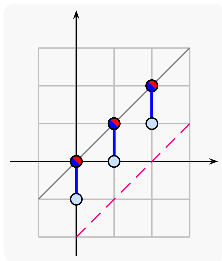
These observations conclude the proof of the *odd identity* (5) for the special case $k = 1$.
The proof of the *even identity* (5) with parameters $K = 2k = 2$ and

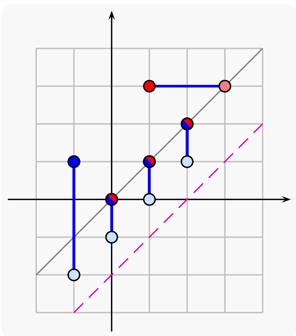
- $M = -m$ and $N = n + m + 1$ for the **lhs**,
- $M = m$ and $N = n$ for the **rhs**,

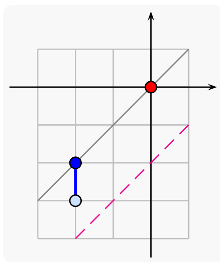
is completely analogous to the odd case, see the following pictures for $m = 2$ (i.e., $M = 1 - k - m = -2$ and $M = 1 - k + m = 2$, respectively, for the **lhs** and the **rhs**):

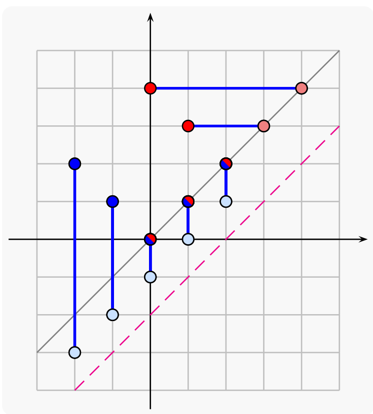


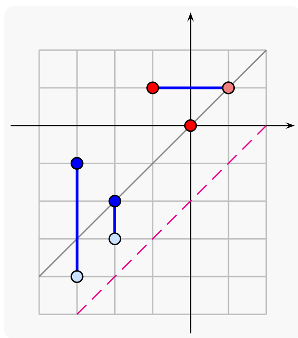












The same reasoning as for the odd identity concludes the proof. □

3. Bijective proof of Cigler's Conjecture for $k > 1$

The rest of this paper is devoted to a bijective proof of Ciglers's Conjecture for $k > 1$. While all single arguments are elementary, the chain of arguments is long and complicated: we will make extensive use of *pictures* to explain them, hoping that the saying "A picture is worth a thousand words" proves to be true.

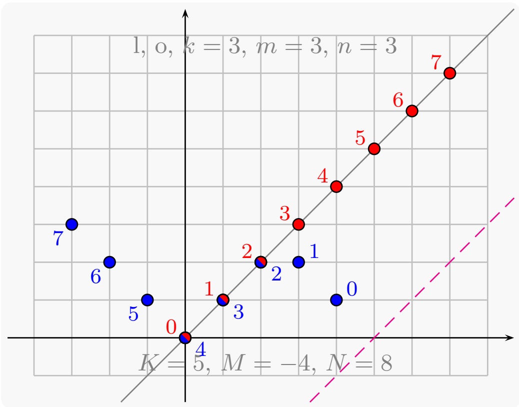
3.1 Encoding permutations of survivors

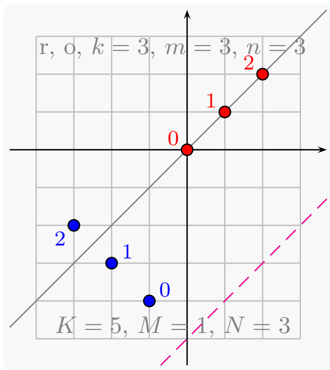
Let us call a permutation π *admissible* if there are survivors (N–tuples of nonintersecting lattice paths) connecting initial point A_i to terminal point $B_{\pi(i)}$ for $i = 0, 1, \dots, N-1$.

Note that for every survivor, the lattice path reaching the terminal point B_i may have length 0 (if B_i is also an enforced initial point: this can only happen for the **lhs**; see subsection 2.5.3), but otherwise must end

- either with an upward step *from below*
- or with a rightwards step *from the left*;

see the following pictures:





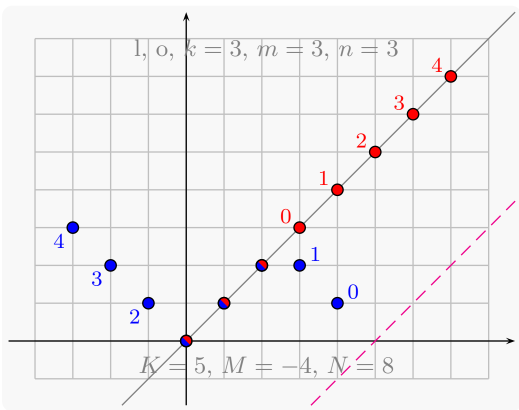
Considering the **lhs**, note that all lattice paths starting at enforced initial points

- *below* the diagonal must end with an upwards step,
- *on* the diagonal have length 0,
- *above* the diagonal must end with a rightwards step.

This means that *all* admissible permutations for the **lhs** must map

- indices $k-1, k, k+1, \dots, k+(m+[\text{even situation}])-2$
- to indices $0, 1, \dots, (m+[\text{even situation}])-1$

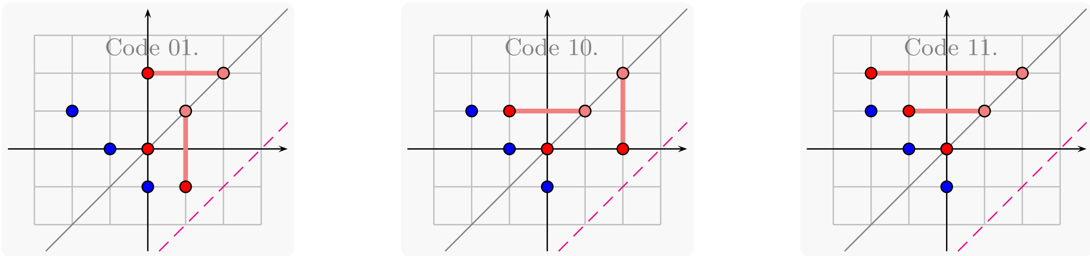
in reverse order (in the above picture: $2 \mapsto 2$, $3 \mapsto 1$ and $4 \mapsto 0$). We may disregard this "enforced subpermutation" and renumber the initial and terminal points as indicated in the following picture:



With this renumbering for the **lhs** (and the original numbering for the **rhs**), we assign to π the 01–code (omitting terminal point B_0)

$$c(\pi) = ([\text{Terminal point } B_i \text{ is reached from the left}])_{i \geq 1}.$$

By considering *enforced terminal segments*, as illustrated in the following pictures, is easy see that the mapping $\pi \mapsto c(\pi)$ is *injective*:



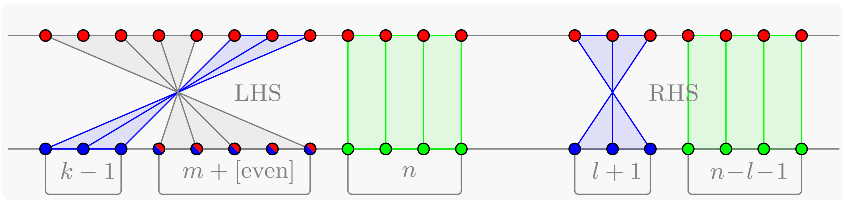
(Note that no admissible permutation is mapped to 01–code 00 in the situation depicted above, since the paths must not intersect the forbidden line.) We call a 01–code *admissible* if it is assigned to an admissible permutation, so (by definition) there is a *bijection* between the set of admissible permutations and the set of admissible codes.

3.2 Relations between signs of admissible permutations and their codes

We want to express the sign of an admissible permutation π in terms of its corresponding 01–code $c(\pi)$: as *base cases*, we consider the 01–codes

- consisting of $k-1$ zeros, followed by n ones, for the **lhs**,
- consisting of $l \geq 0$ zeros, followed by $n-l$ ones, for the **rhs** (note that $l=1$ means that the permutation has precisely one inversion).

The permutations corresponding to these base cases are visualized in the following pictures:



Clearly, the signs of these permutations are

- $(-1)^{\binom{m+[\text{even}]+k-1}{2}}$ for the **lhs** (since the number of inversions of this permutation is $\binom{m+[\text{even}]+k-1}{2}$),
- $(-1)^{\binom{l+1}{2}}$ for the **rhs** (since the permutation has $\binom{l+1}{2}$ inversions (note that this is true also for $l=0$).

For a 01–code $c = (c_i)_{i \geq 1}$, an *inversion* of c is a pair of indices $(i<j)$ such that $c_i > c_j$. Denote by $inv(c)$ the *number* of inversions of c.

If we number the zeros in some 01–code c from left to right (starting with 1) and assume that the i–th zero appears at index z_i ($z_i \geq 1$ by definition) in c, then the number of inversions of c equals the sum

$$inv(c) = \sum_{i=1}^{\#(\text{zeros in } c)} (z_i - i).$$

It is easy to see: a permutation π with code $c(\pi)$

- for the **lhs** has
$$\binom{m+[\text{even}]}{2} + inv(c(\pi)) \tag{8}$$
inversions,

- for the **rhs** has

$$\binom{l+1}{2} + inv\,(c\,(\pi)) = \sum_{i=1}^{l} z_i \tag{9}$$

  inversions, if c contains exactly l zeros.

