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The Order of the (123, 132)-Avoiding Stack Sort

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Abstract: Let s be West's deterministic stack-sorting map. A well-known result is that every length n permutation is sorted with $n-1$ iterations of s. In 2020, Defant introduced the notion of *highly-sorted* permutations—permutations obtained by applying slightly less than $n-1$ iterations of s. In 2023, Choi & Choi extended this notion to generalized stack-sorting maps s_σ, where we relax the condition of becoming sorted to the analogous condition of becoming periodic with respect to s_σ. While periodicity seems counterintuitive in the context of sorting, it arises naturally in these restrictive forms, where the map is studied as an operator with properties rather than a means to "sort" permutations in the usual sense.

In this work, we introduce the notion of *minimally-sorted* permutations as an antithesis to Defant's highly-sorted permutations, and show that the order of the $(123, 132)$-avoiding stack-sorting map is $2\lfloor \frac{n-1}{2} \rfloor$.

Keywords: Periodicity; Permutations; Pattern-avoidance; Stack-sorting
2020 Mathematics Subject Classification: 05A05

1. Introduction

Knuth's *Art of Computer Programming* [17] first introduced the *stack-sorting machine*, in which an input sequence is sorted with a single external stack structure. The elements of the sequence are passed left-to-right through the machine, with two possible operations at every state: *push*, moving the next input element onto the stack, and *pop*, removing the top element from the stack and appending it to the output.

In 1990, West [21] introduced a deterministic version of Knuth's stack-sorting machine as the *stack-sorting map* s, insisting that the stack must always increase from top to bottom and employ a *right-greedy* process: the *push* operation is chosen whenever possible. Since then, various studies have been motivated by Knuth's original machine and West's deterministic s, including pop-stack-sorting [1, 2, 14, 18, 19], stack-sorting Coxeter groups [14, 15], sigma-tau machines [3–5], and stack-sorting of set-partitions [16, 23].

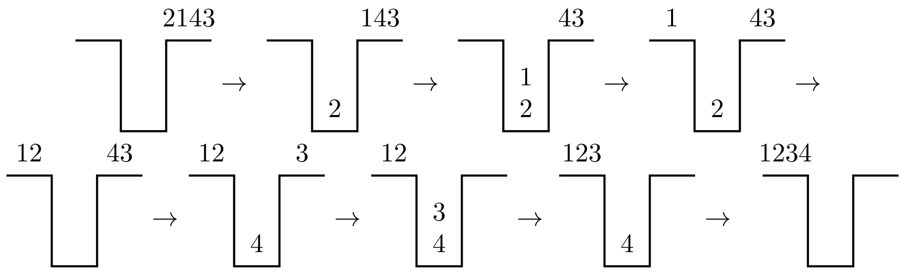

Figure 1: West's deterministic stack-sorting map s on $\pi = 2143$.

In his dissertation, West [21] proved that $s^{n-1}(S_n)$ contains only the identity permutation, justifying repeated applications of s as a correct and terminating sorting algorithm. A natural direction of study, then, is the characterization of *t-stack-sortable* permutations—permutations π such that $s^t(\pi)$ is sorted—for general $t \leq n-1$. Knuth [17] answered the question for $t = 1$, showing that π is 1-stack-sortable if and only if π avoids subsequences of the pattern 231, enumerating the number of such permutations of length n to be $\frac{1}{n+1}\binom{2n}{n}$, the n^{th} Catalan number. In 1990, West [21] characterized the 2-stack-sortable permutations, proving that π is 2-stack-sortable if and only if π avoids subsequences of the pattern 2341 and the barred pattern $3\bar{5}241$. He also conjectured that the number of such permutations of length n is $\frac{2}{(n+1)(2n+1)}\binom{3n}{n}$, which was proven by Zeilberger [24] two years later. West [21, 22] then searched for a polynomial $P(n)$ such that 3-stack-sortable permutations could be enumerated by $\frac{1}{P(n)}\binom{4n}{n}$, but was unsuccessful for $\deg(P(n)) < 7$. In 2012, Úlfarsson [20] characterized 3-stack-sortable permutations with "decorated patterns," but only in 2021, did Defant [13] discover a polynomial-time algorithm to enumerate 3-stack-sortable permutations.

