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Homology of Segre Powers of Boolean and Subspace Lattices

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Abstract: Segre products of posets were defined by Björner and Welker (2005). We investigate the homology representations of the t-fold Segre power $B_n^{(t)}$ of the Boolean lattice B_n. The direct product $\mathfrak{S}_n^{\times t}$ of the symmetric group $\mathfrak{S}_n$ acts on the homology of rank-selected subposets of $B_n^{(t)}$. We give an explicit formula for the decomposition into $\mathfrak{S}_n^{\times t}$-irreducibles of the homology of the full poset, as well as formulas for the diagonal action of the symmetric group $\mathfrak{S}_n$. For the rank-selected homology, we show that the stable principal specialisation of the product Frobenius characteristic of the $\mathfrak{S}_n^{\times t}$-module coincides with the corresponding rank-selected invariant of the t-fold Segre power of the subspace lattice.

Keywords: Ascent; Boolean lattice; Frobenius characteristic; Principal specialization; Subspace lattice; Segre product of posets; Whitney homology
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1. Introduction

Let B_n denote the Boolean lattice of subsets of an n-element set, and let $B_{n,q}$ denote the lattice of subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements.

The *Segre product* of posets was first defined by Björner and Welker, who showed [5, Theorem 1] that this operation preserves the property of being homotopy Cohen-Macaulay. Let $P^{(t)}$ denote the t-fold Segre power $P \circ \cdots \circ P$ (t factors) of a graded poset P. The Segre square $P \circ P$ was studied by the first author in [9] when P is the Boolean lattice B_n or the subspace lattice $B_{n,q}$. Segre powers of the subspace lattice $B_{n,q}$ appear in an early paper of Stanley [13, Ex. 1.2], as an example of binomial posets. See also [15, Ex. 3.18.3].

The symmetric group $\mathfrak{S}_n$ acts on B_n, and hence the Segre power $B_n^{(t)}$ of B_n carries two actions, one for the t-fold direct product $\mathfrak{S}_n^{\times t}$ of $\mathfrak{S}_n$ with itself, and the other for the symmetric group $\mathfrak{S}_n$.

In this paper, we study both these actions on the rank-selected subposets of the Cohen-Macaulay poset $B_n^{(t)}$, giving formulas for the irreducible decomposition on the top homology of $B_n^{(t)}$. For the t-fold Segre power of the subspace lattice, a special case of a theorem of Stanley [13, Theorem 3.1] shows that the Möbius number of $B_{n,q}^{(t)}$ is given by $(-1)^n W_n^{(t)}(q)$ where

$$W_n^{(t)}(q) := \sum_{(\sigma_1,\ldots\sigma_t)\in\mathfrak{S}_n^{\times t}} \prod_{i=1}^{t} q^{\mathrm{inv}(\sigma_i)}.$$

Here $\mathrm{inv}(\tau)$ is the number of inversions of the permutation τ, and the sum is over all t-tuples of permutations in $\mathfrak{S}_n$ with no common ascent, i being an ascent of a permutation σ if $\sigma(i) < \sigma(i+1)$. When $q = 1$ this specialises to the dimension of the homology of $B_n^{(t)}$; it is the number $w_n^{(t)}$ of t-tuples of permutations in the symmetric group $\mathfrak{S}_n$ with no common ascent. The numbers $w_n^{(2)}$ first appear in work of Carlitz, Scoville, and Vaughn [6]. For arbitrary t the numbers $w_n^{(t)}$ have also already appeared in the literature; see Abramson and Promislow [1]. This suggests a deeper connection between the homology modules of the Segre powers of the subspace lattice and the Boolean lattice.

The paper is organised as follows. Prerequisites are reviewed in Section 2. In order to describe the rank-selected homology modules of $B_n^{(t)}$ for the $\mathfrak{S}_n^{\times t}$-action, in Section 3 we develop an extension of the *product Frobenius characteristic* introduced in [9]. In Section 4 we use the Whitney homology technique of [18] to derive recursive formulas for these representations involving symmetric functions in t sets of variables. Section 5 extends these results to the action of $\mathfrak{S}_n^{\times t}$ on the chains and homology of all rank-selected [14] subposets of $B_n^{(t)}$. Finally in Section 6 we use these formulas to investigate the stable principal specialisations of the rank-selected representations of $B_n^{(t)}$, and establish a connection with the corresponding rank-selected invariants of the t-fold Segre power $B_{n,q}^{(t)}$ of the subspace lattice $B_{n,q}$. For the top homology this fact was established in [9], in the case $t = 2$.

Our framework allows us to obtain explicit formulas for the homology representation of $B_n^{(t)}$. A key feature of these formulas is Definition 4.7, where we introduce an injective algebra homomorphism

$$\Phi_t : \Lambda_n(x) \to \otimes_{j=1}^{t} \Lambda_n(X^j)$$

from the algebra of symmetric functions of homogeneous degree n in a single set of variables, to the tensor product of the algebras of degree n-symmetric functions in t sets of variables. We show that Φ_t maps the elementary symmetric function e_n to the product Frobenius characteristic $\beta_n^{(t)}$ of the top homology of $B_n^{(t)}$. By exploiting properties of the homomorphism Φ_t, we obtain the main results of this paper:

1. Theorem 4.13 gives the decomposition into irreducibles of the top homology $\tilde{H}_{n-2}(B_n^{(t)})$ of $B_n^{(t)}$ under the action of $\mathfrak{S}_n^{\times t}$.
2. Theorem 4.20 gives a formula for the irreducible decomposition of the diagonal $\mathfrak{S}_n$-action on $\tilde{H}_{n-2}(B_n^{(t)})$ in terms of Kronecker products, including an explicit formula for the character values.
3. Theorem 5.5 gives a recursive formula for the product Frobenius characteristic of the rank-selected homology, from which one can obtain explicit formulas for the irreducible decomposition.
4. Theorem 6.2 shows that the stable principal specialisation of the product Frobenius characteristic of the rank-selected homology of $B_n^{(t)}$ gives, up to a factor, the corresponding rank-selected invariant for $B_{n,q}^{(t)}$.

All homology in this work is reduced and taken with rational coefficients.

2. Segre powers and rank-selected invariants

We refer the reader to [4, 15, 20] for background on posets and topology.

Recall [15] that the product poset $P \times Q$ of two posets P, Q has order relation defined by $(p, q) \leqslant (p', q')$ if and only if $p \leqslant_P p'$ and $q \leqslant_Q q'$. Segre products are defined in greater generality by Björner and Welker in [5]. This paper is concerned with the following special case.

Definition 2.1 ([5])**.** Let P be a bounded graded poset. The t-fold Segre power $P \circ \cdots \circ P$ (t factors), denoted $P^{(t)}$, is defined for all $t \geqslant 2$ to be the induced subposet of the t-fold product poset $P \times \cdots \times P$ (t factors) consisting of t-tuples $(x_1, \ldots, x_t)$ such that $\text{rank}(x_i) = \text{rank}(x_j), 1 \leqslant i, j \leqslant t$. The cover relation in $P^{(t)}$ is thus $(x_1, \ldots, x_t) \lessdot (y_1, \ldots, y_t)$ if and only if $x_i \lessdot y_i$ in P, for all $i = 1, \ldots, t$. When $t = 1$ we set $P^{(1)}$ equal to P.

It follows that $P^{(t)}$ is also a ranked poset which inherits the rank function of P.

Figure 1 shows the Segre square $P \circ P$ of a poset P, an induced subposet of $P \times P$. Missing in $P \circ P$ are these elements in the product $P \times P$: $(a, c), (a, d), (b, c), (b, d), (c, a), (c, b), (d, a), (d, b)$, as well as all $(\hat{0}, y), (x, \hat{0}), (x, \hat{1}), (\hat{1}, y)$ for $x, y \in P$.

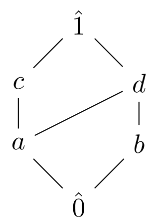
Let P be a finite graded bounded poset of rank n, and let $J \subset [n-1] = \{1, \ldots, n-1\}$ be any subset of nontrivial ranks. Let $P(J)$ denote the rank-selected bounded subposet of P consisting of elements in the rank-set J, together with $\hat{0}$ and $\hat{1}$. Stanley [15, Section 3.13] defined two rank-selected invariants $\tilde{\alpha}_P(J)$ and $\tilde{\beta}_P(J)$ as follows.

- $\tilde{\alpha}_P(J)$ is the number of maximal chains in the rank-selected subposet $P(J)$, and
- $\tilde{\beta}_P(J)$ is the integer defined by the equation

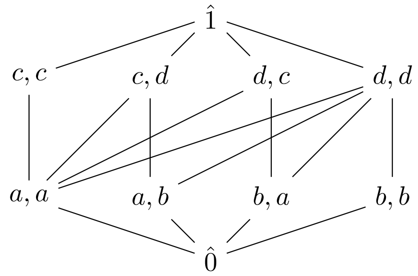
$$\tilde{\beta}_P(J) := \sum_{U \subseteq J} (-1)^{|J|-|U|} \tilde{\alpha}_P(U).$$

Equivalently,

$$\tilde{\alpha}_P(J) = \sum_{U \subseteq J} \tilde{\beta}_P(U).$$



(a) Hasse diagram of P.



(b) Hasse diagram of $P \circ P$

Figure 1: $P \circ P$ is an induced subposet of the product poset $P \times P$.

One also has the formula for the Möbius number of the rank-selected subposet $P(J)$ [15, Eqn. (3.54)]:

$$\tilde{\beta}_P(J) = (-1)^{|J|-1} \mu_{P(J)}(\hat{0}, \hat{1}). \tag{2.1}$$

When the poset P has the recursive structure described in the lemma below, the rank-selected invariants satisfy a pleasing recurrence that we record for later use in Section 6.

Lemma 2.2. *Let P be a graded, bounded poset of rank n, with the property that for any $x \in P$, the poset structure of the interval $(\hat{0}, x)$ depends only on the rank of x. Thus we may write $P_i = (\hat{0}, x_0)$ for any x_0 of rank i. Let $wh_i(P)$ denote the number of elements of P at rank i. Then we have the following recurrence for the rank-selected invariants $\tilde{\beta}_P(J)$ of P, $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\} \subseteq [n-1]$.*

$$\tilde{\beta}_P(J) + \tilde{\beta}_P(J \setminus \{j_r\}) = wh_{j_r}(P) \cdot \tilde{\beta}_{P_{j_r}}(J \setminus \{j_r\})$$

For the full poset P, we have

$$\mu_P(\hat{0}, \hat{1}) = -\sum_{i=0}^{n-1} wh_i(P) \cdot \mu_{P_i}(\hat{0}, \hat{1}).$$

Proof. Observe first that the condition satisfied by P is inherited by every rank-selected subposet Q of P. For such posets Q of rank k, the Möbius function recurrence gives

$$\begin{aligned}\mu_Q(\hat{0}, \hat{1}) &= -\sum_{x:\mathrm{rank}(x)=k-1} \mu_Q(\hat{0}, x) - \sum_{x:\mathrm{rank}(x)\leqslant k-2} \mu_Q(\hat{0}, x) \\ &= -|\{x : \mathrm{rank}(x) = k-1\}| \cdot \mu_Q(\hat{0}, x_0) + \mu_{Q_{\{\leqslant k-2\}}}(\hat{0}, \hat{1}), \\ &\text{where } x_0 \text{ is any fixed element of rank } k-1.\end{aligned}$$

Here $Q_{\{\leqslant k-2\}}$ is the subposet of Q consisting of the bottom $k-2$ ranks.

Applying this to $Q = P(J)$ now gives the result.

By refining the Möbius function recurrence according to the rank, we obtain the special case for the full poset $Q = P$, since

$$\mu_P(\hat{0}, \hat{1}) = -\sum_{i=0}^{n-1} \sum_{x \text{ at rank } i} \mu_P(\hat{0}, x). \qquad \square$$

The condition of the lemma is satisfied by the Boolean lattice B_n and also by the subspace lattice $B_{n,q}$. We can now derive a recurrence for the rank-selected invariants $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$, $J \subseteq [n-1]$, of the t-fold Segre power $B_{n,q}^{(t)}$. From (2.1), these are also the unsigned Möbius numbers of the corresponding rank-selected subposets.

Recall that the number of i-dimensional subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ [15, Proposition 1.7.2] is given by the q-binomial coefficient

$$\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q^i)(1-q^{i-1})\cdots(1-q)\,(1-q^{n-i})(1-q^{n-i-1})\cdots(1-q)}. \tag{2.2}$$

Let $(-1)^{n-2} W_n^{(t)}(q)$ be the Möbius number of the t-fold Segre power of the subspace lattice. When $q = 1$, $B_{n,q}$ specialises to B_n, and hence from Proposition 2.5 we have $W_n^{(t)}(1) = w_n^{(t)}$.