(We will use these facts at the end of our proof.)

Assume that in some 01–code $c = (c_k)_{k\geq 1}$ we have for $1 \leq i < j$

- $c_i = 1$
- and $c_j = 0$,

and let l be the number of components $c_k = 0$, for $i < k < j$: Then *swapping* c_i and c_j

- decreases the number of inversions involving c_i by $l + 1$,
- increases the number of inversions by 1 for each of the $j - i - 1 - l$ ones *between* i and j,

so the change in sign caused by such swap is $(-1)^{|j-i|}$, and the same holds true for $c_i = 0$ and $c_j = 1$: we note this simple observation for later reference.

3.3 Rewrite the identities as equations for sums over 01–codes

If we consider the set of all admissible codes for the **lhs** and for the **rhs** in equations (3) and (5) and write $\mathbf{gf}\,(c)$ for the generating function of survivors corresponding to c (i.e., with permutation π of terminal points corresponding to c), then we may write (3) and (5) in a uniform way:

$$\sum_{C \text{ for LHS}} sgn\,(\pi)\,\mathbf{gf}\,(C) = \sum_{c \text{ for RHS}} sgn\,(\pi)\,\mathbf{gf}\,(c). \tag{10}$$

In general, neither the summation ranges nor the summands of this identity coincide: our proof will involve “reducing” the (larger) sum for the **lhs** to the sum of the **rhs**, by certain cancellations resembling the Lindström–Gessel–Viennot–involution.

3.4 Plan for our bijective proof

The plan for our bijective proof is simple:

1. Find *another* sign–reversing involution ψ on the set of all survivors (i.e., nonintersecting lattice paths) corresponding to the **lhs**.
2. Show that there is a bijection ξ
   - from the set of the fixed points of ψ (i.e., the *survivors of the second cancellation* corresponding to ψ) for the **lhs**
   - to the set of survivors corresponding to the **rhs**
   - which changes the sign by a *constant factor* f (i.e., $sgn\,(\xi\,(o)) = f \cdot sgn\,(o)$), namely
     - $f = (-1)^{\binom{m+k}{2}}$ in the even case,
     - $f = (-1)^{\binom{m+k-1}{2}}$ in the odd case.

As it will turn out, the bijection ξ “respects the summands in (10)” in the following sense: it implies a bijection between

- the summation range (i.e., 01–codes C) of *survivors of the second cancellation* (effected by ψ) appearing in the **lhs** of (10)
- and the summation range (i.e., 01–codes $c = \xi\,(C)$) for the **rhs** of (10),

and it maps

- *survivors of the second cancellation* corresponding to 01–code C in the **lhs**
- to survivors corresponding to 01–code $c = \xi\,(C)$ in the **rhs**.

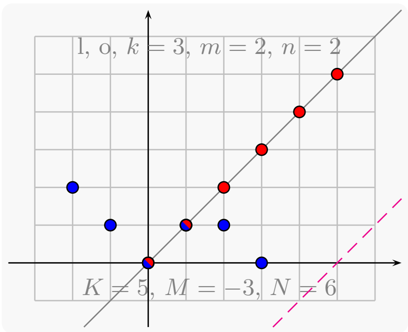
3.5 Step 1: Construct the sign–reversing involution ψ

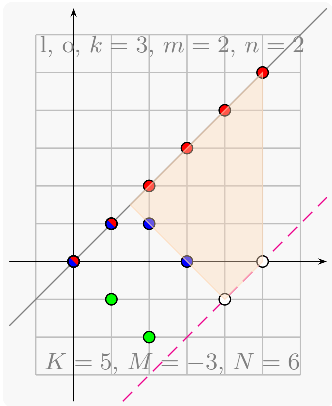
We shall construct the sign–reversing involution ψ on the set of survivors for the **lhs** by combining the Lindström–Gessel–Viennot–idea with simple reflections on the diagonal line $d = (y = x)$.

3.5.1 Reflection for the lhs: the folded situation

If we *reflect* on the diagonal d *all* nonintersecting lattice paths for the **lhs** which lie *above* d, then we call the result of such reflection the *folded situation*. In the following pictures, *reflected* paths will be coloured *green*, as well as their reflected initial points (and all paths and initial points *below* d, which are *not reflected*, will be coloured blue, as before).

We illustrate this for the odd identity with parameters $k = 3$, $m = 2$, and $n = 2$ (i.e., $K = 2k - 1 = 5$, $M = 2 - k - m = -3$, and $N = n + m + k - 1 = 6$). "In terms of initial, two–faced and terminal points", the transition from the *original* situation to the *folded* situation looks like this:



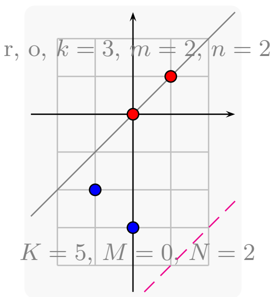


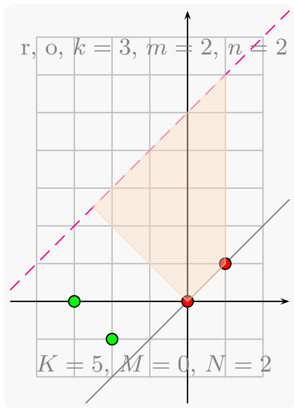
Note that in the right picture above (showing the folded situation), a certain trapezoid is marked in colour, and additional white points on the forbidden line points are shown. We call this the *essential region*: its significance and the meaning of the white points will soon become clear.

3.5.2 Reflection for the rhs: The reflected situation

If we *reflect* on d the forbidden line $y = x - K$ and *all* nonintersecting lattice paths for the **rhs**, then we call the result of such reflection the *reflected situation*.

Again, we illustrate this for the odd identity with parameters $k = 3$, $m = 2$ and $n = 2$ (i.e., $K = 2k - 1 = 5$, $M = 1 - k + m = 0$, and $N = n = 2$). "In terms of initial and terminal points", the transition from the *original* situation to the *reflected* situation looks like this:





Note that in the right picture above (showing the reflected situation), a certain trapezoid is marked in colour again, which we also call the *essential region* (for the reflected situation).

3.5.3 Essential regions

It is easy to see (recall the considerations and illustrations in section 2.5.3) that there is a *translation* which transforms

- the essential regions for the folded situation

- to the essential region for the reflected situation,

which maps bijectively

- green initial points to green initial points
- and white additional points to terminal points,

and which *swaps the roles* of the diagonal and the forbidden line for the folded situation and the reflected situation (see the essential regions in the above pictures).

This congruence of essential regions is crucial for the second step in our proof: Clearly, the reflections explained above give *sign–preserving bijections* (since the permutation π is not affected by the reflections) from the "original situation" to the folded or reflected situation, respectively, so instead of defining a sign–preserving bijection between

- the survivors for the **lhs** which also "survive" the cancellation effected by the *second* sign–reversing involution ψ (yet to be defined)
- and the survivors for the **rhs**,

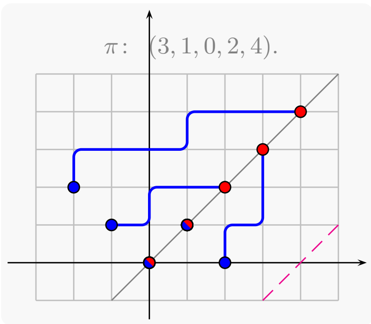
we will define the sign–preserving bijection ξ between

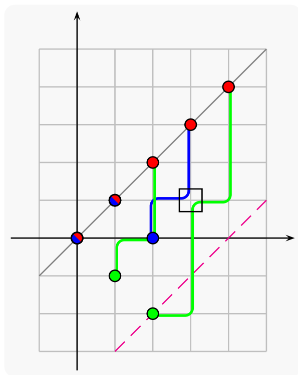
- the "twofold" survivors for the **lhs** *in the folded situation*
- and the survivors for the **rhs** *in the reflected situation*.

The construction of this bijection ξ will involve *only* the essential regions, leaving segments of green paths *outside* the essential regions *unchanged*.

3.5.4 Folded overlays

In the following illustration, the left picture shows nonintersecting lattice paths (for the **lhs**), and the right picture shows the *folded* situation, obtained by the reflection of all paths above the diagonal $d = (y = x)$ on d:





Observe that by this reflection, we get an *overlay* of green and blue paths, such that

- all paths (blue or green) are restricted to the range $y \leq x$,
- blue paths *additionally* are restricted to the range $y > x - K$,
- blue and green paths are *nonintersecting*, i.e., there is no point of intersection of two blue paths or two green paths.

We shall call such overlay of green and blue paths a *folded overlay*. (As already mentioned, folded overlays are in *sign–preserving bijection* with the "original" nonintersecting lattice paths).