In 2020, Defant [11] first defined *t-sorted permutations*, which he considered to be the duals of the t-stack-sortable permutations [12]—permutations in the image of $s^t(S_n)$, a generalization of Bousquet-Mélou's definition [6] of *sorted*. Defant then defined a permutation $\pi \in S_n$ to be *highly-sorted* if π is t-sorted for some t close to n, proving that a t-sorted permutation can contain at most $\lfloor \frac{n-t}{2} \rfloor$ descents [12].

The classical stack-sorting map s has since been generalized to s_σ [7] for permutations σ, where instead of insisting that the stack increases, we insist that the stack avoids top-to-bottom subsequences of the pattern σ. In 2021, Berlow [5] introduced the family of maps s_T, where the stack must avoid top-to-bottom subsequences of every pattern in set T (see Figure 2). In 2023, Choi and Choi [8] generalized Defant's notion of highly-sorted permutations, defining π to be *highly-sorted* with respect to s_σ if π is in the image of s_σ^t for some t close to $\mathrm{ord}_{s_\sigma}(S_n)$, where $\mathrm{ord}_{s_\sigma}(P)$ is the smallest integer k such that every element in $s_\sigma^k(P)$ is periodic under s_σ. We straightforwardly extend this definition to generalized maps s_T.

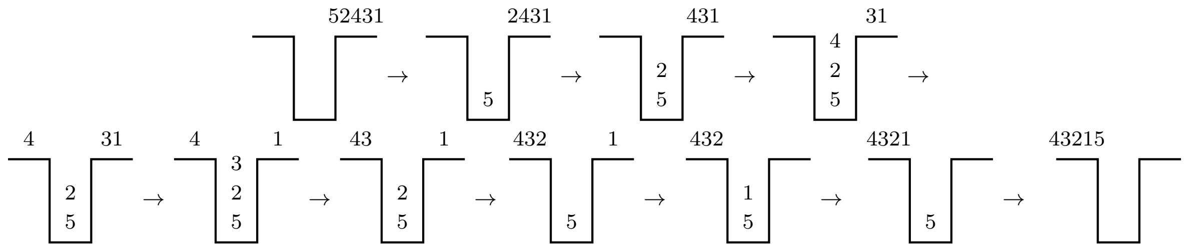

Figure 2: The generalized stack-sorting map $s_{123,132}$ on $\pi = 52431$.

Recently, Choi, Gan, Li, and Zhu [9] studied set partitions that require the maximum number of sorts through an *aba*-avoiding stack. Similarly, we define a permutation π to be *minimally-sorted* with respect to s_T if $\mathrm{ord}_{s_T}(S_n) = \mathrm{ord}_{s_T}(\{\pi\})$, antithetical to Defant's notion of highly-sorted permutations. At the end of this work, we present two conjectures on $\mathfrak{M}_n$, the minimally-sorted permutations with respect to $s_{123,132}$.

In 2021, Berlow [5] studied the periodic points of $s_{123,132}$. She defined a permutation π of length n to be *half-decreasing* if the subsequence $\pi_{n-1}\pi_{n-3}\cdots\pi_{(3-(n \bmod 2))}$ is the identity of length $\lfloor \frac{n-1}{2} \rfloor$. In particular, being order-isomorphic to the identity is not sufficient.

Theorem 1.1 (Berlow [5])**.** *A permutation π is periodic under $s_{123,132}$ if and only if π is half-decreasing.*

Our main result is that we find the exact value of $\mathrm{ord}_{s_{123,132}}(S_n)$, extending Berlow's work on periodic permutations. An analogous result for $s_{321,312}$ follows directly from Theorem 1.2.

Theorem 1.2. *For all positive integers n, we have* $\mathrm{ord}_{s_{123,132}}(S_n) = 2\lfloor \frac{n-1}{2} \rfloor$.