To avoid a profusion of parentheses, for the kth power of the q-binomial coefficient we write $\begin{bmatrix} n \\ i \end{bmatrix}_q^k$.

Proposition 2.3. *We have the following recurrence for the rank-selected invariants of $B_{n,q}^{(t)}$. For the rank-set $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$:*

$$\tilde{\beta}_{B_{n,q}^{(t)}}(J) + \tilde{\beta}_{B_{n,q}^{(t)}}(J \smallsetminus \{j_r\}) = \begin{bmatrix} n \\ j_r \end{bmatrix}_q^t \tilde{\beta}_{B_{j_r}^{(t)}(q)}(J \smallsetminus \{j_r\}) \tag{2.3}$$

For the full poset $B_{n,q}^{(t)}$, we have the recurrence

$$W_n^{(t)}(q) = \sum_{i=0}^{n-1} (-1)^{n-1-i} \begin{bmatrix} n \\ i \end{bmatrix}_q^t W_i^{(t)}(q). \tag{2.4}$$

Proof. We apply Lemma 2.2 to the rank-selected subposet $B_{n,q}^{(t)}(J)$, using the fact that wh_i, the number of elements at rank i in the t-fold Segre power, is precisely $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q^t$. Also, if x_0 is at rank j_r, the interval $(0, x_0)$ in $B_{n,q}^{(t)}(J)$ is poset isomorphic to the rank-selected subposet of $B_{j_r}^{(t)}(q)$ corresponding to the rank-set $J \smallsetminus \{j_r\}$. Using (2.1) for the passage from Möbius numbers to rank-selected invariants, the first recurrence in Lemma 2.2 now gives (2.3).

Similarly, the second recurrence in Lemma 2.2 gives (2.4). □

The recurrence (2.3), in conjunction with an equivariant version of the recurrence in Lemma 2.2 for the Boolean lattice that we derive in Section 5, will be used in Section 6 when we consider the stable principal specialisation.

We conclude this section by explaining more precisely the relevance of the numbers $w_n^{(t)}$ mentioned in the Introduction. We need the following expression due to Stanley for $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ as a polynomial in q with nonnegative coefficients. Let $\mathrm{Asc}(\sigma)$ denote the ascent set of σ, that is, the set $\{i : 1 \leqslant i \leqslant n-1, \sigma(i) > \sigma(i+1)\}$ of ascents of σ.

Theorem 2.4 ([13, Theorem 3.1]). *Let $J \subseteq [n-1]$. Write $J^c = [n-1] \smallsetminus J$. Then for the rank-selected t-fold Segre power of the subspace lattice $B_{n,q}^{(t)}$, one has*

$$\tilde{\beta}_{B_{n,q}^{(t)}}(J) = \sum_{\substack{(\sigma^1,\ldots,\sigma^t) \in \mathfrak{S}_n^{\times t} \\ J^c = \cap_{i=1}^t \mathrm{Asc}(\sigma^i)}} \prod_{i=1}^t q^{\mathrm{inv}(\sigma^i)}$$

In particular, the Möbius number of the t-fold Segre power of the subspace lattice is $(-1)^{n-2} W_n^{(t)}(q)$, where

$$W_n^{(t)}(q) := \sum_{(\sigma_1,\ldots\sigma_t) \in \mathfrak{S}_n^{\times t}} \prod_{i=1}^t q^{\mathrm{inv}(\sigma_i)},$$

and the sum is over all t-tuples of permutations in $\mathfrak{S}_n$ with no common ascent.

Setting $q = 1$ gives the special case of the Segre powers of the Boolean lattice, as mentioned in the Introduction. The generating function below appears in [13]. The numbers $w_n^{(t)}$ also appear in [1].

Proposition 2.5 (See [13, Eqn. (28) and Theorem 3.1]). *The Möbius number of $B_n^{(t)}$ is given by $(-1)^n w_n^{(t)}$, where for $n \geqslant 1$, $w_n^{(t)}$ is the number of t-tuples of permutations in $\mathfrak{S}_n$ with no common ascent. Hence, setting $w_0^{(t)} = 1$, the numbers $w_n^{(t)}$ satisfy the recurrence*

$$\sum_{i=0}^{n} (-1)^i w_i^{(t)} \binom{n}{i}^t = 0. \tag{2.5}$$

Furthermore, we have the generating function

$$\sum_{n \geqslant 0} w_n^{(t)} \frac{z^n}{n!^t} = \frac{1}{f(z)}, \quad \text{where } f(z) = \sum_{n \geqslant 0} (-1)^n \frac{z^n}{n!^t}.$$

More generally, for the rank selection $J \subseteq [n-1]$, the Möbius number $\mu(B_n^{(t)}(J))$ of $B_n^{(t)}(J)$ is given by $(-1)^{|J|-1} w_n^{(t)}(J)$, where $w_n^{(t)}(J)$ is the number of t-tuples of permutations in $\mathfrak{S}_n$ such that their set of common ascents coincides with the complement of J in $[n-1]$.

Proof. The first recurrence is a restatement of the Möbius function recurrence for the lattice $B_n^{(t)}$, and the generating function then follows. The statement for the rank-selected Möbius number is the case $q = 1$ of [13, Theorem 3.1]. □

3. The product Frobenius characteristic

We refer to [11] for all background on symmetric functions and representations of the symmetric group $\mathfrak{S}_n$. See also [16, Chapter 7]. In particular, h_n, e_n and p_n are respectively the homogeneous, elementary, and power sum symmetric functions of degree n, giving rise to basis elements h_λ, e_λ and p_λ indexed by partitions λ of n, in the algebra of symmetric functions of homogeneous degree n, and s_λ is the Schur function indexed by λ.

The action of the symmetric group $\mathfrak{S}_n$ on the Boolean lattice B_n extends naturally to an action of the t-fold direct product $\mathfrak{S}_n^{\times t} := \mathfrak{S}_n \times \cdots \times \mathfrak{S}_n$ (t factors), on the t-fold Segre power of B_n. For the Segre square $B_n \circ B_n$, a *product Frobenius map*, generalizing the well-known ordinary Frobenius characteristic in [11], was defined [9] in order to study the action of $\mathfrak{S}_n \times \mathfrak{S}_n$ on the homology.

The goal of this section is to rigorously define and extend this construction to arbitrary t. We begin by giving an alternative description of the product Frobenius map in [9]. As in [11, Chapter 1, Section 7], let R^n denote the vector space spanned by the irreducible characters of the symmetric group $\mathfrak{S}_n$ over $\mathbb{Q}$, or equivalently the vector space spanned by the class functions of $\mathfrak{S}_n$. Let $R = \oplus_{n\geqslant 0} R^n$. Then R is equipped with the structure of a graded commutative and associative ring with identity element 1 for the group $\mathfrak{S}_0 = \{1\}$, arising from the bilinear map $R^m \times R^n \to R^{m+n}$, defined by $(f, g) \mapsto (f \times g) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}$, the induced character from f and g.

Let $\Lambda^m(X)$ be the ring of symmetric functions in the set of variables X, of homogeneous degree m, and let $\Lambda(X) = \oplus_{m\geqslant 0} \Lambda^m(X)$. Write $\mu \vdash n$ for an integer partition $\mu = (\mu_1 \geqslant \cdots \geqslant \mu_\ell)$ of the integer $n \geqslant 1$, so that $\sum_{i=1}^{\ell} \mu_i = n$, $\mu_i \geqslant 1$ for all i, and $\ell(\mu)$ for the number of parts μ_i of μ. (There is only one integer partition of 0, the empty partition with zero parts.)

The ordinary Frobenius characteristic map ch is defined as follows. For each $f \in R^n$,

$$\mathrm{ch}(f) := \sum_{\mu\vdash n} z_\mu^{-1} f_\mu p_\mu(X), \tag{3.1}$$

where f_μ denotes the value of the class function f on the class indexed by the partition $\mu \vdash n$, z_μ is the order of the centraliser of a permutation of type μ in the symmetric group $\mathfrak{S}_n$, and $p_\mu(X)$ is the power sum symmetric function indexed by μ. In particular when f is the irreducible character χ^λ indexed by the partition λ, then

$$\mathrm{ch}(\chi^\lambda) = s_\lambda(X), \tag{3.2}$$

where $s_\lambda(X)$ is the Schur function indexed by λ. The set of Schur functions $\{s_\lambda(X) : \lambda \vdash n\}$ forms a basis for $\Lambda^n(X)$. Furthermore, ch is a ring isomorphism from R to $\Lambda(X)$, since for $f \in R^m, g \in R^n$,

$$\mathrm{ch}\left((f \times g) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}\right) = \mathrm{ch}(f)\,\mathrm{ch}(g). \tag{3.3}$$

In particular, for $\lambda \vdash m, \mu \vdash n$,

$$\mathrm{ch}\left((\chi^\lambda \times \chi^\mu) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}\right) = s_\lambda(X) s_\mu(X).$$

We wish to generalize this to a t-fold direct product of symmetric groups. The main idea originates in [9], where the case $t = 2$ was treated. We will give a slightly different treatment here, elaborating on details that were omitted in [9]. Let $\underline{\boldsymbol{n}} = (n_1, \ldots, n_t) \in \mathbb{Z}_{\geqslant 0}^t$ be a t-tuple of nonnegative integers, and let $\mathfrak{S}_{\underline{\boldsymbol{n}}}$ be the direct product of symmetric groups $\bigtimes_{i=1}^t \mathfrak{S}_{n_i}$. The irreducible characters of $\mathfrak{S}_{\underline{\boldsymbol{n}}}$ are indexed by t-tuples of partitions $\underline{\boldsymbol{\lambda}} = (\lambda^1, \ldots, \lambda^t)$ where $\lambda^i \vdash n_i$. Let $R^{\underline{\boldsymbol{n}}}$ denote the vector space spanned by the irreducible characters, or equivalently the vector space spanned by the class functions, of the direct product of symmetric groups $\mathfrak{S}_{\underline{\boldsymbol{n}}}$ over $\mathbb{Q}$. Then $R^{\underline{\boldsymbol{n}}} = \otimes_i R^{n_i}$. Let $\underline{R} = \oplus_{\underline{\boldsymbol{n}}\in\mathbb{Z}_{\geqslant 0}^t} R^{\underline{\boldsymbol{n}}}$.

Let (X^i), $i = 1, \ldots, t$ be t sets of variables. For each i we consider the ring of symmetric functions $\Lambda^{n_i}(X^i)$ in the variables (X^i), of homogeneous degree n_i. As in [11, Chapter 1, Section 5, Ex. 25], we identify the tensor product $\bigotimes_{i=1}^t \Lambda^{n_i}(X^i)$ with products of functions of t sets of variables $(X^i)_{i=1}^t$, symmetric in each set separately, i.e., with the vector space spanned by the set of elements

$$\left\{ \prod_{i=1}^t f_{n_i}(X^i) : f_{n_i}(X^i) \in \Lambda^{n_i}(X^i) \right\}.$$

Thus $\bigotimes_{i=1}^t f_{n_i}(X^i) \mapsto \prod_{i=1}^t f_{n_i}(X^i)$.

Definition 3.1 (cf. [9, Definition 3.2])**.** Define the map $\mathrm{Pch} : R^{\underline{\boldsymbol{n}}} \to \bigotimes_{i=1}^t \Lambda^{n_i}(X^i)$ as follows. Let $f_{n_i} \in R^{n_i}$ and define

$$\mathrm{Pch}\left(\bigotimes_{i=1}^t f_{n_i}\right) := \prod_{i=1}^t \mathrm{ch}(f_{n_i})(X^i),$$

where ch denotes the ordinary Frobenius characteristic map on R as in (3.2). This can be extended multilinearly to all of $R^{\underline{\boldsymbol{n}}}$. In particular for the irreducible character $\chi^{\underline{\boldsymbol{\lambda}}} = \bigotimes_{i=1}^t \chi^{\lambda^i}$ indexed by the t-tuple $\underline{\boldsymbol{\lambda}} = (\lambda^1, \ldots, \lambda^t)$, we have

$$\operatorname{Pch}(\chi^{\underline{\boldsymbol{\lambda}}}) = \prod_{i=1}^t s_{\lambda^i}(X^i),$$

a product of Schur functions in t different sets of variables.

Expanding in terms of power sum symmetric functions, we obtain, in analogy with (3.1), for an arbitrary character χ of $\mathfrak{S}_{\underline{\boldsymbol{n}}}$, the formula

$$\operatorname{Pch}(\chi) = \sum_{\underline{\boldsymbol{\mu}}} \chi(\underline{\boldsymbol{\mu}}) \prod_{i=1}^t z_{\mu^i}^{-1} \prod_{i=1}^t p_{\mu^i}(X^i),$$

where we have written $\chi(\underline{\boldsymbol{\mu}})$ for the value of the character χ on the conjugacy class of $\mathfrak{S}_{\underline{\boldsymbol{n}}}$ indexed by the t-tuple $\underline{\boldsymbol{\mu}} = (\mu^1, \ldots, \mu^t)$, $\mu^i \vdash n_i$, and z_μ is the order of the centraliser in $\mathfrak{S}_n$ of an element of cycle-type $\mu \vdash n$.