Note that a folded overlay appears as a directed subgraph of the lattice $\mathbb{Z}^2$ with green and blue edges, which

- have the *normal* direction (rightwards or upwards),
- but might be traversed in the *reversed* direction (leftwards or downwards),

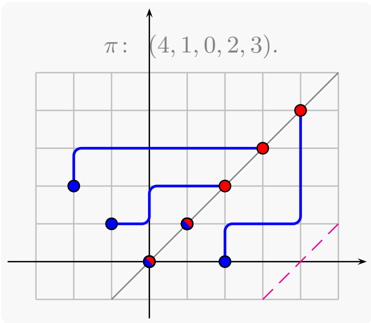
and where double edges (which necessarily must be of opposite colours blue and green) are allowed. We call vertices in this graph which are incident *only* with a green (or blue, respectively) edge a green (or blue, respectively) point, and we call points which are incident with green *and* blue edges *bicoloured*.

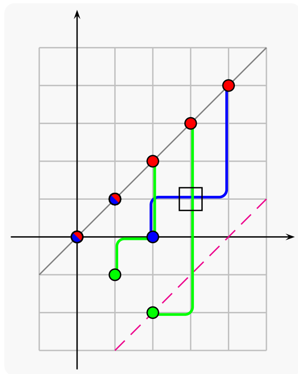
3.5.5 Points of bicoloured intersection and the Lindström–Gessel–Viennot–idea for folded overlays

An *overlay of green and blue paths* may have bicoloured points where some *blue* path intersects some *green* path: in the above right picture, there are three such *points of bicoloured intersection*, the *maximal* (in lexicographic order) of them is indicated by a small square. Clearly,

- every *terminal* point and every initial point which is *not bicoloured* is incident with *precisely one* step which is either green or blue,
- every *bicoloured* point is incident with
  - precisely two blue and precisely two green steps *if* it is not an initial point,
  - precisely one blue and precisely one green step *if* it is an initial point,
- every other point is incident with *precisely two* steps of the *same* colour (either green or blue).

In the above pictures, we immediately see how a sign–reversing involution resembling the Lindström–Gessel–Viennot–method might work — just look at the pictures below, where terminal segments of paths after the maximal point of bicoloured intersection are *exchanged*:





However, we need a slightly more complicated construction.

3.5.6 The significance of essential regions

Clearly, a *point of bicoloured intersection* in some *folded overlay* can only be located

- *weakly above* the forbidden line (since there are no blue steps strictly below)
- and *weakly above* the line containing the blue initial points (since there are no blue steps strictly below)
- and *weakly below* the line $(y = x)$ (since there are no points above)
- and *weakly to the left* of the line $x = M - 1$ (since there are no points to the right).

Note that the region just described is precisely the *essential region* for the folded situation: in the following, we will mostly focus our attention on this essential region.

3.5.7 Bicoloured connections and the sign–reversing involution

For each terminal point B_j in a *folded overlay*, we construct the *bicoloured path* starting at B_j as follows: we start with

- $P = B_j$,
- the *reversed* direction as *current direction*
- and the *unique* colour of the path ending in B_j as *current colour*.

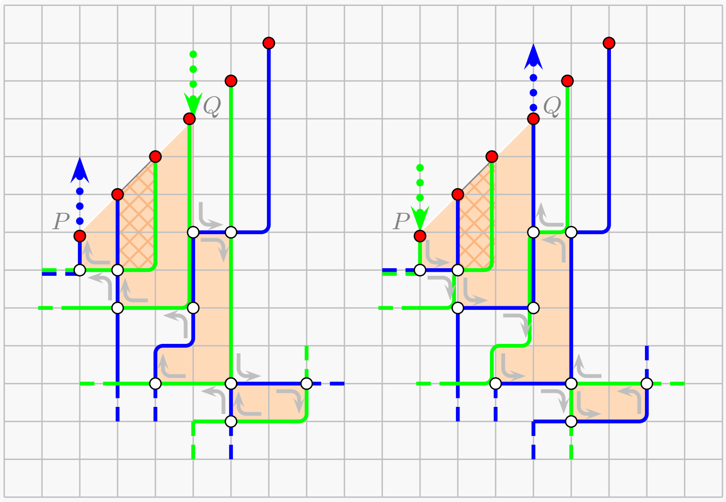
As long as this is possible, we traverse the edge *incident with* P

- of the current colour
- in the current direction,

to its other endpoint Q and set $P = Q$; and if Q is a point of bicoloured intersection, we *swap the current colour (blue/green) and the current direction (normal/reversed).*

It is easy to see that this construction must end in an initial or terminal point Q where we cannot continue (i.e., where there is no incident edge of the current colour and current direction): we call such bicoloured path from B_j to Q a *bicoloured connection of* B_j *and* Q.

The following pictures illustrate this construction: in the left picture, we start at the green terminal point labeled Q, and in the right picture, we start at the green terminal point labeled P.



(Observe that there is also a shorter bicoloured connection, indicated in the pictures by the crosshatched area.)

Now we are in the position to define the involution ψ on the set of all *overlays of green and blue paths* in the folded situation of the **lhs**: Let o be an overlay of green and blue paths: we call a *bicoloured connection* c in o

- which *connects two different* terminal points $B_a \neq B_b$,
- and whose *green* segments *never intersect the forbidden line*

an *involutive bicoloured connection.* (Note that B_a and B_b must have opposite colours, and that an involutive bicoloured connection never leaves the essential region.)

A moment's thought shows that *swapping the colours* (green to blue and vice versa) of *all* edges belonging to an involutive bicoloured connection gives another overlay of green and blue paths o':

- By construction, the blue paths *and* the green paths in o' are *nonintersecting*,
- and the blue paths in o' do not intersect the forbidden line.

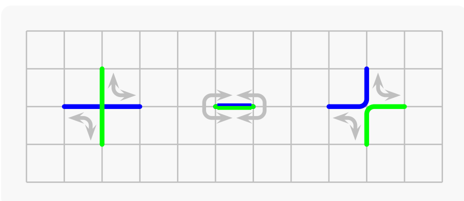
(The right picture above is obtained by this swapping of colours, and vice versa.)

So for all folded overlays o

- which do not contain any involutive bicoloured connection, we simply set $\psi(o) = o$,
- which contain involutive bicoloured connections, we choose the maximal terminal point (in lexicographic order) for which an involutive bicoloured connection c exists and define $\psi(o)$ as the folded overlay o' obtained by the swapping of colours in c (as described above).

In the following pictures, the possible courses of bicoloured connections through a point of bicoloured intersection are indicated by small arrows:

- They show "essentially all" possible situations at *points of intersections* in the network of blue and green paths (modulo swap of colours blue and green, or directions horizontal and vertical)
- and make clear that each *bicoloured* edge would form a (trivial) bicoloured connection of the vertices it is incident with (so bicoloured edges never belong to bicoloured connections of initial or terminal points).



A moment's thought shows that swapping colours in a bicoloured connection

- does not change any *other* bicoloured connection,
- does not destroy any existing bicoloured connection,
- and does not introduce new bicoloured connections:

So the mapping ψ is, in fact, an *involution.*

Moreover, bicoloured connections may intersect, but (by construction) can *never cross* (neither "itself" nor "one another"). Therefore, a bicoloured connection of *terminal points* B_a and B_b forms an "impenetrable barrier" for other bicoloured connections (see the above pictures, where the area enclosed by this "barrier" is coloured). But this implies that the number of terminal points *between* B_a and B_b must be *even*, so for $o \neq \psi(o)$, we have $sgn(\psi(o)) = -\,sgn(o)$, since the swapping of colours amounts to swapping

- a zero at position a
- and a one at position b

(or vice versa) with *odd distance* $|a - b|$ in the 01–code corresponding to o (see the considerations in section 3.1). Therefore, the mapping ψ is, in fact, *sign–reversing.*

So the determinant on the **lhs** in (3) and (5) appears as the generating function of the fixed points of ψ, who "survive" the cancellation effected by ψ. Clearly, these fixed points are the folded overlays which *do not* contain any involutive bicoloured connection: We shall call them *folded survivors.*

The next step in our proof is the exhibition of a bijection ξ between the

- the folded survivors in the **lhs** (in the folded situation)
- and *all* ("normal") survivors in the **rhs** (in the reflected situation)

of identities (3) and (5).

3.6 Step 2: Exhibit the bijection ξ

The bijection ξ we shall exhibit in this section between

- *folded survivors* (i.e, nonintersecting lattice paths from the *folded situation* of the **lhs** which "survive" the cancellation by the sign–preserving involution ψ),
- and ("normal") survivors (i.e., nonintersecting lattice paths) from the *reflected* **rhs**

will leave *unchanged* all segments of paths *outside the essential regions* and employ a "natural transformation" from

- the restriction to the essential region of folded survivors, which have
  - terminal points on the diagonal $y = x$
  - and the forbidden line for blue paths $y = x - K$,
- and the restriction to the essential region of ("normal") survivors, which have
  - terminal points on the diagonal (corresponding by a translation to the forbidden line)
  - and the *reflected* forbidden line $y = x + K$ (corresponding by a translation to the diagonal).

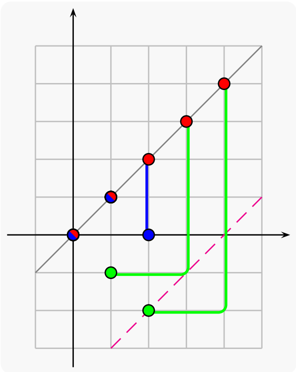
In order to conceive this "natural transformation" inside the essential region, we will show that a folded overlay is a *folded survivor* if and only if it has a certain *simple structure.* This will be the most strenuous part of this section: once we have achieved this, the "natural transformation" will basically be obtained by "inspection of pictures".