2. Preliminaries

We say that $a \in A$ is *periodic* under $f : A \to B$ if there exists a positive integer k such that $f^k(a) = a$. For some ordered set T, we use T_i to denote the ith element of T.

Let $[n]$ denote $\{1, 2, \cdots, n\}$ for positive integers n. A *permutation*, written $\pi = \pi_1\pi_2\cdots\pi_n$, is an ordering of distinct positive integers with *length* $\mathrm{len}(\pi) = n$. We say that $\pi_1, \pi_2, \cdots, \pi_n$ are the *elements* of π, and use $\pi_{[i:j]}$ to denote the subpermutation $\pi_i, \pi_{i+1}, \cdots, \pi_j$. We define $\mathrm{ind}_\pi(i)$, the *index of* i in π, to be j, where $\pi_j = i$. Let S_n be the set of permutations with elements $[n]$. The *reduction* of a permutation π (equivalently, the *standardization* [12]), is the unique permutation $\mathrm{red}(\pi) \in S_n$ such that $\mathrm{red}(\pi)_i = j$ for $1 \le i \le n$, where π_i is the jth smallest number in $\{\pi_1, \pi_2, \cdots, \pi_n\}$. Two permutations π and σ are *order-isomorphic* if $\mathrm{red}(\pi) = \mathrm{red}(\sigma)$, and we write $\pi \cong \sigma$. For instance, $\pi = 57816$ and $\sigma = 48917$ are order-isomorphic, since $\mathrm{red}(\pi) = \mathrm{red}(\sigma) = 24513$. Given permutations π and σ, we say that π *contains* the pattern σ if there exists a sequence of positive integers $a_1 < a_2 < \cdots < a_k$ such that $\pi' = \pi_{a_1}\pi_{a_2}\cdots\pi_{a_k} \cong \sigma$. Otherwise, we say that π *avoids* σ (equivalently, is σ-*avoiding*). For instance, $\pi = 24513$ contains $\sigma = 132$ since $\pi_1\pi_3\pi_5 = 253 \cong \sigma$, but avoids $\tau = 321$. We use $\pi \cdot \tau$ to denote the *concatenation* of π and τ, and let $\mathrm{rev}(\pi)$ denote the *reverse* of π, namely $\pi_n\pi_{n-1}\cdots\pi_1$.

Next, an element π_i of $\pi \in S_n$ is *small* if $\pi_i \le \lfloor \frac{n-1}{2} \rfloor$. An element π_i is a *left-to-right minimum* (equivalently, *ltr-min*) of π if $\pi_i = \min(\pi_{[1:i]})$. Additionally, we say that π_i is a *valley* if π_i is a ltr-min, π_{i+1} (if $i+1 \le n$) is not a ltr-min, and π_{i+2} (if $i+2 \le n$) is a ltr-min. A consecutive subsequence of elements $\pi_{[i:i+j]}$ is a *valley-block* $\overline{v}$ if π_{i+j} is a valley and $\mathrm{red}(\pi_{[1:i+j]})_{[i:i+j]} = j+1, j, \cdots, 1$. We say that the *valley-boundary* of $\pi \in S_n$, denoted $\mathfrak{B}(\pi)$, is the smallest index i such that $\pi_{[i:n]} = \overline{v_1}\pi_{a_1}\overline{v_2}\pi_{a_2}\cdots\overline{v_j}\pi_{a_j}$ for valleys $\overline{v_1}, \cdots, \overline{v_j}$ and elements $\pi_{a_1}, \cdots, \pi_{a_j}$, and set $\mathfrak{B}(\pi) = n$ if no such index exists. The *valley-region* of π is $\pi_{[\mathfrak{B}(\pi):n]}$. For instance, given $\pi = (11, 12, 7, 5, 8, 4, 3, 6, 2, 9, 1, 10)$, the elements $1, 2, 3$, and 5 are valleys and the sets $(7,5), (4,3), (2), (1)$ form 4 valley-blocks in π. Finally, $\mathfrak{B}(\pi) = 3$, since $\pi_{[3:n]} = \overline{7,5}, 8, \overline{4,3}, 6, \overline{2}, 9, \overline{1}, 10$.