When $t = 1$, $\operatorname{Pch}(\chi) = \operatorname{ch}(\chi)$ for all characters χ of $\mathfrak{S}_n$, and the product Frobenius characteristic coincides with the ordinary characteristic map.

There is an inner product on $\bigotimes_{i=1}^t \Lambda^{n_i}(X^i)$ defined by

$$\Big\langle \prod_{i=1}^t f_i, \prod_{i=1}^t g_i \Big\rangle := \prod_{i=1}^t \langle f_i, g_i \rangle_{\Lambda^{n_i}(X^i)}, \tag{3.4}$$

where $\langle f_i, g_i \rangle_{\Lambda^{n_i}(X^i)}$ is the usual inner product [11, Chapter I, Section 7] in the ring of homogeneous symmetric functions $\Lambda^{n_i}(X^i)$ in a single set of variables X^i, corresponding to the inner product of class functions of the symmetric group.

Example 3.2. Let $t = 2$ and consider the regular representation ψ of $\mathfrak{S}_2 \times \mathfrak{S}_3$. Then ψ decomposes into irreducibles as follows:

$$\chi^{((2),(3))} + \chi^{((1^2),(3))} + 2\chi^{((2),(2,1))} + 2\chi^{((1^2),(2,1))} + \chi^{((2),(1^3))} + \chi^{((1^2),(1^3))}.$$

Using X^1 and X^2 for the two sets of variables, we have

$$\begin{aligned} \operatorname{Pch}(\psi) &= s_{(2)}(X^1) s_{(3)}(X^2) + s_{(1^2)}(X^1) s_{(3)}(X^2) + 2 s_{(2)}(X^1) s_{(2,1)}(X^2) + 2 s_{(1^2)}(X^1) s_{(2,1)}(X^2) \\ &\quad + s_{(2)}(X^1) s_{(1^3)}(X^2) + s_{(1^2)}(X^1) s_{(1^3)}(X^2) \\ &= h_1^2(X^1) h_1^3(X^2) \end{aligned}$$

We want Pch to be a ring homomorphism with respect to an induction product akin to (3.3). In [9, Definition 3.6], this induction product was defined to take an ordered pair (ψ, ϕ) where ψ is a character of $\mathfrak{S}_k \times \mathfrak{S}_\ell$ and ϕ is a character of $\mathfrak{S}_m \times \mathfrak{S}_n$, and produce a character of $\mathfrak{S}_{k+m} \times \mathfrak{S}_{\ell+n}$. For the t-fold products, we wish to take a character ψ of $\mathfrak{S}_{\underline{\boldsymbol{m}}} = \bigtimes_{i=1}^t \mathfrak{S}_{m_i}$ and a character ϕ of $\mathfrak{S}_{\underline{\boldsymbol{n}}} = \bigtimes_{i=1}^t \mathfrak{S}_{n_i}$, and map the pair (ψ, ϕ) to a character of $\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}} = \bigtimes_{i=1}^t \mathfrak{S}_{m_i+n_i}$. Here $\underline{\boldsymbol{m}} + \underline{\boldsymbol{n}} = (m_1 + n_1, \ldots, m_t + n_t)$. To do this rigorously, we first take ψ and ϕ to be irreducible characters; the definition will then extend multilinearly in the obvious way.

Definition 3.3. Let $\underline{\boldsymbol{\lambda}} = (\lambda^1, \ldots, \lambda^t), \lambda^i \vdash m_i$ and $\underline{\boldsymbol{\mu}} = (\mu^1, \ldots, \mu^t), \mu^i \vdash n_i$, so that $\chi^{\underline{\boldsymbol{\lambda}}} = \bigotimes_{i=1}^t \chi^{\lambda^i}$ and $\chi^{\underline{\boldsymbol{\mu}}} = \bigotimes_{i=1}^t \chi^{\mu^i}$ are respectively irreducible characters of $\mathfrak{S}_{\underline{\boldsymbol{m}}}$ and $\mathfrak{S}_{\underline{\boldsymbol{n}}}$. The t-fold *induction product* $\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}}$ is then defined to be the induced character

$$\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}} := \bigotimes_{i=1}^t (\chi^{\lambda^i} \otimes \chi^{\mu^i}) \uparrow_{\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i}}^{\mathfrak{S}_{m_i+n_i}},$$

a character of the direct product $\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}$. Note that this defines a bilinear multiplication

$$R^{\underline{\boldsymbol{m}}} \times R^{\underline{\boldsymbol{n}}} \to R^{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}},$$

for all pairs of t-tuples $\underline{\boldsymbol{m}}, \underline{\boldsymbol{n}} \in \mathbb{Z}_{\geqslant 0}^t$, endowing $\underline{R} = \oplus_{\underline{\boldsymbol{n}} \in \mathbb{Z}_{\geqslant 0}^t} R^{\underline{\boldsymbol{n}}}$ with the structure of a commutative and associative graded ring with unity, in exact analogy with [11, p. 112]. The grading is now by t-tuples $\underline{\boldsymbol{n}}$.

We now extend this definition multilinearly to any pair of representations ψ of $\mathfrak{S}_{\underline{\boldsymbol{m}}} = \bigtimes_i \mathfrak{S}_{m_i}$ and ϕ of $\mathfrak{S}_{\underline{\boldsymbol{n}}} = \bigtimes_i \mathfrak{S}_{n_i}$, to produce a new representation $\psi \circ \phi$ of $\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}$. Explicitly, if $\psi = \sum_{\underline{\boldsymbol{\lambda}}} a_{\underline{\boldsymbol{\lambda}}}(\psi) \chi^{\underline{\boldsymbol{\lambda}}}$ and $\phi = \sum_{\underline{\boldsymbol{\mu}}} a_{\underline{\boldsymbol{\mu}}}(\phi) \chi^{\underline{\boldsymbol{\mu}}}$, then we define the induction product of ψ and ϕ to be the following representation of $\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}$:

$$\psi \circ \phi := \sum_{\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\mu}}} a_{\underline{\boldsymbol{\lambda}}}(\psi)\, a_{\underline{\boldsymbol{\mu}}}(\phi)\, (\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}}). \tag{3.5}$$

We note that the dimension of $\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}}$ is $\prod_{i=1}^{t} f^{\lambda^i} f^{\mu^i} \binom{m_i+n_i}{m_i}$, where f^{λ} is the dimension of the irreducible representation of the symmetric group $\mathfrak{S}_{|\lambda|}$ indexed by λ, i.e. the number of standard Young tableaux of shape λ. Hence for the dimension of $\psi \circ \phi$ we have

$$\dim(\psi \circ \phi) = \dim(\psi)\dim(\phi)\prod_i \binom{m_i+n_i}{m_i}.$$

Remark 3.4. One could also form the ordinary induced representations $\psi \uparrow_{\mathfrak{S}_{\underline{\boldsymbol{m}}}}^{\mathfrak{S}_{m_1+\ldots+m_t}}$ of $\mathfrak{S}_{m_1+\ldots+m_t}$ and $\phi \uparrow_{\mathfrak{S}_{\underline{\boldsymbol{n}}}}^{\mathfrak{S}_{n_1+\ldots+n_t}}$ of $\mathfrak{S}_{n_1+\ldots+n_t}$ as well as the induced representation $\psi \otimes \phi$ from $\mathfrak{S}_{\underline{\boldsymbol{m}}} \times \mathfrak{S}_{\underline{\boldsymbol{n}}}$ to $\mathfrak{S}_{\sum m_i + \sum n_j}$. However, it is the above definition that proves useful in studying the Segre product of Boolean lattices.

Proposition 3.5. *The map* Pch *is a bijective ring homomorphism, with respect to the induction product* $\circ$ *in* $\underline{R}$*, from* $\underline{R}$ *to* $\bigotimes_{i=1}^{t} \Lambda(X^i)$*. Explicitly, if* $\underline{\boldsymbol{m}}, \underline{\boldsymbol{n}} \in \mathbb{Z}_{\geqslant 0}^{t}$ *and* ψ *and* ϕ *are characters of* $\mathfrak{S}_{\underline{\boldsymbol{m}}}$ *and* $\mathfrak{S}_{\underline{\boldsymbol{n}}}$ *respectively, then*

$$\mathrm{Pch}(\psi \circ \phi) = \mathrm{Pch}(\psi) \cdot \mathrm{Pch}(\phi).$$

Proof. It is clear that Pch is an isomorphism of vector spaces, since by Definition 3.1 it maps basis elements to basis elements. First, we establish that

$$\mathrm{Pch}(\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}}) = \mathrm{Pch}(\chi^{\underline{\boldsymbol{\lambda}}})\,\mathrm{Pch}(\chi^{\underline{\boldsymbol{\mu}}})$$

for irreducible characters $\chi^{\underline{\boldsymbol{\lambda}}}$ of $\mathfrak{S}_{\underline{\boldsymbol{m}}}$ and $\chi^{\underline{\boldsymbol{\mu}}}$ of $\mathfrak{S}_{\underline{\boldsymbol{n}}}$. We have

$$\begin{aligned}
\mathrm{Pch}(\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}}) &= \mathrm{Pch}\left(\bigotimes_{i=1}^{t} (\chi^{\lambda^i} \otimes \chi^{\mu^i}) \uparrow_{\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i}}^{\mathfrak{S}_{m_i+n_i}}\right) \quad \text{from Definition 3.3} \\
&= \prod_{i=1}^{t} \mathrm{ch}\left((\chi^{\lambda^i} \otimes \chi^{\mu^i}) \uparrow_{\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i}}^{\mathfrak{S}_{m_i+n_i}}\right) \quad \text{from Definition 3.1} \\
&= \prod_{i=1}^{t} s_{\lambda^i}(X^i) s_{\mu^i}(X^i), \ \text{since ch is a ring homomorphism on } R \text{ for} \\
&\quad \text{the ordinary induction of characters} \\
&= \prod_{i=1}^{t} s_{\lambda^i}(X^i) \cdot \prod_{i=1}^{t} s_{\mu^i}(X^i) \\
&= \mathrm{Pch}(\bigotimes_{i=1}^{t} \chi^{\lambda^i}) \cdot \mathrm{Pch}(\bigotimes_{i=1}^{t} \chi^{\mu^i}) \ \text{again using Definition 3.1} \\
&= \mathrm{Pch}(\chi^{\underline{\boldsymbol{\lambda}}}) \cdot \mathrm{Pch}(\chi^{\underline{\boldsymbol{\mu}}}).
\end{aligned}$$

Now let ψ and ϕ be arbitrary characters of $\mathfrak{S}_{\underline{\boldsymbol{m}}}$ and $\mathfrak{S}_{\underline{\boldsymbol{n}}}$ respectively, with irreducible decompositions $\psi = \sum_{\underline{\boldsymbol{\lambda}}} a_{\underline{\boldsymbol{\lambda}}}(\psi)\chi^{\underline{\boldsymbol{\lambda}}}$ and $\phi = \sum_{\underline{\boldsymbol{\mu}}} a_{\underline{\boldsymbol{\mu}}}(\phi)\chi^{\underline{\boldsymbol{\mu}}}$. From (3.5) and the linearity of the map Pch we have

$$\begin{aligned}
\mathrm{Pch}(\psi \circ \phi) &= \sum_{\underline{\boldsymbol{\lambda}},\underline{\boldsymbol{\mu}}} a_{\underline{\boldsymbol{\lambda}}}(\psi)\, a_{\underline{\boldsymbol{\mu}}}(\phi)\ \mathrm{Pch}(\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}}) \\
&= \sum_{\underline{\boldsymbol{\lambda}},\underline{\boldsymbol{\mu}}} a_{\underline{\boldsymbol{\lambda}}}(\psi)\, a_{\underline{\boldsymbol{\mu}}}(\phi)\, \mathrm{Pch}(\chi^{\underline{\boldsymbol{\lambda}}})\, \mathrm{Pch}(\chi^{\underline{\boldsymbol{\mu}}}) \\
&= \sum_{\underline{\boldsymbol{\lambda}}} a_{\underline{\boldsymbol{\lambda}}}(\psi)\, \mathrm{Pch}(\chi^{\underline{\boldsymbol{\lambda}}}) \cdot \sum_{\underline{\boldsymbol{\mu}}} a_{\underline{\boldsymbol{\mu}}}(\phi)\, \mathrm{Pch}(\chi^{\underline{\boldsymbol{\mu}}}) \\
&= \mathrm{Pch}(\psi) \cdot \mathrm{Pch}(\phi),
\end{aligned}$$

which finishes the proof. $\square$

The next proposition shows the equivalence of the above definition of induction product with the one in [9]. Before proving this proposition, some explanation is in order. It is clear that $\bigtimes_{i=1}^{t}(\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i})$ is a direct product of Young subgroups in $\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}$. The key point here is that we can also view $\mathfrak{S}_{\underline{\boldsymbol{m}}} \times \mathfrak{S}_{\underline{\boldsymbol{n}}}$ as a subgroup of $\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}$. In order to do this, we view all the permutations as acting on $2t$ disjoint sets of symbols $(\sqcup_{i=1}^{t} M_i) \sqcup (\sqcup_{j=1}^{t} N_j)$, where $|M_i| = m_i$ and $|N_i| = n_i$, $1 \leqslant i \leqslant t$. (Here $\sqcup$ denotes disjoint union.)