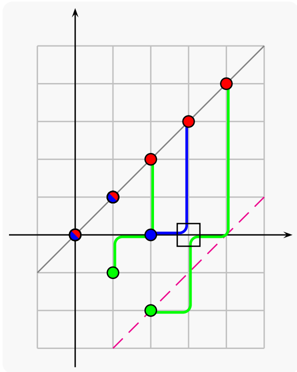
3.6.1 Examples of folded survivors

By definition, a *folded survivor* is a folded overlay which *does not* have an *involutive bicoloured connection*, i.e.,

- either there is *no point of intersection at all*
- or each bicoloured connection of terminal points contains some green segment which intersects the forbidden line $y = x - k$.

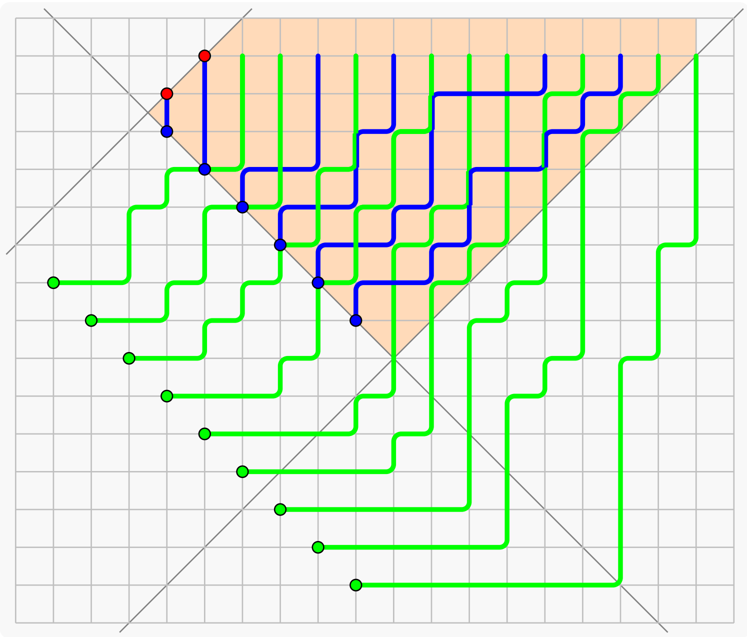
The following pictures illustrate this:





It is hard to conceive the structure of folded survivors in small examples, so we shall consider larger ones: in order to limit the size of the corresponding pictures, we observe that for any folded overlay $\mathcal{F}$ there are constants $z \in \mathbb{Z}$ such that $\mathcal{F}$ does not have points of bicoloured intersection (x, y) with $y > z$, so we will "cut away" the uninteresting part above such level z in the following graphical illustrations.

The following picture shows a folded survivor in the odd case with parameters $k = 7$, $m = 4$ and $n = 9$ (which gives $K = 13$, $M = -9$ and $N = 19$ for the **lhs**), where we have "cut away" the part above level 5:



Here, the four two–faced points and only two of the terminal points are shown (the others are "cut away"), and the *essential region* (which, too is cut) is indicated as coloured area.

It is easy to verify "by inspection" that this folded overlay has no involutive bicoloured connection. We shall use this folded survivor as a running example in this section.

Since the transformation we want to present only involves the part *inside the essential region* (where points of bicoloured intersections might be located), we shall omit the green segments of paths *outside* the essential region in the following pictures.

3.6.2 The planar graph corresponding to a folded overlay

Let $\mathcal{F}$ be some folded overlay: if we

- consider only points and steps *inside the essential region*,

- disregard the orientation (but not the colour!) of all edges,
- and "merge" all double (undirected) edges thus obtained into *bicoloured* single edges (having colours blue *and* green; we shall call the edges of *unique* colour *unicoloured*),

then we obtain a simple *planar* graph. Note that all "interior" vertices of this graph have degree 2, 3 or 4, and "loose ends" (vertices of degree 1) can only appear on the "boundary", i.e., they are

- either terminal points
- or initial blue points
- or green points on the forbidden line.

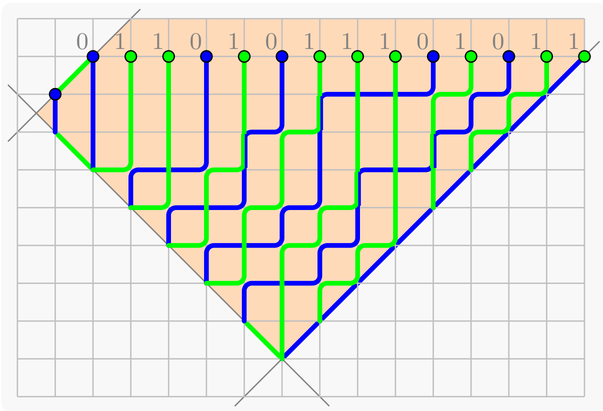
We can "close such loose ends" by introducing *artificial slanted edges* connecting neighbouring "loose ends", with the following "colouring rule":

- slanted edges on the line ($y = x$) have the same colour as their left point,
- slanted edges on the line ($y = -x - 2M - K + 2$) (this is the line containing the blue initial points) are coloured green,
- slanted edges on the forbidden line ($y = x - K$) are coloured blue;

see the following picture for an illustration, where we also added blue and green terminal points (shifted downwards, to limit the height of the picture) together with the entries in the corresponding 01–code: recall that, by definition,

- blue terminal points are reached *from below*,
- green terminal points are reached *from the left*

in the original (unfolded) situation (the rightmost isolated green point on the forbidden line indicates the 9–th green path, all of whose steps are completely "cut away") in this picture:



The only reason for introducing these *artificial slanted edges* is to avoid the tedious distinction between "partially open" and "closed areas", which now uniformly appear as *finite faces* in the *simple planar graph* thus obtained: We call it the *planar graph corresponding to* $\mathcal{F}$ and denote it by $\overline{\mathcal{F}}$

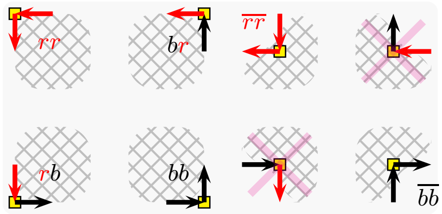
3.6.3 Faces of the planar graph corresponding to a folded overlay

It is obvious that $\overline{\mathcal{F}}$ is *connected*, and that the *boundary* ∂f of each finite face f of $\overline{\mathcal{F}}$ is a *closed curve* in $\mathbb{R}^2$, which we may traverse in counterclockwise orientation. But it is *not so obvious* that this closed curve must correspond to a *circle* in the graph–theoretical sense (i.e., passes through every point at most once).

When traversing (in counterclockwise orientation) the boundary ∂f of a finite face f of of some *general* planar subgraph G (without vertices of degree 1) of the lattice $\mathbb{Z}^2$, there are edges

- which are traversed rightwards or upwards: call those the *black edges*,
- which are traversed leftwards or downwards: call those the *red edges*.

Now observe that, in general, there can be precisely eight *types of kinks* in ∂f, as shown in the following picture (where the crosshatched areas indicate the finite face f whose boundary contains the kink):



But a moment's thought shows that two of these kinks are *impossible* if f is a finite face of the planar graph $\overline{\mathcal{F}}$ corresponding to some folded overlay $\mathcal{F}$ (these are "crossed out" in the pictures above), so there are only

- two types of kinks consisting of two red edges: denote them by rr and $\overline{rr}$,
- two types of kinks consisting of two black edges: denote them by bb and $\overline{bb}$
- and one type of kink where a red edge is followed by a black edge (denote this by rb,
- and one type of kink where a black edge is followed by a red edge (denote this by br),

which can occur in the boundary ∂f of some finite face f of $\overline{\mathcal{F}}$. Note that when traversing such boundary in counterclockwise orientation,

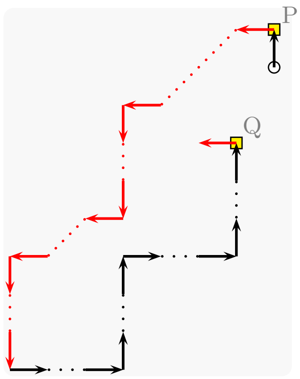
- rr may follow $\overline{rr}$, and vice versa; and rr may also be followed by rb,
- bb may follow $\overline{bb}$, and vice versa; and bb may also be followed by br,
- br *can only be followed* by rr,
- rb *can only be followed* by bb,

and there is no other possible succession of types of kinks.

Clearly, the boundary ∂f must contain at least

- one red path segment (consisting only of red edges)
- and one black path segment (consisting only of black edges)

(since otherwise ∂f would be just a *path* in $\mathbb{Z}^2$), and red and black segments must be "joined" by kinks br or rb. Now consider the maximal kink br (in lexicographic order) in ∂f: the following picture (where the maximal kink br is labeled P) makes clear that there cannot be *another* kink br in ∂f.



So the boundary ∂f of every finite face of $\overline{\mathcal{F}}$ consists *precisely* of

- one black segment (we shall call it the *right boundary* of f)
- and one red segment (we shall call it the *left boundary* of f).