We conclude by noting that permutation indices will be considered modulo n for the duration of this paper. In particular, let $\pi_i := \pi_j$, where j is the unique element of $[n]$ such that $i \equiv j \pmod{n}$.

3. Proof of the Main Result

We preface this section with two propositions, immediate from the preliminaries.

Proposition 3.1. *Given $\sigma, \tau \in S_3$, it holds that $(s_{\sigma,\tau}(\pi))_n = \pi_1$ for all $\pi \in S_n$ and $n \ge 1$.*

Proposition 3.2. *Let $\overline{v_1}, \cdots, \overline{v_i}$ be the valley-blocks of π from left to right, and let $\mathrm{len}(v_j) = l_j$ for all j. Then, the permutation $\overline{v_1} \cdot \overline{v_2} \cdot \cdots \cdot \overline{v_i}$ is the reverse of the identity of length $\sum l_i$.*

We now begin the proof of Theorem 1.2 with several auxiliary lemmas that demonstrate the monovariant movement of valley-blocks under $s_{123,132}$.

Lemma 3.1. *For any $\pi \in S_n$ and ltr-min π_i with $i > 1$, let $j \le n$ be the largest index such that $\pi_i = \min(\pi_{[1:j]})$. It holds that $s_{123,132}(\pi)_{j-1} = \pi_i$.*

Proof. Since π_i is a ltr-min, just before π_i enters the stack, π_1 must be the only element in the stack. After the elements $\pi_{[i+1:j]}$ have all entered the stack, π_i and π_1 necessarily remain in the stack since $\pi_{i+1}, \cdots, \pi_j > \pi_i$. Additionally, since $\pi_{j+1} < \pi_i$, just before π_{j+1} enters the stack, π_j must exit the stack. At this moment, the $j-1$ elements $\pi_2, \pi_3, \cdots, \pi_j$ have been the only elements to exit the stack, with π_i being the last, so $s_{123,132}(\pi)_{j-1} = \pi_i$. □

Lemma 3.2. *Given a valley-block $\bar{v} = \pi_{[i:i+j]}$ of π, we have $s_{123,132}(\pi)_{i+j} = \pi_{i+j}$ and $s_{123,132}(\pi)_{k-1} = \pi_k$ for $i \le k < i+j$.*

Proof. Just before π_i enters the stack, π_1 must be the only element in the stack. Since $\overline{v}$ consists of the $j+1$ smallest elements of $\pi_{[1:i+j]}$ in descending-order, just before any element of $\overline{v}$ enters the stack, the previous element must exit. Hence, $k-2$ elements exit before π_k for $i \le k < i+j$, and thus $s_{123,132}(\pi)_{k-1} = \pi_k$. Finally, by Lemma 3.1, π_{i+j} is a fixed point. □

Next, we show that $s_{123,132}$ preserves the elements in the valley-region of π.

Lemma 3.3. *Suppose $\pi_{[i:j]}$ and $\pi_{[j+2:k]}$ are two valley-blocks of π. Then, $s_{123,132}(\pi)_{j-1} = \pi_{j+1}$.*

Proof. Right before π_j enters the stack, the only element remaining must be π_1. Now, since $\pi_{j+1} > \pi_j$, the stack will read $\pi_{j+1}\pi_j\pi_1$ top to bottom just after π_{j+1} enters. Finally, since π_{j+2} is also a ltr-min, just before it enters, π_{j+1} and π_j must have left the stack. Hence, every element in $\pi_{[1:j]}$ exits the stack before π_{j+1} except π_1 and π_j, yielding $s(\pi)_{j-1} = \pi_{j+1}$. □

Lemma 3.4. *If π_i is in the valley-region of π, then π_i is also in the valley-region of $s_{123,132}(\pi)$.*