Let $\sigma_i \in \mathfrak{S}_{m_i} = \mathfrak{S}(M_i), \tau_i \in \mathfrak{S}_{n_i} = \mathfrak{S}(N_i)$, $i = 1, \ldots, t$, where $\mathfrak{S}(M_i)$ is the set of permutations on the letters in the set M_i, etc. Thus the t-tuple $(\sigma_1, \ldots, \sigma_t) \in \mathfrak{S}_{\underline{\boldsymbol{m}}}$ may be viewed as a product of commuting permutations

$\prod_{i=1}^t \sigma_i$, and similarly for a t-tuple $(\tau_1, \dots, \tau_t) \in \mathfrak{S}_{\underline{\boldsymbol{n}}}$. Likewise, the t-tuple of ordered pairs $((\sigma_1, \tau_1), \dots, (\sigma_t, \tau_t))$ in $\bigtimes_{i=1}^t (\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i})$ can be identified with the product of commuting permutations $\prod_{i=1}^t \sigma_i \tau_i$. With this identification, $\mathfrak{S}_{\underline{\boldsymbol{m}}} \times \mathfrak{S}_{\underline{\boldsymbol{n}}}$ and $\bigtimes_{i=1}^t (\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i})$ coincide as subgroups of $\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}$, which is identified with the direct product $\bigtimes_{i=1}^t \mathfrak{S}(M_i \sqcup N_i)$.

Proposition 3.6. *Let $\chi^{\underline{\boldsymbol{\lambda}}} = \bigotimes_{i=1}^t \chi^{\lambda^i}$ and $\chi^{\underline{\boldsymbol{\mu}}} = \bigotimes_{i=1}^t \chi^{\mu^i}$ be irreducible characters of $\mathfrak{S}_{\underline{\boldsymbol{m}}}$ and $\mathfrak{S}_{\underline{\boldsymbol{n}}}$, as in Definition 3.3. Then*

$$\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}} = \left(\chi^{\underline{\boldsymbol{\lambda}}} \otimes \chi^{\underline{\boldsymbol{\mu}}}\right) \Big\uparrow_{(\bigtimes_{i=1}^t \mathfrak{S}_{m_i}) \times (\bigtimes_{i=1}^t \mathfrak{S}_{n_i})}^{\bigtimes_{i=1}^t \mathfrak{S}_{m_i+n_i}}, \tag{3.6}$$

which can be rewritten

$$\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}} = \left(\chi^{\underline{\boldsymbol{\lambda}}} \otimes \chi^{\underline{\boldsymbol{\mu}}}\right) \Big\uparrow_{\mathfrak{S}_{\underline{\boldsymbol{m}}} \times \mathfrak{S}_{\underline{\boldsymbol{n}}}}^{\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}}.$$

More generally, if ψ is a character of $\mathfrak{S}_{\underline{\boldsymbol{m}}}$ and ϕ is a character of $\mathfrak{S}_{\underline{\boldsymbol{n}}}$, then

$$\psi \circ \phi = (\psi \otimes \phi) \Big\uparrow_{\mathfrak{S}_{\underline{\boldsymbol{m}}} \times \mathfrak{S}_{\underline{\boldsymbol{n}}}}^{\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}}. \tag{3.7}$$

Proof. A straightforward computation (e.g., using the definition of an induced module via cosets) shows that if V_i is a representation of a subgroup H_i of a group G_i, $1 \leqslant i \leqslant t$, then

$$\bigotimes_{i=1}^t (V_i \big\uparrow_{H_i}^{G_i}) \simeq (\bigotimes_{i=1}^t V_i) \big\uparrow_{\times_{i=1}^t H_i}^{\times_{i=1}^t G_i}. \tag{3.8}$$

We now have

$$\begin{aligned} \chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}} &= \bigotimes_{i=1}^t (\chi^{\lambda^i} \otimes \chi^{\mu^i}) \big\uparrow_{\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i}}^{\mathfrak{S}_{m_i+n_i}} \\ &= \left(\bigotimes_{i=1}^t (\chi^{\lambda^i} \otimes \chi^{\mu^i})\right) \Big\uparrow_{\bigtimes_{i=1}^t (\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i})}^{\bigtimes_{i=1}^t \mathfrak{S}_{m_i+n_i}} \text{ by (3.8).} \end{aligned}$$

Now we claim that

$$\left(\bigotimes_{i=1}^t (\chi^{\lambda^i} \otimes \chi^{\mu^i})\right) \Big\uparrow_{\bigtimes_{i=1}^t (\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i})}^{\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}} = (\chi^{\underline{\boldsymbol{\lambda}}} \otimes \chi^{\underline{\boldsymbol{\mu}}}) \big\uparrow_{\mathfrak{S}_{\underline{\boldsymbol{m}}} \times \mathfrak{S}_{\underline{\boldsymbol{n}}}}^{\mathfrak{S}_{\underline{\boldsymbol{m}}+\underline{\boldsymbol{n}}}}. \tag{3.9}$$

Write f for the character $\bigotimes_{i=1}^t (\chi^{\lambda^i} \otimes \chi^{\mu^i})$ of $\bigtimes_{i=1}^t (\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i})$ and g for the character $\chi^{\underline{\boldsymbol{\lambda}}} \otimes \chi^{\underline{\boldsymbol{\mu}}} = (\bigotimes_{i=1}^t \chi^{\lambda^i}) \otimes (\bigotimes_{i=1}^t \chi^{\mu^i})$ of $\mathfrak{S}_{\underline{\boldsymbol{m}}} \times \mathfrak{S}_{\underline{\boldsymbol{n}}}$. In view of the identifications made immediately preceding Proposition 3.6, we have, for $\sigma_i \in \mathfrak{S}_{m_i}, \tau_i \in \mathfrak{S}_{n_i}$, $1 \leqslant i \leqslant t$,

$$\begin{aligned} f((\sigma_1, \tau_1), \dots, (\sigma_t, \tau_t)) &= \prod_{i=1}^t (\chi^{\lambda^i} \otimes \chi^{\mu^i})(\sigma_i, \tau_i) = \prod_{i=1}^t \chi^{\lambda^i}(\sigma_i) \chi^{\mu^i}(\tau_i), \\ g((\sigma_1, \dots, \sigma_t), (\tau_1, \dots, \tau_t)) &= \chi^{\underline{\boldsymbol{\lambda}}}(\sigma_1, \dots, \sigma_t) \cdot \chi^{\underline{\boldsymbol{\mu}}}(\tau_1, \dots, \tau_t) = \prod_{i=1}^t \chi^{\lambda^i}(\sigma_i) \prod_{i=1}^t \chi^{\mu^i}(\tau_i), \end{aligned}$$

which shows that the two expressions coincide, establishing (3.6). Finally (3.7) follows by multilinearity. □

We illustrate these ideas with the case $t = 2$. Let ψ be a character of $\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}$, and ϕ a character of $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$. Then by definition of the induction product and Proposition 3.6, $\psi \circ \phi$ is the induced module $(\psi \otimes \phi) \big\uparrow_{(\mathfrak{S}_{m_1} \times \mathfrak{S}_{m_2}) \times (\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})}^{\mathfrak{S}_{m_1+n_1} \times \mathfrak{S}_{m_2+n_2}}$. Again, the key point established in the proof of (3.9) above is that $\psi \otimes \phi$ can be viewed as a character of $(\mathfrak{S}_{m_1} \times \mathfrak{S}_{n_1}) \times (\mathfrak{S}_{m_2} \times \mathfrak{S}_{n_2})$.

We will use the following important special case in the next section.

Corollary 3.7. *Let ψ be a character of the t-fold direct product $\mathfrak{S}_r^{\times t}$ and let ϕ be a character of the t-fold direct product $\mathfrak{S}_{n-r}^{\times t}$. Then*

$$(\psi \otimes \phi) \big\uparrow_{\mathfrak{S}_r^{\times t} \times \mathfrak{S}_{n-r}^{\times t}}^{\mathfrak{S}_n^{\times t}} = \psi \circ \phi, \text{ and hence } \operatorname{Pch}\left((\psi \otimes \phi) \big\uparrow_{\mathfrak{S}_r^{\times t} \times \mathfrak{S}_{n-r}^{\times t}}^{\mathfrak{S}_n^{\times t}}\right) = \operatorname{Pch}(\psi) \operatorname{Pch}(\phi).$$

4. The actions of $\mathfrak{S}_n^{\times t}$ and $\mathfrak{S}_n$ on the homology of $B_n^{(t)}$

In this section, we begin by generalizing [9, Theorem 4.1] to the t-fold Segre power of Boolean lattices $B_n^{(t)}$ for any t.

From work of Björner and Welker [5, Theorem 1, Corollary 9] it is known that the Segre product preserves the property of being (homotopy) Cohen-Macaulay, and hence by iteration, from the well-known fact that the Boolean lattice B_n is homotopy Cohen-Macaulay, we know that the same is true for $B_n^{(t)}$. In particular every open interval $(x, y) = ((x_1, \dots, x_t), (y_1, \dots, y_t))$, where $0 \leqslant |x_i| = r < s = |y_j| \leqslant n$ for all $1 \leqslant i, j \leqslant t$, has the homotopy type of a wedge of spheres in the top dimension $s - r - 2$, and hence the homology of (x, y) vanishes in all but this top dimension.

We will use the Whitney homology technique of [18] to determine the action of $\mathfrak{S}_n^{\times t}$ on the top homology module $\tilde{H}_{n-2}(B_n^{(t)})$.

Theorem 4.1 ([18, Lemma 1.1 and Theorem 1.2])**.** *Let Q be a bounded and ranked Cohen-Macaulay poset of rank n, and let G be a finite group of automorphisms of Q. Let $WH_r(Q)$ denote its rth Whitney homology, defined by*

$$WH_r(Q) = \bigoplus_{x \in Q,\, \mathrm{rank}(x) = r} \tilde{H}_{r-2}(\hat{0}, x).$$

Then $WH_r(Q)$ is a G-module, and as virtual G-modules one has the identity

$$\tilde{H}_{n-2}(Q) = \sum_{r=0}^{n-1} (-1)^{n-r+1} WH_r(Q).$$

Theorem 4.2. *Fix $t \geqslant 1$. Set $\beta_0^{(t)} = 1$, and for $n \geqslant 1$ denote by $\beta_n^{(t)}$ the product Frobenius characteristic $\mathrm{Pch}(\tilde{H}_{n-2}(B_n^{(t)}))$ of the top homology of the t-fold Segre power $B_n^{(t)}$. Then $\beta_n^{(t)}$ satisfies the recurrence*

$$\sum_{i=0}^{n} (-1)^i \beta_i^{(t)} \prod_{j=1}^{t} h_{n-i}(X^j) = 0.$$

Proof. When $t = 1$, $B_n^{(t)} = B_n$ and $\beta_n^{(1)} = \beta_n$ is simply the Frobenius characteristic of the top homology of the Boolean lattice, so $\beta_n^{(1)} = \beta_n = e_n$. In this case, the above equation reduces to the well-known symmetric function identity $e_n = \sum_{i=0}^{n-1} (-1)^{n-i-1} e_i h_{n-i}$.

We apply Theorem 4.1 to the Cohen-Macaulay poset $Q = B_n^{(t)}$, which has rank n. We need to compute the $\mathfrak{S}_n^{\times t}$-module structure of the Whitney homology $WH_r(Q)$. Let x_0 be the t-tuple $([r], [r], \dots, [r])$, where $[r] = \{1, 2, \dots, r\}$; x_0 has rank r in Q. The stabiliser of x_0 at rank r, and hence of the interval $(\hat{0}, x_0)$, is $(\mathfrak{S}_r \times \mathfrak{S}_{n-r})^{\times t}$. The orbit of x_0 under the action of $\mathfrak{S}_n^{\times t}$ generates all other elements at rank r, and hence $WH_r(Q)$ is $\mathfrak{S}_n^{\times t}$-isomorphic to a module induced from the direct product $(\mathfrak{S}_r \times \mathfrak{S}_{n-r})^{\times t}$. The copies of $\mathfrak{S}_{n-r}$ act trivially on x_0. More precisely, if $((\sigma_1, \tau_1), \dots, (\sigma_t, \tau_t))$ is an element of $(\mathfrak{S}_r \times \mathfrak{S}_{n-r})^{\times t}$, then the t-tuple $(\sigma_1, \dots, \sigma_t)$ acts like the representation $\tilde{H}_{r-2}(B_r^{(t)})$, with product Frobenius characteristic $\beta_r^{(t)}$, and the t-tuple $(\tau_1 \dots, \tau_t)$ acts trivially, i.e., like the representation $\bigotimes_{i=1}^{t} \mathbb{1}_{\mathfrak{S}_{n-r}}$.