3.6.4 Vertical strips

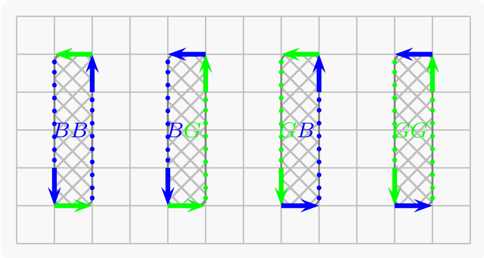
For our further considerations we will need a very simple special case: We call a finite face f of $\overline{\mathcal{F}}$ a *vertical strip* if its left and right boundaries both contain *only one* horizontal or slanted step, and we call the *vertical segment* of the left (or right, respectively) boundary of f the *left side* (or *right side*, respectively) of f.

It is easy to see that for any vertical strip f in $\overline{\mathcal{F}}$,

- the last (vertical) edge e_v of the left (or right) side of f and the horizontal edge e_h immediately following e_v must have *opposite* colours, whence *both* e_v and e_h must be *unicoloured*,

- and all edges of the left (or right) side of f must have the *same* colour as e_v (but some of them might have *both* colours);

see the following picture, which shows the four possible combinations of the colours of the left and right side of vertical strips, which we denote by BB, BG, GB and GG:



Note that the horizontal edges in the pictures above could also be *artificial slanted edges*, as explained above.

3.6.5 Adjacency of vertical strips

We say that a pair (f_l, f_r) of vertical strips is *adjacent* if the intersection

- of the right side of f_l
- with the left side of f_r

is *not empty*, but *does not* contain a horizontal edge, and we call f_l the left strip and f_r the right strip of the adjacent pair. The following pictures show all the *relative positions* ("modulo the swap of colours blue and green") of left and right strips of an adjacent pair (f_l, f_r), and make clear that not all of these relative positions are possible in the planar graph $\overline{\mathcal{F}}$ corresponding to some folded overlay $\mathcal{F}$: recall that according to the above considerations,

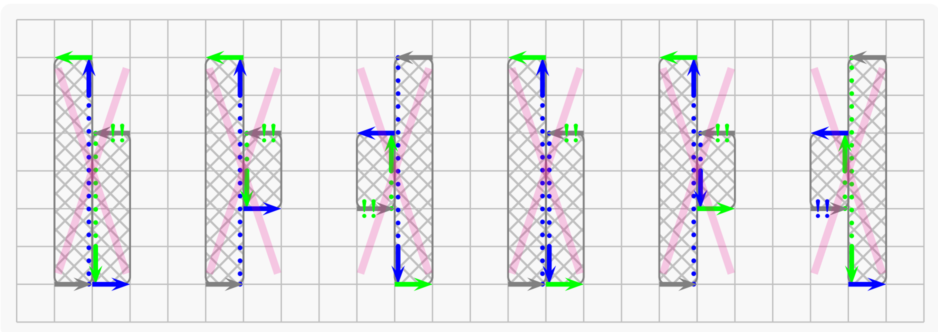
- *all* horizontal (or artificial slanted) edges are *necessarily unicoloured* edges,
- and
  - the upper horizontal (or artificial slanted) edge of the left strip f_l and the uppermost vertical edge of the right side of f_l
  - as well as the the lower horizontal (or artificial slanted) edge of the right strip f_r and the lowest vertical edge of the left side of f_r

  are both *unicoloured* and of *different* colours.

In the following pictures, unicoloured horizontal edges whose colour is not already determined (as just explained) are drawn in gray. However, in some situations, their colour is *implied* by the situation of the adjacency: this is shown by marking such edges with double exclamation marks of the respective colour in the following pictures.

3.6.6 Impossible adjacencies

A careful inspection reveals that the following situations of adjacencies are impossible (and therefore are crossed out in the pictures):

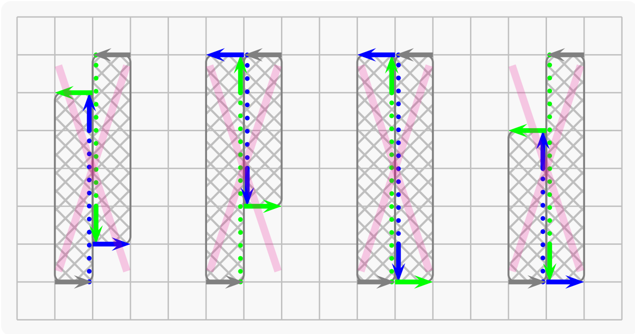


To see this, just note that in a folded overlay

- a point is never incident with two outgoing (or incoming) steps of the *same colour* as is the case in the first three of the above pictures,

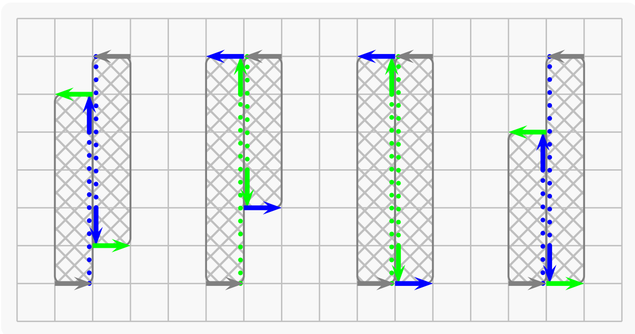
- and a path never starts (or ends) on the side of a vertical strip, as is the case in the last three of the above pictures.

Also, the following situations of adjacencies are impossible (note that the vertical sides are of *different* colour here):



3.6.7 Possible adjacencies: Ascending and descending

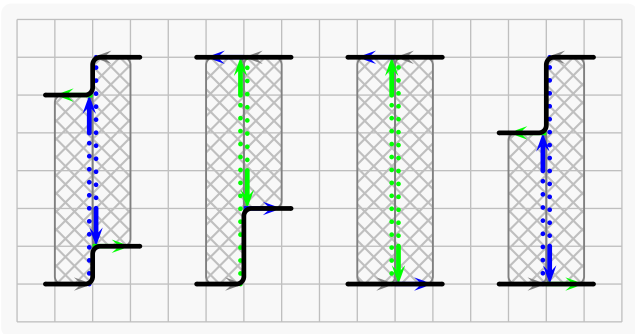
However, similar situations with vertical sides of the *same* colour *are* possible: we say that an adjacent pair in any of these relative positions is *ascending*.



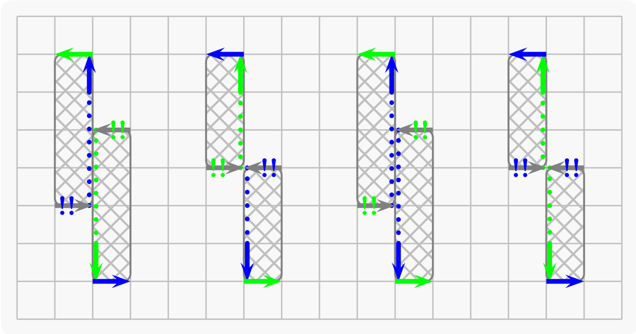
Note for later reference that

- the horizontal edges involved in an ascending adjacent pair of vertical strips
- together with their adjacent vertical edges

may be viewed as (segments of) nonintersecting lattice paths, see the following picture for an illustration, where these segments are drawn in black:



Finally, also the following situations of adjacencies are all possible: we say that an adjacent pair in any of the relative positions shown below is *descending*.



The following observation is an immediate consequence of the above considerations:

- For any *descending adjacent* pair, the left strip and the right strip must be of the *same type* BB, BG, GB or GG.
- If in an *ascending adjacent* pair the left strip is of type BB or GB, then the right strip must be of the type BB or BG.
- If in an *ascending adjacent* pair the left strip is of type BG or GG, then the right strip must be of the type GB or GG.

3.6.8 Rows of columns of vertical strips

Note that *if* every finite face of $\overline{\mathcal{F}}$ is a vertical strip, *then* we can *partition* the family of these faces in *maximal chains of descending adjacent vertical strips*, i.e.,

- sequences $(v_1, v_2, \dots, v_l)$ of vertical strips *of the same type*
- where (v_i, v_{i+1}) are *descending adjacent* (as explained above) for $i = 1, 2, \dots, l-1$,
- which cannot be "extended" to a longer sequence with the same properties.

We call such maximal chains *columns*.

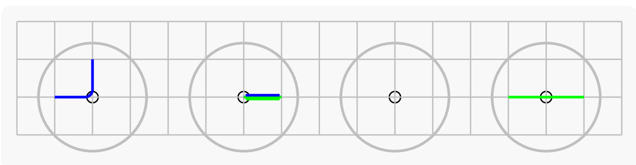
Note that these columns may be ordered such that (the vertical strips belonging to) two *consecutive* columns are *ascending* adjacent: we call such succession of ascending adjacent columns a *row of columns*.

3.6.9 Free kinks and other configurations

We consider the following four situations that might occur in the simple planar graph $\overline{\mathcal{F}}$ corresponding to some folded overlay $\mathcal{F}$:

- A *free kink* is a *unicoloured* path segment consisting of a horizontal step followed by a vertical step, whose "middle vertex" has degree 2 (i.e., is *not incident* to another edge)
- a *horizontal bicoloured edge* (as defined above),
- an *isolated vertex* is a point in the essential region that does not belong to any path,
- an *uncrossed double step* is a *unicoloured* path segment consisting of two horizontal steps.