Proof. Let $\pi_{[\mathfrak{B}(\pi):n]} = \overline{v_1}\pi_{a_1}\overline{v_2}\cdots\overline{v_j}\pi_{a_j}$, the valley-region of π, and let $\mathrm{len}(\overline{v_i}) = l_i$ for $1 \le i \le j$. Then, by Lemma 3.1 and Lemma 3.2, we have that $s_{123,132}(\pi)$ ends with the suffix

$$(v_{1[1:l_1-1]}) \cdot \pi_{b_1} \cdot (v_{1[l_1]} \cdot v_{2[1:l_2-1]}) \cdot \pi_{b_2} \cdot (v_{2[l_2]} \cdot v_{3[1:l_3-1]}) \cdots (v_{j-1}{}_{[l_{j-1}]} \cdot v_{j[1]}) \cdot \pi_{b_{j-1}} \cdot (v_{j[l_j]}) \cdot \pi_{b_j}$$

for some elements $\pi_{b_1}, \pi_{b_2}, \cdots, \pi_{b_j}$. By Proposition 3.2, this suffix is of the form $\overline{w_1}\pi_{b_{c_1}}\cdots\overline{w_k}\pi_{b_{c_k}}$, where $\pi_{b_{c_1}}, \cdots, \pi_{b_{c_k}}$ are the elements of $\{\pi_{b_1}, \cdots, \pi_{b_j}\}$ that are not ltr-mins. Hence, this suffix is fully contained in the valley-region of $s_{123,132}(\pi)$. However, it also contains all the elements in valley-blocks in $\pi_{[\mathfrak{B}(\pi):n]}$, and all the elements in between valley-blocks in $\pi_{[\mathfrak{B}(\pi):n]}$ by Lemma 3.3, which fully encompass all of elements in the valley-block, finishing the proof. □

Lemma 3.5. *Let $\pi_i = \min(\pi_{[1:\mathfrak{B}(\pi)-1]})$. If π_i is small, then π_i is in the valley-region of $s_{123,132}(\pi)$.*

Proof. If $i = 1$, then the claim follows from Proposition 3.1. Otherwise, just before π_i enters the stack, π_1 must be the only element remaining in the stack, since π_i is a ltr-minimum. Then, after $\pi_{i+1}, \cdots, \pi_{\mathfrak{B}(\pi)-1}$ have all entered the stack, π_i will remain in the stack. However, when $\pi_{\mathfrak{B}(\pi)}$ enters the stack, π_i will necessarily leave, since $\pi_{\mathfrak{B}}$ is part of a valley-block to the right of π_i, so $\pi_{\mathfrak{B}(\pi)} < \pi_i$. Thus, since every other element in $\pi_1, \cdots, \pi_{\mathfrak{B}(\pi)-1}$ was popped out before π_i, except for π_1, we have $s(\pi)_{\mathfrak{B}(\pi)-2} = \pi_i$. However, since $\pi_i = \min(\pi_1, \cdots, \pi_{\mathfrak{B}(\pi)-1})$, the proof of Lemma 3.4 shows that π_i is in the valley-region of $s_{123,132}(\pi)$. □

By Lemma 3.4, elements never leave the valley-region, and by Lemma 3.5, a small element is always added to the valley-region every iteration, implying the following result.

Corollary 3.1. *For any $\pi \in s_{123,132}^{\lfloor \frac{n-1}{2} \rfloor}(S_n)$, it holds that $i \geq \mathfrak{B}(\pi)$ for all small elements π_i.*

Corollary 3.1 gives a characterization of the $\lfloor \frac{n-1}{2} \rfloor$-sorted permutations under a $s_{123,132}$ map. We continue by showing that these permutations become periodic with at most $\lfloor \frac{n-1}{2} \rfloor$ further passes.