This is the induction product of Definition 3.3, and by Proposition 3.6 and Corollary 3.7, the induced module $WH_{r-2}(Q)$ coincides with $\tilde{H}_{r-2}(B_r^{(t)}) \circ (\mathbb{1}_{\mathfrak{S}_{n-r}})^{\otimes t}$, whose product Frobenius characteristic is precisely $\beta_r^{(t)} \prod_{j=1}^{t} h_{n-r}(X^j)$. □

The dimension of the rth Whitney homology is given by

$$\dim \left(\tilde{H}_{r-2}(B_r^{(t)}) \right) \binom{n}{r}^t = (-1)^r \mu(B_r^{(t)}) \binom{n}{r}^t = w_r^{(t)} \binom{n}{r}^t,$$

where $w_r^{(t)}$ is the number defined in Proposition 2.5. Hence Theorem 4.2 is the group-equivariant version of (2.5).

Corollary 4.3. *Let $n \geqslant 1$. In the $\mathfrak{S}_n^{\times t}$-module $\tilde{H}_{n-2}(B_n^{(t)})$, the multiplicity of*

1. *the trivial representation $\bigotimes_{j=1}^{t} \chi^{(n)}$ is zero unless $n = 1$, in which case it is 1;*

2. *the sign representation $\bigotimes_{j=1}^{t} \chi^{(1^n)}$ is 1 for all $n \geqslant 1$.*

Proof. We deduce this from the recurrence of Theorem 4.2, which we rewrite as

$$\beta_n^{(t)} = \sum_{i=0}^{n-1} (-1)^{n-1-i} \beta_i^{(t)} \prod_{j=1}^{t} h_{n-i}(X^j). \tag{4.1}$$

Now $\prod_{j=1}^t h_n(X^j)$ and $\prod_{j=1}^t e_n(X^j)$ are respectively the product Frobenius characteristics of the trivial and sign representations of $\mathfrak{S}_n^{\times t}$. We use the inner product (3.4) to compute the multiplicities. We have, for all n, using the Kronecker delta $\delta_{n,1}$ which equals 1 if $n = 1$ and zero otherwise:

$$\langle \prod_{j=1}^t h_n(X^j), \prod_{j=1}^t h_n(X^j) \rangle = \prod_{j=1}^t \langle h_n(X^j), h_n(X^j) \rangle = 1,$$
$$\langle \prod_{j=1}^t h_n(X^j), \prod_{j=1}^t e_n(X^j) \rangle = \prod_{j=1}^t \langle h_n(X^j), e_n(X^j) \rangle = \delta_{n,1}.$$

We compute

$$\begin{aligned}
\langle \beta_r^{(t)} \prod_{j=1}^t h_{n-r}(X^j), \prod_{j=1}^t h_n(X^j) \rangle &= \langle \beta_r^{(t)} \prod_{j=1}^t h_{n-r}(X^j), \prod_{j=1}^t h_r(X^j) \prod_{j=1}^t h_{n-r}(X^j) \rangle \\
&= \langle \beta_r^{(t)}, \prod_{j=1}^t h_r(X^j) \rangle \cdot \langle \prod_{j=1}^t h_{n-r}(X^j), \prod_{j=1}^t h_{n-r}(X^j) \rangle \\
&= \langle \beta_r^{(t)}, \prod_{j=1}^t h_r(X^j) \rangle \prod_{j=1}^t \langle h_{n-r}(X^j), h_{n-r}(X^j) \rangle.
\end{aligned}$$

Since $\langle \beta_r^{(t)}, \prod_{j=1}^t h_r(X^j) \rangle = 1$ for $r = 0, 1$, by induction the recurrence (4.1) gives Item (1).
Similarly, for $0 \leqslant r \leqslant n-1$, we have

$$\langle \beta_r^{(t)} \prod_{j=1}^t h_{n-r}(X^j), \prod_{j=1}^t e_n(X^j) \rangle = \langle \beta_r^{(t)}, \prod_{j=1}^t e_r(X^j) \rangle \delta_{n-r,1}.$$

Hence the recurrence (4.1) now gives, for $n \geqslant 3$, $\langle \beta_n^{(t)}, \prod_{j=1}^t e_n(X^j) \rangle = \langle \beta_{n-1}^{(t)}, \prod_{j=1}^t e_{n-1}(X^j) \rangle$.

By direct calculation, the multiplicity of the sign representation is zero in $\beta_n^{(t)}$ when $n = 0, 1$, and it is 1 for $n = 2$. Item (2) now follows from (4.1) by induction. ☐

Note the agreement with the case $t = 1$, when $B_n^{(t)}$ is simply the Boolean lattice B_n, and so its homology carries the sign representation of $\mathfrak{S}_n$.

Example 4.4. Let $t = 2$. We use the recurrence to compute some of the symmetric functions $\beta_n^{(2)}$ in two sets of variables X^1, X^2. We have $\beta_0^{(2)} = 1$ and $\beta_1^{(2)} = h_1(X^1)h_1(X^2)$, the product characteristic of the trivial representation of $\mathfrak{S}_1 \times \mathfrak{S}_1$. Then the recurrence gives $\beta_2^{(2)} = \beta_1^{(2)}(h_1(X^1)h_1(X^2)) - h_2(X^1)h_2(X^2) = h_1^2(X^1)h_1^2(X^2) - h_2(X^1)h_2(X^2)$, and so

$$\beta_2^{(2)} = e_2(X^1)h_2(X^2) + h_2(X^1)e_2(X^2) + e_2(X^1)e_2(X^2).$$

Similarly,

$$\begin{aligned}
\beta_3^{(2)} &= \beta_2^{(2)} h_1(X^1)h_1(X^2) - \beta_1^{(2)} h_2(X^1)h_2(X^2) + \beta_0^{(2)} h_3(X^1)h_3(X^2) \\
&= h_1^3(X^1)h_1^3(X^2) - 2h_2(X^1)h_2(X^2)h_1(X^1)h_1(X^2) + h_3(X^1)h_3(X^2) \\
&= (h_3(X^1)e_3(X^2) + e_3(X^1)h_3(X^2)) + e_3(X^1)e_3(X^2) + s_{(2,1)}(X^1)s_{(2,1)}(X^2) \\
&\quad + 2\left(s_{(2,1)}(X^1)e_3(X^2) + e_3(X^1)s_{(2,1)}(X^2)\right).
\end{aligned}$$

By definition of the product Frobenius characteristic and the fact that $\tilde{H}_{n-1}(B_n^{(2)})$ is a true $(\mathfrak{S}_n \times \mathfrak{S}_n)$-module, it follows that $\beta_n^{(2)}$ must have a positive expansion in the basis $\{s_\lambda(X^1)s_\mu(X^2) : \lambda, \mu \vdash n\}$. This is confirmed by the above examples.

Definition 4.5. Define $Z_i^{(t)} := \prod_{j=1}^t h_i(X^j)$, and define the degree of $Z_i^{(t)}$ to be i.

Also define, for each $\lambda \vdash n$, $Z_\lambda^{(t)} = \prod_j Z_{\lambda_j}^{(t)}$. Thus $Z_\lambda^{(t)} = \prod_{j=1}^t h_\lambda(X^j)$.

Lemma 4.6. *The symmetric function $\beta_n^{(t)}$ can be written as a polynomial of homogeneous degree n in $\{Z_i^{(t)} : 1 \leqslant i \leqslant n\}$ such that for $n \geqslant 2$, the sum of the coefficients is 0.*

Proof. The $\beta_n^{(t)}$ are defined recursively by $\beta_1^{(t)} = Z_1^{(t)}$, $\beta_0^{(t)} = 1$ and

$$\beta_n^{(t)} = \sum_{i=0}^{n-1} (-1)^{n-1-i} \beta_i^{(t)} Z_{n-i}^{(t)}. \tag{4.2}$$

The statement follows by induction using Corollary 4.3, since each monomial in $Z_i^{(t)}$ contributes 1 to the multiplicity of the trivial representation, while the contribution from $\beta_n^{(t)}$ is 0 except for $\beta_1^{(t)}$ and $\beta_0^{(t)}$, where the contribution is 1. □

Recall the recurrence for the Frobenius characteristic of the top homology of the Boolean lattice, namely, the symmetric function identity in $\Lambda_n(X)$ (in one set of variables X) given by [11]

$$e_n = \sum_{i=0}^{n-1} (-1)^{n-1-i} e_i h_{n-i}. \tag{4.3}$$

This is equivalent to the generating function identity [11]

$$\sum_{n\geqslant 0} u^n e_n = (\sum_{u\geqslant 0} u^n (-1)^n h_n)^{-1}. \tag{4.4}$$

Definition 4.7. Define a map $\Phi_t : \Lambda(X) \to \bigotimes_{j=1}^t \Lambda(X^j)$ by setting

$$\Phi_t(h_n) := \prod_{j=1}^t h_n(X^j) = Z_n^{(t)},$$

and extending multiplicatively and linearly to all of $\Lambda(X)$. This is well defined since the h_n are algebraic generators for $\Lambda(X)$.

Proposition 4.8. *The map $\Phi_t : \Lambda(X) \to \bigotimes_{j=1}^t \Lambda(X^j)$ is an injective degree-preserving algebra homomorphism such that*

$$\Phi_t(e_n) = \beta_n^{(t)}.$$

Moreover, $\{\beta_n^{(t)}\}_n$ is an algebraically independent set.

Proof. Since the $\{h_\lambda(X^j)\}_\lambda$ are linearly independent in $\Lambda_n(X^j)$, the set $\{Z_\lambda^{(t)}\}_{\lambda\vdash n}$ is linearly independent in $\bigotimes_{j=1}^t \Lambda_n(X^j)$, and so the $Z_n^{(t)}$ are algebraically independent in $\bigotimes_{j=1}^t \Lambda(X^j)$. The h_n, $n \geqslant 1$, are algebraically independent generators for the ring $\Lambda(X)$ [11]. Thus Φ_t extends to an injective degree-preserving algebra homomorphism $\Lambda(X) \to \bigotimes_{j=1}^t \Lambda(X^j)$, so that $\Phi_t(\Lambda_n(X)) \subset \bigotimes_{j=1}^t \Lambda_n(X^j)$.

In particular $\Phi_t(h_\lambda) = Z_\lambda^{(t)}$. Applying Φ_t to (4.3) gives precisely the recurrence (4.2), with the same initial conditions, since $\Phi_t(e_1) = \Phi_t(h_1) = Z_1^{(t)}$, $\Phi_t(e_0) = 1 = Z_0^{(t)}$. Hence $\Phi_t(e_n) = \beta_n^{(t)}$.

The injectivity of Φ_t now implies that the $\beta_n^{(t)}$ must be algebraically independent. □

Example 4.9. To illustrate the workings of the map Φ_t, we compute $\Phi_2(s_{(n-1,1)})$ for $t = 2$. Since $s_{(n-1,1)} = h_{n-1}h_1 - h_n$, we have, for the two sets of variables X^1, X^2 as in Example 4.4,

$$\begin{aligned}
\Phi_2(s_{(n-1,1)}) &= \Phi_2(h_{n-1}h_1) - \Phi_2(h_n)\\
&= \Phi_2(h_{n-1})\Phi_2(h_1) - \Phi_2(h_n)\\
&= h_{n-1}(X^1)h_{n-1}(X^2)\, h_1(X^1)h_1(X^2) - h_n(X^1)h_n(X^2)\\
&= (s_{(n-1,1)}(X^1) + s_{(n)}(X^1))\,(s_{(n-1,1)}(X^2) + s_{(n)}(X^2)) - h_n(X^1)h_n(X^2)\\
&= s_{(n-1,1)}(X^1)s_{(n-1,1)}(X^2) + (s_{(n-1,1)}(X^1)s_{(n)}(X^2) + s_{(n-1,1)}(X^2)s_{(n)}(X^1)).
\end{aligned}$$

For $t = 3$ we would similarly have, for the three sets of variables X^i, $i = 1, 2, 3$,

$$\begin{aligned}
\Phi_3(s_{(n-1,1)}) &= \Phi_3(h_{n-1})\Phi_3(h_1) - \Phi_3(h_n)\\
&= h_{n-1}(X^1)h_{n-1}(X^2)h_{n-1}(X^3)\, h_1(X^1)h_1(X^2)h_1(X^3) - h_n(X^1)h_n(X^2)h_n(X^3)\\
&= s_{(n-1,1)}(X^1)s_{(n-1,1)}(X^2)s_{(n-1,1)}(X^3) + (s_{(n-1,1)}(X^1)s_{(n)}(X^2)s_{(n-1,1)}(X^3)\\
&+ s_{(n-1,1)}(X^2)s_{(n)}(X^1)s_{(n-1,1)}(X^3) + s_{(n)}(X^1)s_{(n)}(X^2)s_{(n-1,1)}(X^3)\\
&+ s_{(n-1,1)}(X^1)s_{(n-1,1)}(X^2)s_{(n)}(X^3)\\
&+ s_{(n-1,1)}(X^1)s_{(n)}(X^2)s_{(n)}(X^3) + s_{(n-1,1)}(X^2)s_{(n)}(X^1)s_{(n)}(X^3).
\end{aligned}$$

Example 4.9 is a special case of Item (1) of Proposition 4.10 below. Using the Jacobi-Trudi expansion [11, Eqns. (3,4), (3,5)] of the Schur function s_λ in the basis of homogeneous symmetric functions h_μ, from Definition 4.5 we obtain

Proposition 4.10. *For $\lambda \vdash n$, let λ' denote the conjugate partition of λ. Then*

1. $\Phi_t(s_\lambda) = \det(Z^{(t)}_{\lambda_i - i + j})_{1 \leqslant i,j \leqslant \ell(\lambda)}$, *where* $Z^{(t)}_0 = 1$ *and we set* $Z^{(t)}_m = 0$ *if* $m < 0$.
2. $\Phi_t(s_{\lambda'}) = \det(\beta^{(t)}_{\lambda_i - i + j})_{1 \leqslant i,j \leqslant \ell(\lambda)}$, *where* $\beta^{(t)}_0 = 1$ *and we set* $\beta^{(t)}_m = 0$ *if* $m < 0$.