The pictures below illustrate these simple notions:



3.6.10 Folded overlays down to some given level y_0

Let $\overline{\mathcal{F}}$ be the planar graph corresponding to some folded overlay $\mathcal{F}$. For some fixed $z \in \mathbb{Z}$, we may *remove* all vertices and edges which lie *strictly below* the horizontal line $(y = z)$ in both $\overline{\mathcal{F}}$: we denote by $\overline{\mathcal{F}_z}$ the result of this operation.

This operation might introduce "new loose ends" (vertices of degree 1) in $\overline{\mathcal{F}_z}$: As before, we "close" such "loose ends" by adding edges of appropriate colour (again, these artificial edges are only introduced "for convenience", in order to simplify the description). Note that the boundaries of all finite faces in $\overline{\mathcal{F}}_y$ again consist of a right segment and a left segment, as described in section 3.6.3

Lemma 3.1. *Let $\mathcal{F}$ be a folded overlay, and let $z \in \mathbb{Z}$. If $\overline{\mathcal{F}}_{z+1}$ does not contain a* free kink*, then $\overline{\mathcal{F}}_z$ does contain*

- *neither an isolated vertex*
- *nor a horizontal bicoloured edge*
- *nor an uncrossed double step.*

Proof. Consider the set

$$S = \left\{\zeta \in \mathbb{Z}\colon\ \overline{\mathcal{F}}_\zeta \text{ does contain a horizontal step}\right\} \subset \mathbb{Z}.$$

If S is empty, then $\overline{\mathcal{F}}$ clearly does contain neither a horizontal bicoloured edge nor an uncrossed double step, and since in this case the whole graph appears as a "curtain of vertical paths, pending from the terminal points without any gaps", there also is no isolated vertex: So the assertion is true in this case.

If $S \neq \emptyset$, then consider $z_0 = \max(S) \in \mathbb{Z}$.

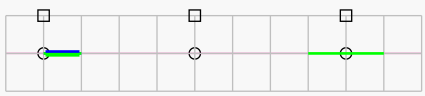
For $z \geq z_0$, $\overline{\mathcal{F}}_z$ might be empty: in this case the assertion is trivially true. Otherwise, $\overline{\mathcal{F}}_z$ consists of a "curtain of vertical paths" (as above), and some (*unicoloured*!) horizontal steps on level z_0 (only if $z = z_0$): so every vertex (x, y) in $\overline{\mathcal{F}}_z$ is *adjacent* to a vertical step, and this implies that $\overline{\mathcal{F}}_z$ does contain

- neither an isolated vertex
- nor a horizontal bicoloured edge b (since the vertical *upwards* step incident with the left point of this edge b would have the *same* colour as b)
- nor an uncrossed double step.

Now assume that the assertion is *false*, and let $z_1 < z_0$ be the maximal level for which this is the case, i.e., z_1 is the *maximal* integer such that $\overline{\mathcal{F}}_{z_1+1}$ does not contain a free kink, but $\overline{\mathcal{F}}_{z_1}$ contains

- either an isolated vertex
- or a horizontal bicoloured edge
- or an uncrossed double step.

But a close look at these situations reveals that this is impossible:



Note that in all three situations at level z_1 depicted above, the vertex directly above (indicated by a small box in the pictures) *cannot* be entered by a vertical step from below. But it is not an isolated vertex (since there are none on level $z_1 + 1$ by assumption), so

- it must be entered from the left by a horizontal step *of unique colour*, since there are no bicoloured edges on level $z_1 + 1$,
- and it must be left by a vertical step (of the same colour, of course), since there are no uncrossed horizontal steps at level $z_1 + 1$.

But this would give a free kink on level $z_1 + 1$, a contradiction. □

Lemma 3.2. *Let $\mathcal{F}$ be a folded overlay, and let $z \in \mathbb{Z}$. If $\overline{\mathcal{F}}_{z+1}$ does not contain a free kink, then every* finite *face of $\overline{\mathcal{F}_z}$ is a* vertical strip*.*

Proof. Let f be an arbitrary finite face of the planar simple graph $\overline{\mathcal{F}_z}$ and consider the left side l of f, which we traverse now in the *normal* direction (i.e., rightwards and upwards).

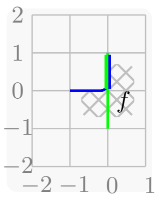
Recall that l starts with a *vertical step*. If l does not contain a horizontal step, it must end with a *single* slanted step.

By Lemma 3.1, l does contain

- neither a horizontal bicoloured edge
- nor an uncrossed double step:

So every horizontal step h belonging to l is *unicoloured*: if h is not the *last* step in l, then it must be followed by a *vertical edge* v in l, and since l starts with a *vertical* step, this horizontal step h must appear on some level $> z$.

Since $\overline{\mathcal{F}}_{z+1}$ does not contain a free kink, v must be *bicoloured*. But the segment (h, v) cannot belong to the left side of f, as the following picture shows:



So *only the last step* in l may be horizontal or slanted: in both cases, the right side of f is a path from the initial point (x, y) of l to its terminal point $(x + 1, y + h)$ (for some integer h) which starts with a horizontal or slanted step, and f is a vertical strip. □

Lemma 3.3. *If $\mathcal{F}$ is a folded* survivor*, then $\overline{\mathcal{F}}$ cannot contain free kinks* strictly above *the forbidden line (i.e., the "middle vertex" of any free kink* must lie on the forbidden line*).*

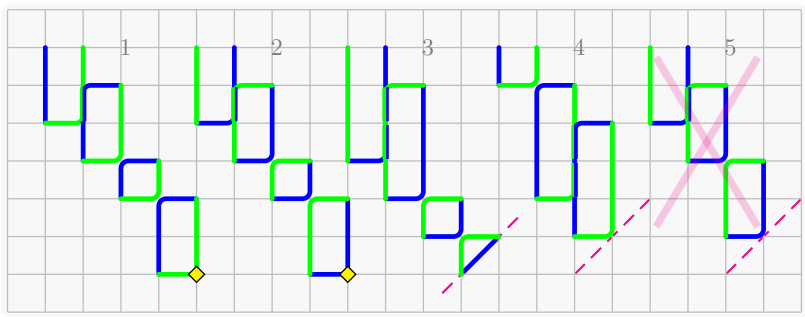
Proof. Consider again the set

$$S = \left\{\zeta \in \mathbb{Z}\colon\ \overline{\mathcal{F}}_\zeta \text{ does contain a horizontal step}\right\} \subset \mathbb{Z}.$$

If S is empty, then $\overline{\mathcal{F}}$ clearly does not contain a free kink: So the assertion is true in this case.

If $S \neq \emptyset$, then consider $z_0 = \max(S) \in \mathbb{Z}$: clearly, $\overline{\mathcal{F}}_{z_0+1}$ does not contain a free kink. Now assume that the assertion is false, and let $z \leq z_0$ be the maximal number such that $\overline{\mathcal{F}_z}$ does contain a free kink *above* the forbidden line.

By Lemma 3.2, every finite face of $\overline{\mathcal{F}_z}$ is a vertical strip, and we may arrange the family of these faces in a row of columns. Clearly, a free kink can only occur in columns of type BG or GB. The following picture shows the possibilities for such columns:



Picture 5 (counted from the left) shows an impossible chain (since blue paths never intersect the forbidden line) and is thus crossed out.

Picture 4 shows a column of type BG which ends with a (green) free kink on the forbidden line.

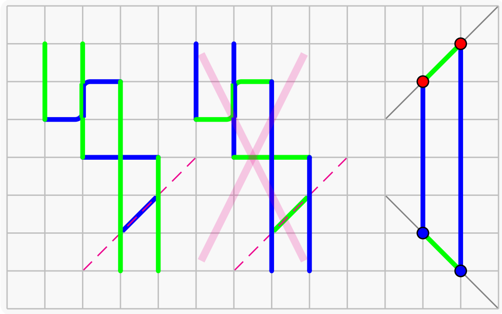
Picture 3 shows a column of type GB which does not end in a free kink.

Pictures 1 and 2 show free kinks strictly above the forbidden line (marked by small rhombi in the pictures): it is easy to see that these chains would give *involutive bicoloured connections* of two terminal points of different colour, which contradicts our assumption that $\mathcal{F}$ is a folded *survivor.* □

Lemma 3.4. *If $\mathcal{F}$ is a folded* survivor *which contains a column c of type GB, then there cannot be a column c' of type BB which lies* to the right of *c.*

In particular, any column which is ascending adjacent "to the right" of c must be of type *BG.*

Proof. If c would end above the forbidden line, then it would have a free kink above the forbidden line and thus yield an involutive bicoloured connection: So c ends on the forbidden line, and the same must be true for c': But it is impossible for a column of type BB to run into the forbidden line, see the middle picture in the following illustration:



(It is no problem for a column of type GG to run into the forbidden line, see the left picture above, and a column of type BB must end in blue initial points, see the right picture above.)

By the considerations in section 3.6.5, a column c' which is ascending adjacent to the right of c must be of type BG or BB, and we just made clear that type BB is impossible. □

Our considerations make clear that certain distributions of colours for the terminal points (which are not also initial points) are *impossible* for folded survivors:

Corollary 3.1. *If $\mathcal{F}$ is a folded* survivor*, then the sequence of terminal points which are not also initial points*

- *may start with a sequence of consecutive blue points,*
- *and every blue point not belonging to this starting sequence must be immediately followed by a green point.*

(In other words: after the first green terminal point, there can never follow a pair of consecutive blue terminal points.)