Lemma 3.6. *For $\pi \in S_n$ and small element i, if $\pi_{n-2i+2} = i$ and i is in the valley-region of π, then $s_{123,132}(\pi)_{n-2i+1} = i$.*

Proof. Suppose for the sake of contradiction that π_{n-2i+2} is directly in between two valley-blocks, so that $\pi_{[j:n-2i+1]}$ is a valley-block for some $j \leq n-2i$. By definition, π_{n-2i+1} is a valley, and by Lemma 3.1, $s^k(\pi)_{n-2i+1} = \pi_{n-2i+1}$ for all k. But this contradicts Theorem 1.1, since we have $s^k(\pi)_{n-2i+1} \neq \pi_{n-2i+2} = i$. Now, suppose that π_{n-2i+2} is itself a valley. This similarly contradicts Theorem 1.1, since we have $s^k(\pi)_{n-2i+2} = \pi_{n-2i+2}$ for all k by Lemma 3.1.

Since π_{n-2i+2} is in the valley-region of π, the only remaining possibility is that π_{n-2i+2} is part of a valley-block but not a valley. Hence, by Lemma 3.4, we have $s_{123,132}(\pi)_{n-2i+1} = i$, as desired. □

Lemma 3.7. *For some positive integer $i \leq \lfloor \frac{n-1}{2} \rfloor$ and $\pi \in S_n$, let $\sigma = s_{123,132}^{i+\lfloor \frac{n-1}{2} \rfloor}(\pi)$. Then, the permutation $\sigma_{n-1}\sigma_{n-3}\cdots\sigma_{n-2i+1}$ is the identity of length i.*

Proof. We induct on i. The base case $i = 1$ is immediate—in particular, $s_{123,132}^{\lfloor \frac{n-1}{2} \rfloor}(\pi)_{n-1} = 1$, which becomes a fixed element by Lemma 3.1, since otherwise $\mathfrak{B}(\pi) = n$ which contradicts Corollary 3.1.

Now suppose that for some $1 < j \leq \lfloor \frac{n-1}{2} \rfloor$, it holds that for all π and $i < j$, the permutation $\sigma_{n-1}\sigma_{n-3}\cdots\sigma_{n-2i+1}$ is the identity of length i, where $\sigma = s_{123,132}^{i+\lfloor \frac{n-1}{2} \rfloor}(\pi)$. First, we note that by Lemma 3.2 and Lemma 3.3, if an element π_i is in the valley-region of π, we have $s_{123,132}(\pi)_x = \pi_i$ for some $x \in \{i-2, i-1, i\}$. Next, consider some $\pi \in S_n$, and let

$$\mathfrak{Z} = \left\{ \operatorname{ind}_{s^k_{123,132}(\pi)}(j) \;\middle|\; \left\lfloor \frac{n-1}{2} \right\rfloor \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor + j \right\}.$$

By Lemma 3.1, if $\mathfrak{Z}_l - \mathfrak{Z}_{l+1} = 0$ for some $l \leq j$, we must have $s_{123,132}^{\lfloor \frac{n-1}{2} \rfloor + l}(\pi)_{n-2j+1} = j$, or equivalently $\mathfrak{Z}_l \leq n-2j+1$. Similarly, if $\mathfrak{Z}_l - \mathfrak{Z}_{l+1} = 1$, we have by Lemma 3.2 and Lemma 3.3 that the element j must be in a valley-block (but not a valley) of $s_{123,132}^{\lfloor \frac{n-1}{2} \rfloor + l-1}(\pi)$, so by the inductive hypothesis, $\mathfrak{Z}_l \leq n-j-l+2$. Otherwise, $\mathfrak{Z}_l - \mathfrak{Z}_{l+1} = 2$, so we conclude recursively that $\mathfrak{Z}_{j+1} \leq n-2j+1$. But combining Lemma 3.1, Lemma 3.6, and the fact that $\mathfrak{Z}_l - \mathfrak{Z}_{l+1} \leq 2$ for all l, we derive $\mathfrak{Z}_{j+1} = n-2j+1$, or equivalently $s_{123,132}^{\lfloor \frac{n-1}{2} \rfloor + j}(\pi)_{n-2j+1} = j$. Hence, for all π and $i < j+1$, the permutation $\sigma_{n-1}\sigma_{n-3}\cdots\sigma_{n-2i+1}$ is the identity of length i, where $\sigma = s_{123,132}^{i+\lfloor \frac{n-1}{2} \rfloor}(\pi)$, completing the induction.