Proof. The two items are clear from the definitions. □

For an integer partition $\lambda \vdash n$ of n, we write $\ell(\lambda)$ for the total number of parts of λ, and $m_i(\lambda)$ for the number of parts equal to i. Also, let $K_{\mu,\nu}$ be the Kostka number, i.e., the number of semistandard Young tableaux of shape $\mu \vdash n$ and weight $\nu \vdash n$ [11, Eqn. (5.12)]. In particular, $f^\mu = K_{\mu,(1^n)}$ is the number of standard Young tableaux of shape μ.

Definition 4.11. For $\lambda \vdash n$ with $m_i(\lambda)$ parts of size i and number of parts $\ell(\lambda)$, define c_λ to be the integer

$$c_\lambda = (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_i m_i(\lambda)!}.$$

The integers c_λ play an important role in the irreducible decomposition of $\beta^{(t)}_n$, as we prove next. First, we record a fact that will be needed in the proof.

Lemma 4.12. *The multiplicity of the $\mathfrak{S}_n^{\times t}$-irreducible indexed by the t-tuple of partitions $\underline{\boldsymbol{\mu}} = (\mu^1, \ldots, \mu^t)$, $\mu^j \vdash n$, $1 \leqslant j \leqslant t$, in the $\mathfrak{S}_n^{\times t}$-module with product Frobenius characteristic $Z^{(t)}_\lambda$ is*

$$\textstyle\prod_{j=1}^t K_{\mu^j,\lambda}.$$

Proof. We use the well-known expansion [11] $h_\lambda = \sum_{\mu \vdash n} K_{\mu,\lambda}\, s_\mu$. This gives, for each set of variables X^j, $1 \leqslant j \leqslant t$, $h_\lambda(X^j) = \sum_{\mu^j \vdash n} K_{\mu^j,\lambda}\, s_{\mu^j}(X^j)$.

Since by definition, $Z^{(t)}_\lambda = \prod_{j=1}^t h_\lambda(X^j)$, the result follows. □

If $\{u_\lambda\}, \{v_\lambda\}$ are two sets of bases for the ring of symmetric functions $\Lambda_n(X)$, as in [11, Ch1, Sec 6] we write $\mathcal{M}(u,v)$ for the transition matrix from the basis $\{u_\lambda\}$ to the basis $\{v_\lambda\}$. More precisely,

$$u_\lambda = \sum_\mu \mathcal{M}(u,v)_{\lambda,\mu}\, v_\mu.$$

Theorem 4.13. *For the product Frobenius characteristic $\beta^{(t)}_n$ of the top homology of $B^{(t)}_n$, we have:*

1. $\beta^{(t)}_n = \sum_{\lambda \vdash n} c_\lambda Z^{(t)}_\lambda$.
2. $\sum_{n \geqslant 0} u^n \beta^{(t)}_n = (\sum_{u \geqslant 0} u^n (-1)^n Z^{(t)}_n)^{-1}$.
3. *The multiplicity of the $\mathfrak{S}_n^{\times t}$-irreducible indexed by the t-tuple of partitions $\underline{\boldsymbol{\mu}} = (\mu^1, \ldots, \mu^t)$, $\mu^j \vdash n$, $1 \leqslant j \leqslant t$, in $\tilde{H}_{n-2}(B^{(t)}_n)$ equals*
$$c^t_{\underline{\boldsymbol{\mu}}} = \sum_{\lambda \vdash n} c_\lambda \prod_{j=1}^t K_{\mu^j,\lambda}.$$
4. *Let $\mathcal{M}(s,h)$ denote the transition matrix from the basis of Schur functions to the basis of homogeneous symmetric functions.*

   The multiplicity of the $\mathfrak{S}_n^{\times t}$-irreducible indexed by the t-tuple of partitions $(\mu^1, \ldots, \mu^t)$, $\mu^i \vdash n$, $1 \leqslant i \leqslant t$, in the (possibly virtual) module whose product Frobenius characteristic is
$$\langle \Phi_t(s_\lambda), \prod_{j=1}^t s_{\mu^j}(X^j) \rangle = \sum_{\nu \vdash n} \mathcal{M}(s,h)_{\lambda,\nu} \prod_{j=1}^t K_{\mu^j,\nu}. \tag{4.5}$$

Proof. Let $\mathcal{M}(e,h)$ denote the transition matrix from the basis of elementary symmetric functions to the basis of homogeneous symmetric functions [11, Chapter 1, Section 6]. Then

$$e_n = \sum_{\lambda \vdash n} \mathcal{M}(e,h)_{(n),\lambda} h_\lambda,$$

and hence, applying Φ_t and using Proposition 4.8,

$$\beta_n^{(t)} = \sum_{\lambda \vdash n} \mathcal{M}(e,h)_{(n),\lambda} Z_\lambda^{(t)}.$$

Since $\mathcal{M}(s,h)$ is the transition matrix from the basis of Schur functions to the basis of homogeneous symmetric functions, its inverse $\mathcal{M}(h,s)$ is the transpose of the Kostka matrix $K = (K_{\lambda,\mu})$, and $\mathcal{M}(e,h)_{(n),\lambda} = \mathcal{M}(s,h)_{(1^n),\lambda} = (K^{-1})_{\lambda,(1^n)}$. Eğecioğlu and Remmel [7, Corollary 1, Part (iv)] give an explicit formula for the signed numbers $(K^{-1})_{\lambda,(1^n)}$, namely, $(K^{-1})_{\lambda,(1^n)} = (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_i m_i(\lambda)!}$. These are precisely the numbers c_λ in Definition 4.11. This establishes (1).

Similarly, applying Φ_t to (4.4) establishes (2).

Item (3) is now immediate from (1) by applying Lemma 4.12.

Item (4) results from applying Φ_t to the identity

$$s_\lambda = \sum_{\nu \vdash n} \mathcal{M}(s,h)_{\lambda,\nu} h_\nu,$$

and using Lemma 4.12 again. □

The following example illustrates Equation (4.5).

Example 4.14. Let $\lambda = (3,2,1)$. The Jacobi-Trudi determinant expands to give $s_\lambda = h_{321} - h_{33} - h_{411} + h_{51}$. Applying Φ_t, we obtain, for the multiplicity of the t-tuple of partitions $(\mu^1, \ldots, \mu^t)$ of 6 in the module with product Frobenius characteristic $\Phi_t(s_\lambda)$, the expression

$$\langle \Phi_t(s_\lambda), \prod_{j=1}^t s_{\mu^j}(X^j) \rangle = \prod_{j=1}^t K_{\mu^j,321} - \prod_{j=1}^t K_{\mu^j,33} - \prod_{j=1}^t K_{\mu^j,411} + \prod_{j=1}^t K_{\mu^j,51}.$$

Example 4.15. Computing $\beta_3^{(2)}$ using the above formulas gives $\beta_3^{(2)} = Z_{(3)}^{(2)} - 2Z_{(2,1)}^{(2)} + Z_{(1^3)}^{(2)}$, since $e_3 = h_3 - 2h_2h_1 + h_1^3$. Since $h_2h_1 = s_{(3)} + s_{(2,1)}$ and $h_1^3 = s_{(3)} + 2s_{(2,1)} + s_{(1^3)}$, we obtain

$$\begin{aligned}\beta_3^{(2)} &= Z_{(3)}^{(2)} - 2Z_{(2,1)}^{(2)} + Z_{(1^3)}^{(2)} \\ &= s_{(3)}(X^1)s_{(3)}(X^2) - 2(s_{(3)}(X^1) + s_{(2,1)}(X^1))(s_{(3)}(X^2) + s_{(2,1)}(X^2)) \\ &\quad + (s_{(3)}(X^1) + 2s_{(2,1)}(X^1) + s_{(1^3)}(X^1))(s_{(3)}(X^2) + 2s_{(2,1)}(X^2) + s_{(1^3)}(X^2)),\end{aligned}$$

which, upon simplification, agrees with Example 4.4.

Item (1) generalises the result of Eğecioğlu and Remmel [7, Corollary 1, Part (iv)] asserting that

$$e_n = \sum_{\lambda \vdash n} c_\lambda h_\lambda. \tag{4.6}$$

Corollary 4.16. *Fix $t \geqslant 1$. Then the following hold.*

1. *For any fixed t-tuple of partitions $(\mu_1, \ldots, \mu_t)$ of n, the sum*

$$c_{\underline{\mu}}^t = \sum_{\lambda \vdash n} c_\lambda \prod_{j=1}^t K_{\mu^j,\lambda} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_i m_i(\lambda)!} \prod_{j=1}^t K_{\mu^j,\lambda}$$

   is a nonnegative integer.

2. *For $n \geqslant 2$, $\sum_{\lambda \vdash n} c_\lambda = 0$.*

3. $\sum_{\lambda \vdash n} c_\lambda \frac{n!}{\prod_{i \geqslant 1} \lambda_i!} = 1.$

Proof. The first part follows from Theorem 4.13, Item (3). Items (2) and (3) are obtained from (4.6), by respectively taking the multiplicity of the trivial representation, and dimensions. □

The results of Corollary 4.3 can now be recovered from Item (1) of Theorem 4.13, since $\langle h_\lambda, e_n\rangle = \delta_{\lambda,1^n}$ and $\langle h_\lambda, h_n\rangle = 1$ for all n.

Remark 4.17. In contrast to Corollary 4.16, the expressions in Item (4) of Theorem 4.13 may be negative integers; for arbitrary λ, the $\mathfrak{S}_n^{\times t}$-module whose product Frobenius characteristic is $\Phi_t(s_\lambda)$ may NOT be a true module. Using SageMath reveals the following counterexamples.

1. If $\lambda = 322$, then for the 2-tuple $(\mu^1, \mu^2) = (43, 61)$, the multiplicity of (μ^1, μ^2) in the module whose product Frobenius characteristic is $\Phi_2(s_{322})$ is -1. Moreover, this implies that the module $\Phi_t(s_{322})$ does not correspond to a true module for any $t \geqslant 2$.

2. If $\lambda = 2221$, the multiplicity of the 2-tuple $(\mu^1, \mu^2) = (331, 511)$ in the module corresponding to $\Phi_2(s_{2221})$ is -2.

However, we have the following.

Proposition 4.18. *Let $\lambda \vdash n$. Then $\Phi_t(h_\lambda)$ and $\Phi_t(e_\lambda)$ are the product Frobenius characteristics of true $\mathfrak{S}_n^{\times t}$-modules.*

In addition, $\Phi_t(s_\lambda)$ is the product Frobenius characteristic of a true $\mathfrak{S}_n^{\times t}$-module if λ has at most two parts.

Proof. From Definition 4.7, $\Phi_t(h_n)$ is the characteristic of the trivial $\mathfrak{S}_n^{\times t}$-module. Since Φ_t is defined multiplicatively, we have $\Phi_t(h_\lambda) = \prod_j \Phi_t(h_{\lambda_j}) = Z_\lambda^{(t)}$, which is the characteristic of a true module.