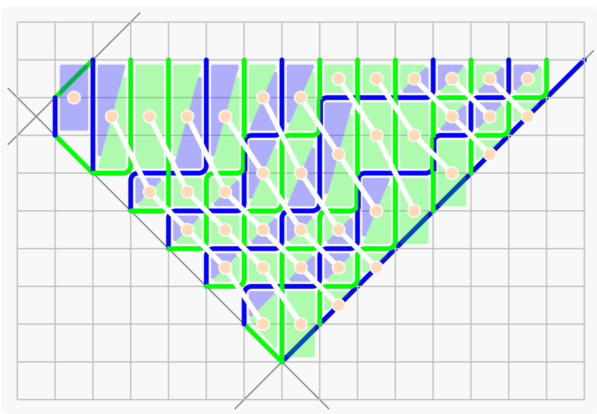
3.6.11 The mapping ξ from the lhs to the rhs

We showed that the planar graph $\overline{\mathcal{F}}$ corresponding to a *folded survivor* $\mathcal{F}$ necessarily appears as a row of columns (where consecutive columns are *ascending adjacent*): It is easy to see that every row of columns

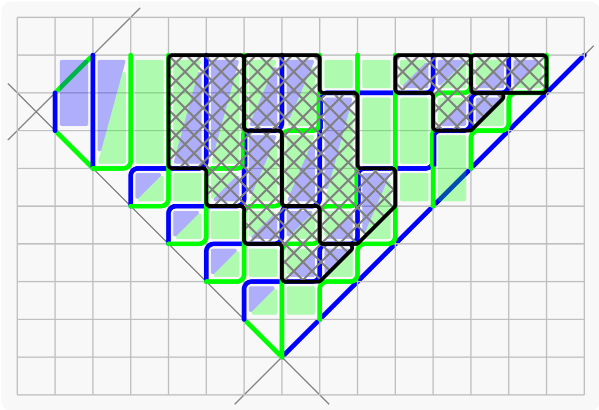
- corresponds bijectively to a planar graph $\overline{\mathcal{F}}$ of some folded overlay $\mathcal{F}$
- which does not have an involutive bicoloured connection (in fact, bicoloured connections correspond *precisely* to columns of type BG or GB in a folded survivor, and these are *not* involutive since they end on the forbidden line),

so we have the following *characterization*: a folded overlay $\mathcal{F}$ is a *folded survivor* if and only if its corresponding planar graph $\overline{\mathcal{F}}$ appears as a row of columns.

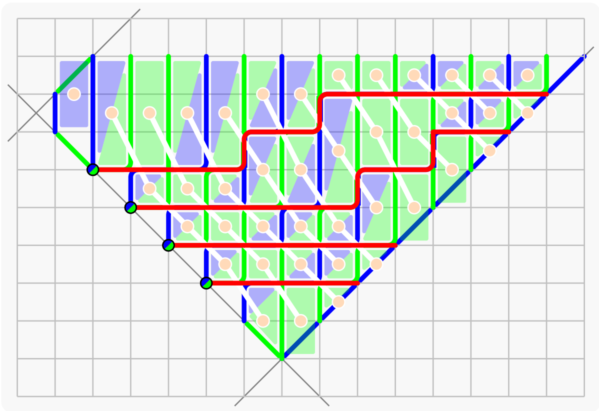
The mapping ξ from the **lhs** to the **rhs** now is best conceived by looking at a sequence of pictures: the first picture shows our running example, where the maximal descending chains of vertical strips of the same type are indicated by white paths:



In the next picture, chains of type GB with their mandatory "right neighbour" of type BG are indicated as crosshatched:



The next picture visualizes that the row of columns may be viewed as (segments of) lattice paths (drawn in red):

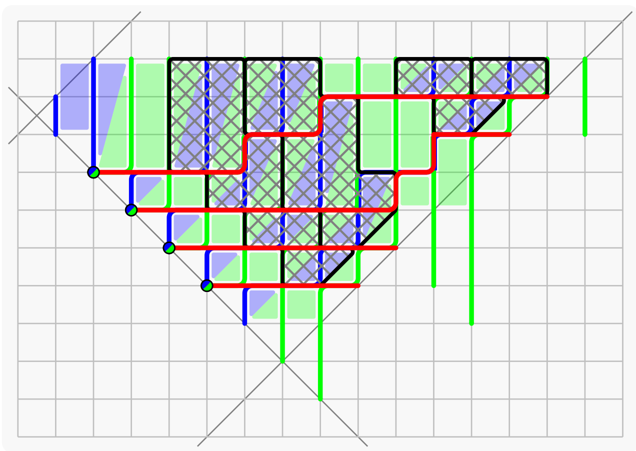


Note that these red paths

- are nonintersecting,
- and end on the forbidden line,
- and can never intersect the diagonal ($y = x$):

So now the diagonal plays the role of the forbidden line for these red path segments, and the forbidden line plays the role of the diagonal (in the sense that it contains the terminal points for these paths).

In the next picture, the artificial slanted edges are removed and enforced segments of green paths reaching the forbidden line *from below* are added:



This shows how the colour distribution of green or blue terminal points for a folded survivor *determines* the green points on the forbidden line which must be reached

- from below (by the green segments of paths just added),
- or from the left (by the red segments of paths).

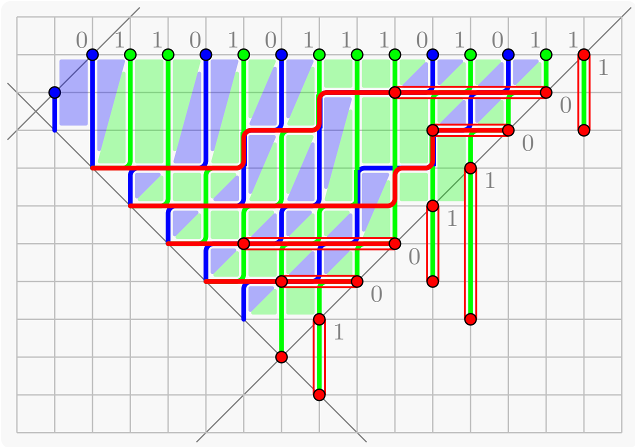
Altogether, this shows that each row of columns which corresponds to some folded survivor *uniquely determines* segments of nonintersecting lattice paths (the red paths in the above picture) connecting

- certain blue initial points
- with certain points on the forbidden line,

(as illustrated in the above pictures).

In the following picture, we added

- the 01–code for the **lhs** (corresponding to the colours of the blue and green terminal points)
- and the *enforced terminal segments* for the red and green paths ending on the forbidden line together with the corresponding *reflected* 01–code for these terminal segments:



This picture shows that

- the (original) diagonal ($y = x$) plays the role of the forbidden line
- and the (original) forbidden line plays the role of the diagonal (in the sense that it contains the terminal points)

for the collection of red and green segments of paths.

3.6.12 The effect of ξ on 01–codes and their signs

The above picture also shows

- the 01–code C for the **lhs** (implied by the blue and green terminal points, in our example: 01101011101011)
- *and* the *reflected* 01–code c corresponding to the black and red or green paths reaching the forbidden line, i.e.
  - 1 for points reached *from below*,
  - 0 for points reached *from the left*

  (in our example: 10011001).

(This *swap* of zeros and ones corresponds to the fact that we consider the *reflected* situation for the **rhs** here.)

The bijective correspondence between these two 01–codes is easy to see: For the i–th zero in the 01–code C corresponding to the green and blue terminal points, we count the number z_i of ones to its left (in our example, these numbers are $(0, 2, 3, 6, 7)$) and omit all zeros among these numbers (in our example: $(2, 3, 6, 7)$). Then these numbers are *precisely* the *indices* (starting with one) of the zeros in the *reflected* 01–code c. But this simple observation already proves that the mapping ξ *changes the sign* of (the permutations corresponding to) these 01–codes by a constant factor: Recall (see section 3.1) that the number of inversions of the permutation

- corresponding to some 01–code c for the **rhs** (which is *reflected* in our case, but we already accounted for the reflection by swapping ones and zeros) is equal to the sum of the indices of the zeros in c (in our example: the permutation corresponding to 10011001 has $2 + 3 + 6 + 7 = 18$ inversions), see (9),
- corresponding to some 01–code C for the **lhs** is

$$\binom{m + k - 1 + [\text{even}]}{2} + inv\,(C)\,,$$

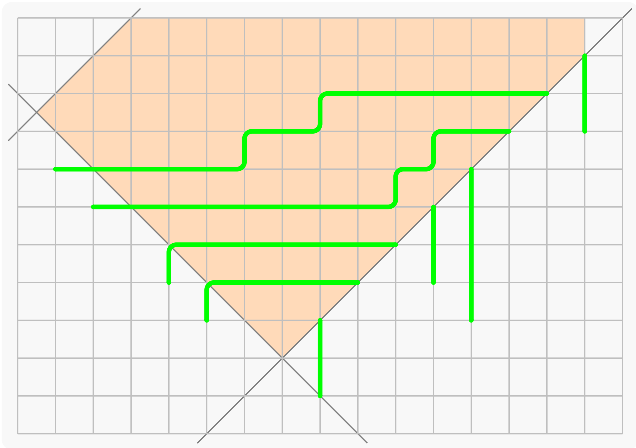
  see (8), and $inv\,(C)$ (by definition) equals the sum of the numbers $\sum_i z_i$ defined above (in our example: $0 + 2 + 3 + 6 + 7 = 18$).