□

In particular, any $\pi \in s_{123,132}^{2\lfloor \frac{n-1}{2} \rfloor}(S_n)$ is half-decreasing, which implies the following by Theorem 1.1.

Corollary 3.2. *For all positive integers n, we have $\operatorname{ord}_{s_{123,132}}(S_n) \leq 2\lfloor \frac{n-1}{2} \rfloor$.*

Finally, we present a family of minimally-sorted permutations to show that precisely $2\lfloor \frac{n-1}{2} \rfloor$ iterations are required to sort all of S_n. Define

$$\gamma_n = \left(\frac{n+1}{2}, 2, 3, \cdots, \frac{n-1}{2}, \frac{n+3}{2}, \cdots, n-2, 1, n-1, n \right)$$

for odd $n \geq 5$ and $\gamma_n = \gamma_{n-1} \cdot n$ for even $n \geq 6$. It is immediate that $\operatorname{ord}_{s_{123,132}}([n]) = 2\lfloor \frac{n-1}{2} \rfloor$ for $n \leq 4$. Hence, we consider $n \geq 5$. Let δ_n denote the permutation $\operatorname{rev}((\gamma_n)_{[2:n-3]})$ when n is odd and $\operatorname{rev}((\gamma_n)_{[2:n-4]})$ when n is even.

n	γ_n
5	$(3,2,1,4,5)$
6	$(3,2,1,4,5,6)$
7	$(4,2,3,5,1,6,7)$
8	$(4,2,3,5,1,6,7,8)$
9	$(5,2,3,4,6,7,1,8,9)$

Table 1: The first few γ_n for $n \geq 5$.

Lemma 3.8. *For positive integers $n \geq 5$ and $k \leq \lfloor \frac{n-1}{2} \rfloor$, we have $s^k_{123,132}(\gamma_n)_{[1:n-2k-2]} = (\delta_n)_{[k:n-k-3]}$ for odd n and $s^k_{123,132}(\gamma_n)_{[1:n-2k-3]} = (\delta_n)_{[k:n-k-4]}$ for even n. Furthermore, $\zeta_{n-1}\zeta_{n-3}\cdots\zeta_{n-2k+1}$ is the identity permutation of length k, where $\zeta = s^k_{123,132}(\gamma_n)$.*

Proof. We induct on k. For brevity, we will prove the lemma for when n is odd—the proof for even n is directly analogous. For the base case $k = 1$, we have $s_{123,132}(\gamma_n)_n = (\gamma_n)_1 = \frac{n+1}{2}$ by Proposition 3.1. Since $(\gamma_n)_{[2:n-3]}$ is strictly increasing, these elements are popped out in reverse order just before 1 enters the stack. Hence, $s_{123,132}(\gamma_n)_{[1:n-4]} = \delta_n = (\delta_n)_{[1:n-4]}$. Finally, $s_{123,132}(\gamma_n)_{n-1} = 1$ by Lemma 3.1, completing the base case.

Next, suppose $s^k_{123,132}(\gamma_n)_{[1:n-2k-2]} = (\delta_n)_{[k:n-k-3]}$ for some k and $\zeta_{n-1}\zeta_{n-3}\cdots\zeta_{n-2k+1}$ is the identity of length k where $\zeta = s^k_{123,132}(\gamma_n)$. By Proposition 3.1, we have $s^{k+1}_{123,132}(\gamma_n)_n = s^k_{123,132}(\gamma_n)_1$, and since $s^k_{123,132}(\gamma_n)_{[1:n-2k-2]}$ is strictly decreasing, it follows that these elements will exit the stack in the same order, giving $s^{k+1}_{123,132}(\gamma_n)_{[1:n-2k-4]} = (\delta_n)_{[k+1:n-k-4]}$ by the inductive hypothesis. Finally, by Lemma 3.1, we have $s^{k+1}_{123,132}(\gamma_n)_{n-2k-1} = k+1$, completing the induction. □