Theorem 4.13 tells us that $\Phi_t(e_n)$ is the product Frobenius characteristic $\beta_n^{(t)}$ of the top homology module of $B_n^{(t)}$. Again we have $\Phi_t(e_\lambda) = \prod_j \Phi_t(e_{\lambda_j}) = \prod_j \beta_{\lambda_j}^{(t)}$, and the result follows for $\Phi_t(e_\lambda)$.

Now let $\lambda = (a, b)$ where $a \geqslant b \geqslant 1$. Consider the t-tuple $(\mu^1, \ldots, \mu^t)$ of partitions of n. The coefficient of $\prod_{j=1}^t s_{\mu^j}(X^j)$ in $\Phi_t(s_\lambda)$ is given by

$$\prod_{j=1}^t K_{\mu^j,(a,b)} - \prod_{j=1}^t K_{\mu^j,(a+1,b-1)}. \tag{4.7}$$

We claim that for any partition μ of $a+b$, $K_{\mu,(a,b)} \geqslant K_{\mu,(a+1,b-1)}$.

Recall that the Kostka number $K_{\mu,\nu}$ equals the inner product $\langle s_\mu, h_\nu\rangle$. Hence, by the Jacobi-Trudi identity,

$$K_{\mu,(a,b)} - K_{\mu,(a+1,b-1)} = \langle s_\mu, h_a h_b - h_{a+1}h_{b-1}\rangle = \langle s_\mu, s_{(a,b)}\rangle,$$

so that the difference $K_{\mu,(a,b)} - K_{\mu,(a+1,b-1)}$ is 1 if $\mu = (a, b)$, and 0 otherwise. The claim follows.

This shows that, in (4.7), every factor in the second product is less than or equal to the corresponding factor in the first product. Consequently, the coefficient (4.7) is nonnegative, finishing the proof. □

Remark 4.19. Take dimensions in Item (3) of Theorem 4.13. We obtain the following curious enumerative identity.

$$w_n^{(t)} = \sum_{\lambda \vdash n} \sum_{\substack{\mu^i \vdash n \\ 1 \leqslant i \leqslant t}} (-1)^{n-\ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda), m_(\lambda), \ldots} \prod_{j=1}^t K_{\mu^j,(1^n)} K_{\mu^j,\lambda}.$$

Table 1 is compiled from OEIS A212855.

$t\backslash n$	0	1	2	3	4	5
$t=1$	1	1	1	1	1	1
2	1	1	3	19	211	3651
3	1	1	7	163	8983	966751
4	1	1	15	1135	271375	
5	1	1	31	7291	7225951	
6	1	1	63	45199	182199871	

Table 1: The numbers $w_n^{(t)}$ for $0 \leqslant n \leqslant 5$, $1 \leqslant t \leqslant 6$

4.1 The diagonal action of $\mathfrak{S}_n$ on $B_n^{(t)}$

Since the symmetric group $\mathfrak{S}_n$ itself acts diagonally on the t-fold Segre power $B_n^{(t)}$, we can ask for a description of this diagonal action. Recall the definition of $c^t_{\underline{\boldsymbol{\mu}}}$ from Theorem 4.13. Also let $g^\lambda_{\underline{\boldsymbol{\mu}}}$ denote the Kronecker coefficient [11] $\langle \chi^\lambda, \prod_{j=1}^t \chi^{\mu_j}\rangle$, that is, the multiplicity of the $\mathfrak{S}_n$-irreducible χ^λ in the tensor product of $\mathfrak{S}_n$-irreducibles χ^{μ^j}, $1 \leqslant j \leqslant t$. Finally, recall [11] that $*$ denotes the internal product in the ring of symmetric functions $\Lambda^n(X)$ in a single set of variables X.

From Theorem 4.13 we deduce the following.

Theorem 4.20. *For the diagonal $\mathfrak{S}_n$-action on $\tilde{H}_{n-2}(B_n^{(t)})$:*

1. *The (ordinary) Frobenius characteristic is this signed sum of characteristics of permutation modules:*
$$\mathrm{ch}\ \tilde{H}_{n-2}(B_n^{(t)}) = \sum_{\lambda \vdash n} c_\lambda\,(h_\lambda)^{*t},$$
writing $(h_\lambda)^{*t}$ for $\underbrace{h_\lambda * h_\lambda * \cdots * h_\lambda}_{t}$.

2. *The multiplicity of the $\mathfrak{S}_n$-irreducible indexed by λ is $\sum_{\underline{\boldsymbol{\mu}}} c^t_{\underline{\boldsymbol{\mu}}} g^\lambda_{\underline{\boldsymbol{\mu}}}$, where the sum is over all t-tuples of partitions $\underline{\boldsymbol{\mu}} = (\mu^1, \dots, \mu^t)$, $\mu^j \vdash n$, $1 \leqslant j \leqslant t$. Equivalently,*
$$\mathrm{ch}\ \tilde{H}_{n-2}(B_n^{(t)}) = \sum_{\underline{\boldsymbol{\mu}}} c^t_{\underline{\boldsymbol{\mu}}} g^\lambda_{\underline{\boldsymbol{\mu}}}\, s_\lambda = \sum_{\underline{\boldsymbol{\mu}}} c^t_{\underline{\boldsymbol{\mu}}}\, s_{\mu^1} * \cdots * s_{\mu^t}.$$

3. *The trace of an element $\sigma \in \mathfrak{S}_n$ is $\sum_{\lambda \vdash n} c_\lambda (\chi^{M^\lambda}(\sigma))^t$, where χ^{M^λ} is the character of the permutation module corresponding to h_λ.*
It is also equal to $\sum_{\underline{\boldsymbol{\mu}}} c^t_{\underline{\boldsymbol{\mu}}} \prod_{j=1}^t \chi^{\mu^j}(\sigma)$.

Proof. The first item follows directly from Item (1) of Theorem 4.13.

Item (3) of Theorem 4.13 gives us the following decomposition for the $\mathfrak{S}_n^{\times t}$-action on $\tilde{H}_{n-2}(B_n^{(t)})$:
$$\sum_{\underline{\boldsymbol{\mu}}} c^t_{\underline{\boldsymbol{\mu}}} \bigotimes_{j=1}^t \chi^{\mu^j}.$$

The action of $\mathfrak{S}_n$ is obtained by restricting the $\mathfrak{S}_n^{\times t}$-action to its diagonal subgroup which is isomorphic to $\mathfrak{S}_n$, giving the following for the trace of $\sigma \in \mathfrak{S}_n$ on $\tilde{H}_{n-2}(B_n^{(t)})$:
$$\sum_{\underline{\boldsymbol{\mu}}} c^t_{\underline{\boldsymbol{\mu}}} \bigotimes_{j=1}^t \chi^{\mu^j}(\sigma).$$

By definition of the internal product $*$ and Kronecker coefficients, $\mathrm{ch}(\bigotimes_{j=1}^t \chi^{\mu^j}) = s_{\mu^1} * \cdots * s_{\mu^t}$. The remaining items now follow. ☐

Remark 4.21. By a theorem of Eğecioğlu and Remmel [8, p. 109-111, Part (9)] (see also [2, Transition Matrices]), the coefficient of the power sum p_μ in the expansion of h_λ is $|OB_{\mu,\lambda}|/z_\mu$, where $OB_{\mu,\lambda}$ counts the number of *ordered μ-brick tabloids of shape* λ. We refer the reader to [8, pp.108-111] for definitions. It follows that the value of the character of χ^{M^λ} on the conjugacy class μ is the nonnegative integer $|OB_{\mu,\lambda}|$, giving the expression
$$\sum_{\lambda \vdash n} c_\lambda\,(|OB_{\mu,\lambda}|)^t$$
for the character value of $\tilde{H}_{n-2}(B_n^{(t)})$ on the conjugacy class indexed by μ.

Taking dimensions in Item (1) of Theorem 4.20, we obtain:

Corollary 4.22. *For $t \geqslant 1$, $w_n^{(t)} = \sum_{\lambda \vdash n} c_\lambda \left(\frac{n!}{\prod_{i\geqslant 1} \lambda_i!}\right)^t$.*

Proposition 4.23. *We have the following explicit formulas for $n = 2$ and $n = 3$.*

1. *For $n=2$: $\tilde{H}_0(B_2^{(t)}) = 2^{t-1}\chi^{(1^2)} + (2^{t-1}-1)\chi^{(2)}$, in agreement with the case $t=1$. In fact, it is e-positive:*

$$\mathrm{ch}\ \tilde{H}_0(B_2^{(t)}) = (2^{t-1}-1)e_{(1,1)} + e_2.$$

The dimension is $w_2^{(t)} = 2^t - 1$.

2. *For $n=3$: $\tilde{H}_1(B_3^{(t)}) = 2^{t-1}\chi^{(1^3)} + (2^{t-1}-1)\chi^{(3)} + (2^t-2)\sum_{r=1}^{t}\binom{t}{r}(\chi^{(2,1)})^{\otimes r}$, again agreeing with the case $t=1$, and it is e-positive:*

$$\mathrm{ch}\ \tilde{H}_1(B_3^{(t)}) = (6^{t-1}-3^{t-1})e_{(1,1,1)} + e_3.$$

The dimension is $w_3^{(t)} = 6(6^{t-1}-3^{t-1})+1$. (See **OEIS** *A248225, A127222 for the sequence $\{6^t-3^t\}$.)*

3. *For $n=4$: $\mathrm{ch}\ \tilde{H}_2(B_4^{(t)}) = e_4 - (2^{t-1}-1)e_{2,1,1} + (2^{t-1}-1)e_{2,2} + \gamma_4\, e_{1,1,1,1}$ where $\gamma_4(t) = \frac{(4^{t-1}-1)}{3} + \frac{3(6^{t-2}-2^{t-2})}{2} - 17\cdot 12^{t-2} + 2^{t-2} + 24^{t-1}$.*

   The dimension is $w_4^{(t)} = 1 - 6(2^{t-1}-1) + \gamma_4(t)$.

Proof. We sketch the proof for Item (3), omitting details of the brute-force computations involving internal products. The case $n=2$ is also easily computed directly, since $B_2^{(t)}$ is a rank 2 poset.

We compute the right-hand side of Item (1) of Theorem 4.20. First observe that the definition of c_λ gives $c_{(4)} = -1, c_{(31)} = 2, c_{(22)} = 1, c_{(211)} = -3$ and $c_{(1^4)} = 1$.

We repeatedly use the fact that for H a subgroup of a group G and V, W respectively H- and G-modules, one has the G-module isomorphism $V\uparrow_H^G \otimes W \simeq (V\otimes W\downarrow_H^G)\uparrow_H^G$.

This gives $(h_1^4)^{*t} = 24^t h_1^4$. Also $(h_4)^{*t} = h_4$.

From [19, Lemma 6.1], and also [16, Ch. 7, Supplementary Problems 137 (a)], we have

$$(h_{31})^{*t} = h_{31} + (2^{t-1}-1)h_{21^2} + (S(t,3)+S(t,4))h_{1^4},$$

where $S(n,k)$ is the Stirling number of the second kind; also $S(t,3)+S(t,4) = \frac{4^{t-1}+2}{6} - 2^{t-2}$.

An inductive argument also gives the following closed formulas.

1. $(h_{22})^{*t} = 2^{t-1}h_{22} + (6^{t-2} + \frac{6^{t-2}-2^{t-2}}{2})h_{1^4}$.
2. $(h_{21^2})^{*t} = 2^{t-1}h_{21^2} + (12^{t-2}\cdot 5 + 2^{t-2}(6^{t-2}-1))h_{1^4}$.

Since

$$\mathrm{ch}\ \tilde{H}_2(B_4^{(t)}) = c_{(4)}(h_4)^{*t} + c_{(31)}(h_{31})^{*t} + c_{(22)}(h_{22})^{*t} + c_{(21^2)}(h_{21^2})^{*t} + c_{(1^4)}(h_{1^4})^{*t},$$

putting these facts together gives the stated expansion in the elementary symmetric functions. □

The first few values of $\gamma_4(t)$ are $0, 9, 375, 11309, 01085, 7591669, 186637045$.

Sage data for $n, t \leqslant 7$ supports the following conjectures for the diagonal action.

Conjecture 4.24. *The coefficient of e_n in the elementary basis expansion of $\mathrm{ch}\ \tilde{H}_{n-2}(B_n^{(t)})$ is always 1.*

Conjecture 4.25. *For $t \geqslant 1$, there are $\mathfrak{S}_n$-equivariant inclusions $\tilde{H}_{n-2}(B_n^{(t)}) \hookrightarrow \tilde{H}_{n-2}(B_n^{(t+1)})$.*

The next conjecture may be viewed as a stability result for the set of irreducibles appearing in $\tilde{H}_{n-2}(B_n^{(t)})$.