3.6.13 Completion of the mapping ξ

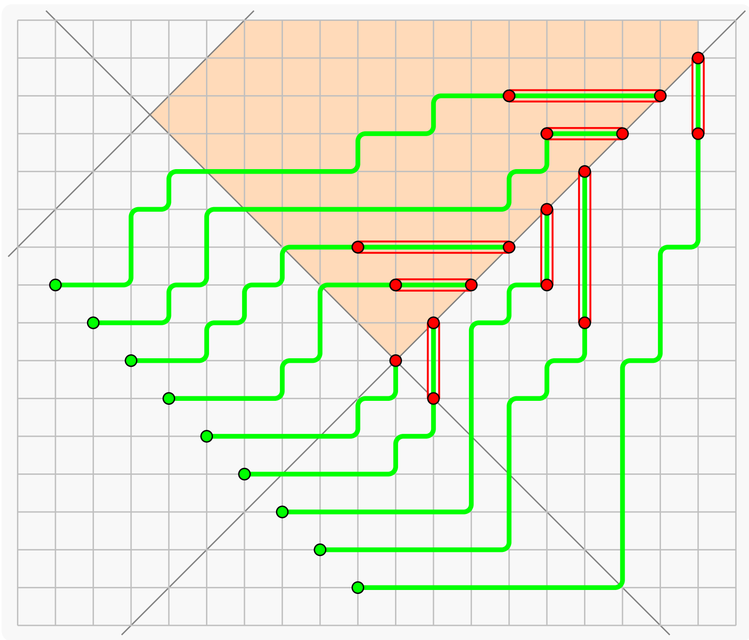
Now by simply

- omitting the folded survivor's overlay of green and blue paths
- and changing the colours of the red paths to green

in the above picture, we obtain the restriction to the essential region of some survivors from the *reflected* **rhs** (in the picture, also segments of green paths *outside* the essential regions are shown, in order to indicate that we view the essential region now as part of a bigger picture):



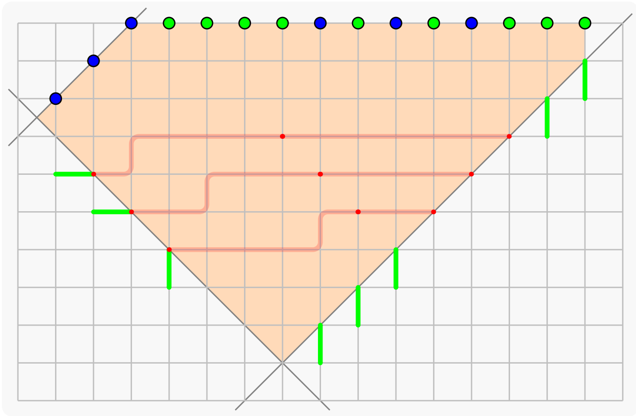
If we *restore* the segments of green paths *outside* the essential regions (see the first picture of our running example), then we obtain the following "normal" survivor:



It is easy to see that the above picture corresponds to the situation of the reflected **rhs** with 01–code 01100110, and it is obvious that the mapping ξ thus constructed is *injective*.

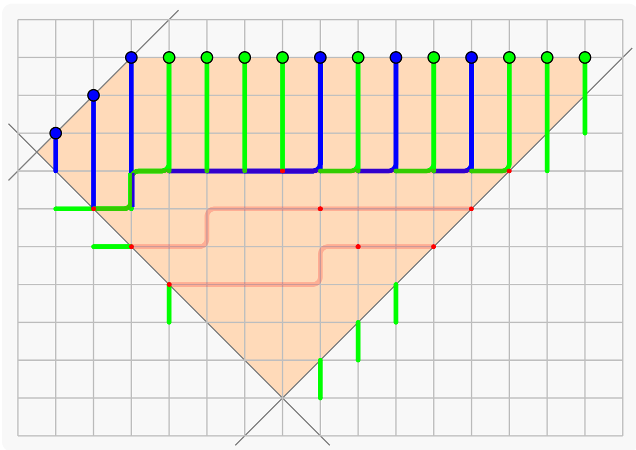
3.6.14 Surjectivity of the mapping ξ

To see that ξ is also *surjective*, we shall exhibit its inverse mapping: consider a "normal" survivor with 01–code c from the *reflected* **rhs**, and *interpret* the segments of its green paths inside the essential region as *red paths* of a *folded survivor* with code C corresponding to c (as described above). Then a *unique* row of columns (and thus a unique *folded survivor*) is obtained "step by step", starting at the diagonal ($y = x$), as illustrated in the following example:

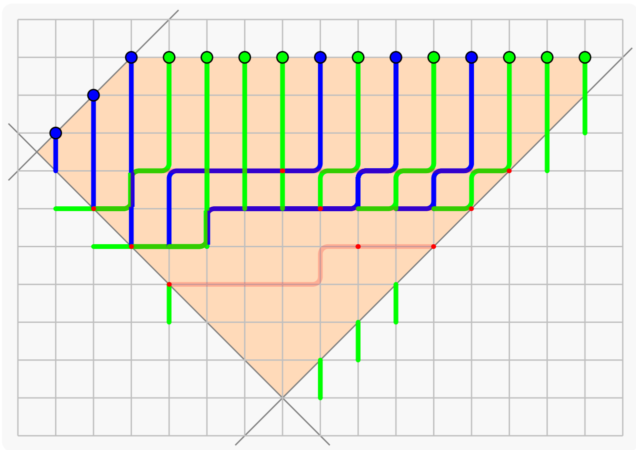


The picture shows a section of a ("normal") survivor for the *reflected* **rhs** with (reflected) 01–code 00011100, where green path segments in the essential region are coloured red. According to the considerations in section 3.6.12, the colouring of terminal points for the corresponding folded survivor is uniquely determined by this 01–code (since the picture is cut at level 6, the terminal blue and green points *above* level 6 are "shifted down").

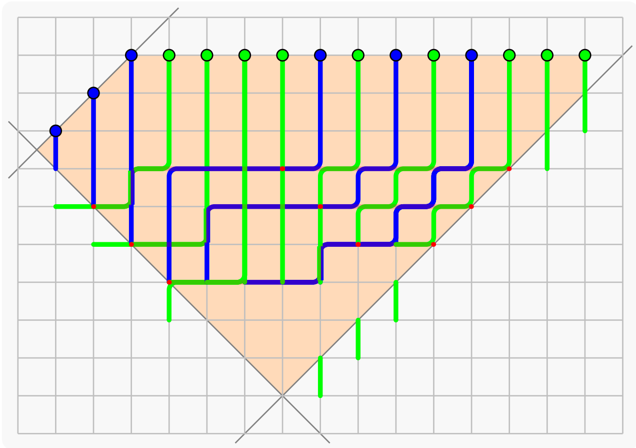
If we draw the row of columns "pending from the blue and green terminal points" down to the uppermost red path, then we obtain the following picture:



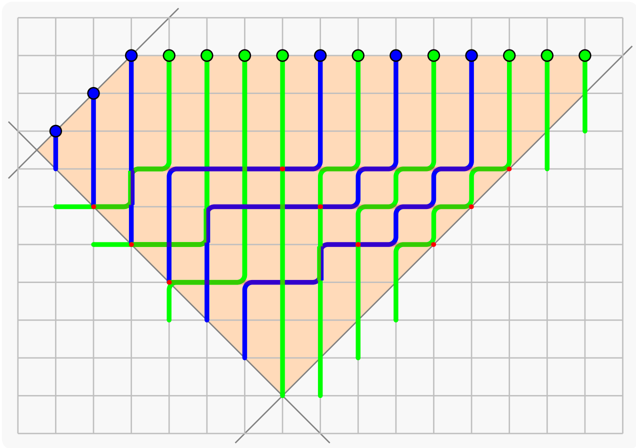
Now recall that in a folded survivor, every descending chain must contain vertical strips *of the same type*: so the second step of completing the descending chains "rightwards–downwards" yields the following situation:



The third step yields:



And the fourth step finishes the construction, and yields a row of columns which we may view as the restriction to the essential region of a *folded survivor* :



So the mapping ξ is indeed a bijection that changes the sign of survivors by the constant factor

$$(-1)^{\binom{\frac{m+k-1+[\text{even}]}{2}}{}}$$

This finishes our proof of (3) and (5).

References

[1] D. André, *Solution directe du probleme résolu par M. Bertrand*, C. R. Acad. Sci. Paris 105(436) (1887), 7.

[2] G. Andrews and J. Wimp, *Some q-orthogonal polynomials and related Hankel determinants*, Rocky Mountain J. Math. 32(2) (2002), 429–442.

[3] J. Cigler, *Shifted Hankel determinants of Catalan numbers and related results II: Backward shifts*, arXiv:2306.07733.

[4] J. Cigler, *Some experimental observations about Hankel determinants of convolution powers of Catalan numbers*, arXiv:2308.07642v2.

[5] J. Cigler, *A special class of Hankel determinants*, arXiv:1302.4235.

[6] J. Cigler, *Hankel determinants of convolution powers of Catalan numbers revisited*, arXiv:2403.11244.

[7] I. M. Gessel and X. Viennot, *Determinants, paths, and plane partitions*, preprint, 1989.

[8] B. Lindström, *On the vector representation of induced matroids*, Bull. London Math. Soc. 5 (1973), 85–90.