Lemma 3.9. *For all positive integers n, we have $\mathrm{ord}_{s_{123,132}}(S_n) \geq 2\lfloor \frac{n-1}{2} \rfloor$.*

Proof. It follows from Lemma 3.8 that $s^{\lfloor \frac{n-1}{2} \rfloor - 1}_{123,132}(\gamma_n)_1 = \lfloor \frac{n-1}{2} \rfloor$. By Proposition 3.1 and Lemma 3.3, we have $\mathrm{ind}_{s^k_{123,132}(\gamma_n)}(\lfloor \frac{n-1}{2} \rfloor) = n - 2(k - \lfloor \frac{n-1}{2} \rfloor)$ for $k \geq \lfloor \frac{n-1}{2} \rfloor$. Hence, $k = 2\lfloor \frac{n-1}{2} \rfloor$ is the minimal k such that $s^k_{123,132}(\gamma_n)$ is half-decreasing, giving us the desired bound. □

Finally, we conclude that exactly $2\lfloor \frac{n-1}{2} \rfloor$ iterations are required to sort S_n.

Proof of Theorem 1.2. Corollary 3.2 and Lemma 3.9 directly imply $\mathrm{ord}_{s_{123,132}}(S_n) = 2\lfloor \frac{n-1}{2} \rfloor$. □

4. Future Directions

To study Defant's notion of highly-sorted permutations and our newly-introduced notion of minimally-sorted permutations, characterizing the periodic permutations under generalized stack-sorting maps is a prerequisite. We state a conjecture on the periodic points of other $s_{\sigma,\tau}$ stack-sorting maps for three pairs of (σ, τ), and restate a conjecture from Berlow.

Conjecture 4.1. *For $(\sigma, \tau) = (123, 213), (132, 312), (231, 321)$, the map $s_{\sigma,\tau}$ is a bijection from S_n to itself, and all permutations are periodic.*

Conjecture 4.2 (Berlow [5])**.** *For $(\sigma, \tau) = (213, 231), (132, 213), (231, 312)$, the only periodic points of $s_{\sigma,\tau}$ are the identity permutation and its inverse.*

Recall that $\mathfrak{M}_n$ is the set of minimally-sorted permutations under $s_{123,132}$. We conjecture several properties of elements in $\mathfrak{M}_n$. However, these conditions are not sufficient for $n \geq 7$.

Conjecture 4.3. *For $\pi \in \mathfrak{M}_n$, the following conditions hold true:*

- $\pi_1 \geq \lfloor \frac{n+1}{2} \rfloor$.
- ***For odd*** *n: $\pi_{n-2} = 1$ and $\pi_{n-1}, \pi_n \geq \lfloor \frac{n+1}{2} \rfloor$.*
- ***For even*** *n: $\pi_{n-3} = 1$ and $\pi_{n-2}, \pi_{n-1}, \pi_n \geq \lfloor \frac{n+1}{2} \rfloor$.*

Next, an enumerative conjecture on $\mathfrak{M}_n$, computationally verified for $n \leq 6$.

Conjecture 4.4. *For all positive integers n, we have $|\mathfrak{M}_{2n}| = (n+1)|\mathfrak{M}_{2n-1}|$.*

Finally, we conclude with an enumerative conjecture on $\mathrm{Sort}_{t,n}(123, 132)$, the set of length n permutations that are t-stack-sortable under $s_{123,132}$.

Conjecture 4.5. *For any positive integer t and $n \geq 2t+1$, we have:*

- $|\mathrm{Sort}_{t,n}(123, 132)| = \frac{n+3}{2}|\mathrm{Sort}_{t,n-2}(123, 132)|$ *if n is odd.*
- $|\mathrm{Sort}_{t,n}(123, 132)| = \frac{n+4}{2}|\mathrm{Sort}_{t,n-2}(123, 132)|$ *if n is even.*

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