Conjecture 4.26. *For $t \geqslant 3$, any irreducible appearing in $\tilde{H}_{n-2}(B_n^{(t)})$ also appears in $\tilde{H}_{n-2}(B_n^{(2)})$.*

5. Rank-selection in $B_n^{(t)}$

For a fixed subset J of the nontrivial ranks $[1, n-1]$, the rank-selected subposet $B_n^{(t)}(J)$ is defined to be the bounded poset $\{x \in B_n^{(t)} : \mathrm{rank}(x) \in J\}$ with the top and bottom elements $\hat{0}, \hat{1}$ appended. Since rank-selection preserves the property of being Cohen-Macaulay [3, Theorem 6.4] (see also the references in [18, Page 226]), these posets have at most one nonvanishing homology module, which is in the top dimension. The study of the homology of rank-selected subposets was initiated in [14].

Denote by $\beta_n^{(t)}(J)$ the product Frobenius characteristic of the top homology $\tilde{H}_{k-1}(B_n(J))$ of the rank-selected subposet of $B_n^{(t)}$ corresponding to the rank-set J. Our strategy to determine the rank-selected representations of $B_n^{(t)}$ will be to use the known results [14] for rank-selection in the Boolean lattice, and then apply the homomorphism Φ_t to obtain the corresponding results for the Segre product $B_n^{(t)}$.

Stanley's theory of rank-selected invariants for Cohen-Macaulay posets, discussed in Section 2, takes the following group-equivariant form, first described in [14]. Note that dropping the Cohen-Macaulay condition results in a similar formulation with the top homology module of each rank-selected subposet being replaced by the alternating sum of homology modules, i.e., the Lefschetz module (see [17]).

Theorem 5.1 ([14]). *Let G be a finite group of automorphisms of a bounded and ranked Cohen-Macaulay poset P. For each subset J of the nontrivial ranks, let $P(J)$ denote the bounded rank-selected subposet $\{x \in P : \mathrm{rank}(x) \in J\} \cup \{\hat{0}, \hat{1}\}$. Then G is a group of automorphisms of the Cohen-Macaulay subposet $P(J)$, and consequently the maximal chains in $P(J)$ and the top homology of $P(J)$ carry representations of G. Let $\alpha_P(J)$ and $\beta_P(J)$ respectively denote these representations. Then*

$$\alpha_P(J) = \sum_{U \subseteq J} \beta_P(U),$$
$$\beta_P(J) = \sum_{U \subseteq J} (-1)^{|J|-|U|} \alpha_P(U).$$

In the notation of Section 2, the dimensions of the representations $\alpha_P(J)$, $\beta_P(J)$ are respectively $\tilde{\alpha}_P(J)$, $\tilde{\beta}_P(J)$.

Next, we review the specific results for the Boolean lattice. Let $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$ be a subset of the nontrivial ranks $[n-1]$ of B_n. The $\mathfrak{S}_n$-module afforded by the maximal chains of $B_n(J)$ is easily seen to have Frobenius characteristic $\alpha_n(J)$ where

$$\alpha_n(J) = h_{j_1} h_{j_2-j_1} \cdots h_{j_r-j_{r-1}} h_{n-j_r}, \tag{5.1}$$

since it is the permutation action on the cosets of the Young subgroup $\mathfrak{S}_{j_1} \times \mathfrak{S}_{j_2-j_1} \times \cdots \times \mathfrak{S}_{j_r-j_{r-1}} \times \mathfrak{S}_{n-j_r}$: the chains are permuted transitively by $\mathfrak{S}_n$, and the stabiliser of a chain is that Young subgroup.

Let $\beta_n(J)$ denote the homology of the rank-selected subposet $B_n(J)$ of the Boolean lattice B_n. For a standard Young tableau τ of shape $\lambda \vdash n$ (see [16] for definitions), the descent set $\mathrm{Des}(\tau)$ of τ is the set of entries i such that $i+1$ appears in a row strictly below i. Using Theorem 5.1, the permutation module of Equation (5.1), and Robinson-Schensted insertion (see [16]), Stanley [14, Section 4] now deduces the following result, originally due to Solomon.

Theorem 5.2 ([12, 14]). *For any subset $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$ of the nontrivial ranks of B_n, one has:*

1. $\alpha_n(J) = \sum_{\lambda \vdash n} s_\lambda \ |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) \subseteq J\}|$;
2. $\beta_n(J) = \sum_{\lambda \vdash n} s_\lambda \ |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) = J\}|$.

Now consider the $\mathfrak{S}_n^{\times t}$-module M_J of maximal chains in the rank-selected t-fold Segre power $B_n^{(t)}(J)$. Denote its product Frobenius characteristic by $\alpha_n^{(t)}(J)$. Again it is clear that the chains are transitively permuted by $\mathfrak{S}_n^{\times t}$. An element x of the chain c is of the form $x = (J_1, J_2, \ldots, J_t)$ where every J_i has the same cardinality n_x for some $n_x \in J$. The stabiliser of the chain c is thus $(\mathfrak{S}_{j_1} \times \mathfrak{S}_{j_2-j_1} \times \cdots \times \mathfrak{S}_{j_r-j_{r-1}} \times \mathfrak{S}_{n-j_r})^{\times t}$, and hence, by definition of the product Frobenius characteristic map Pch, we have

$$\alpha_n^{(t)}(J) = \mathrm{Pch}\ M_J = \prod_{i=1}^{t} (h_{j_1} h_{j_2-j_1} \cdots h_{j_r-j_{r-1}} h_{n-j_r})(X^i).$$

Finally, we obtain the analogue of Theorem 5.2 for the t-fold Segre power $B_n^{(t)}$. Recall that Theorem 4.13 gives a possibly virtual expression for the decomposition of $\Phi_t(s_\lambda)$ into irreducibles. However, by Theorem 5.1, $\alpha_n^{(t)}(J)$ and $\beta_n^{(t)}(J)$ are indeed the product Frobenius characteristics of true $\mathfrak{S}_n^{\times t}$-modules, namely the module of maximal chains of the rank-selected subposet $B_n^{(t)}(J)$ and the top homology module of $B_n^{(t)}(J)$.

Theorem 5.3. *For any subset J of nontrivial ranks of $B_n^{(t)}$, the homomorphism Φ_t maps the Frobenius characteristic of the $\mathfrak{S}_n$-action on the chains of $B_n(J)$ to the product Frobenius characteristic of the $\mathfrak{S}_n^{\times t}$-action on the chains of $B_n^{(t)}(J)$. More precisely, we have:*

1. $\alpha_n^{(t)}(J) = \Phi_t(\alpha_n(J)) = \sum_{\lambda \vdash n} \Phi_t(s_\lambda) \ |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) \subseteq J\}|$;
2. $\beta_n^{(t)}(J) = \Phi_t(\beta_n(J)) = \sum_{\lambda \vdash n} \Phi_t(s_\lambda) \ |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) = J\}|$.

Proof. It follows immediately from the preceding discussion, by applying the homomorphism Φ_t to (5.1), that

$$\alpha_n^{(t)}(J) = \prod_{i=1}^{t} (h_{j_1} h_{j_2-j_1} \cdots h_{j_r-j_{r-1}} h_{n-j_r})(X^i) = \Phi_t(\alpha_n(J)).$$

Hence from Theorem 5.1 we obtain

$$\beta_n^{(t)}(J) = \sum_{U\subseteq J} \alpha_n^{(t)}(U) = \sum_{U\subseteq J} \Phi_t(\alpha_n(U)) = \Phi_t(\beta_n(J)).$$

Invoking Theorem 5.2 now finishes the proof. □

A skew-shape is said to be *connected* [16, p. 345] if every two consecutive rows in its Ferrers diagram overlap in at least one square. Recall that a ribbon (also called a border strip [11, 16], or rim hook) is a connected skew-shape with no 2 by 2 square. Equivalently, a ribbon is a skew-shape where two consecutive rows overlap in exactly one square.

It follows from Solomon's result [12] that if $J = \{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n-1\}$, then $\beta_n(J)$ is the (Frobenius characteristic of the) $\mathfrak{S}_n$-module indexed by the ribbon whose Ferrers diagram has rows of lengths $j_1, j_2 - j_1, \ldots, n - j_r$, top to bottom. In particular when J consists of k consecutive ranks starting at the first rank, $J = [k]$, then $\beta_n(J)$ corresponds to the $\mathfrak{S}_n$-irreducible indexed by the hook shape $(n-k, 1^k)$.

We can now make the following addition to Proposition 4.18.

Corollary 5.4. *Let $f = s_{(n-k,1^k)}$ be the Schur function indexed by a hook $(n-k, 1^k)$, or more generally the Schur function indexed by a ribbon corresponding to the set $J = \{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n-1\}$, that is, whose rows top to bottom are $j_1, j_2 - j_1, \ldots, n - j_r$. Then $\Phi_t(f)$ is the product Frobenius characteristic of a true $\mathfrak{S}_n^{\times t}$-module.*

Proof. This is immediate from the preceding remarks and the fact that $\Phi_t(\beta_n(J)) = \beta_n^{(t)}(J)$. □

Theorem 5.3 can also be deduced by explicitly deriving a recurrence for the rank-selected homology, using the Whitney homology technique of [18]. Indeed, for the Boolean lattice itself (cf. [18, Theorem 1.10, Example 1.11]), if $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$ is a subset of nontrivial ranks in B_n, we have that

$$\begin{aligned}\beta_n(J) + \beta_n(J \setminus \{j_r\}) &= \operatorname{ch} \mathrm{WH}_{j_r}(B_n(J)) \\ &= \operatorname{ch} \bigoplus_{x:|x|=j_r} \tilde{H}(\hat{0}, x)_{B_n(J)}.\end{aligned}$$

Since the interval $(\hat{0}, x)_{B_n(J)}$ is isomorphic to the rank-selected subposet $B_{j_r}(J \setminus \{j_r\})$, checking stabilisers gives the recurrence

$$\beta_n(J) + \beta_n(J \setminus \{j_r\}) = \beta_{j_r}(J \setminus \{j_r\})\, h_{n-j_r}. \tag{5.2}$$

Similarly, for the rank-selected homology representation $\beta_n^{(t)}(J)$ of the t-fold Segre power $B_n^{(t)}$, we obtain, by an analysis completely analogous to the above, and the one carried out in the proof of Theorem 4.2, or alternatively by simply applying the homomorphism Φ_t to (5.2):

Theorem 5.5. *Let $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$ be a subset of nontrivial ranks in $B_n^{(t)}$. The product Frobenius characteristic* Pch $\tilde{H}(B_n^{(t)}(J))$ *of the rank-selected subposet of $B_n^{(t)}$ satisfies the following recurrence.*

$$\beta_n^{(t)}(J) + \beta_n^{(t)}(J \setminus \{j_r\}) = \beta_{j_r}^{(t)}(J \setminus \{j_r\}) \prod_{i=1}^{t} h_{n-j_r}(X^i).$$

This is precisely the $\mathfrak{S}_n^{\times t}$-equivariant version of (2.3) of Proposition 2.3, a connection that will be exploited in the next section.

6. The t-fold Segre power of the subspace lattice: rank-selection and stable principal specialisation

We will show in this section that the surprising relationship discovered in [9] between the homology representation of the Boolean Segre square $B_n \circ B_n$ and the Möbius number of the subspace lattice Segre square $B_{n,q} \circ B_{n,q}$ holds in greater generality, namely for all rank-selected subposets of the t-fold Segre powers in each case.

Recall that $\beta_n^{(t)} := \mathrm{Pch}(\tilde{H}_{n-2}(B_n^{(t)}))$, where $B_n^{(t)}$ is the t-fold Segre power of B_n. The recurrence for $\beta_n^{(t)}$ is

$$\beta_n^{(t)} = \sum_{i=0}^{n-1} (-1)^{n-1-i}\, \beta_i^{(t)} \prod_{i=1}^{t} h_{n-i}(X^i) \tag{6.1}$$

The *stable principal specialisation* [16, Chapter 7, Section 8] of a symmetric function f in variables $x_1, x_2, \dots$ is the function of q obtained from f by means of the substitution $x_i \to q^{i-1}, i \geqslant 1$. Similarly, we consider the stable principal specialisation of a function in $\otimes_{i=1}^{t}\Lambda(X^i)$ to be the function of q obtained by replacing each set of variables X^i by the set $\{1, q, q^2, \dots\}$. Using the stable principal specialisations $\mathrm{ps}\,\beta_n^{(t)}$ and $\mathrm{ps}\,h_n := h_n(1, q, q^2, \dots) = \prod_{i=1}^{n}(1-q^i)^{-1}$ [16, Proposition 7.8.3], from (6.1) we have the following identity for $t \geqslant 1$.

$$\mathrm{ps}\,\beta_n^{(t)} = \sum_{i=0}^{n-1} (-1)^{n-1-i} \left(\prod_{j=1}^{n-i} (1-q^j)^{-1} \right)^t \mathrm{ps}\,\beta_i^{(t)}$$

The extension of [9, Theorem 4.2] to t-fold products is now immediate. Note the particular case $t = 1$, for which $W_n^{(1)}(q) = q^{\binom{n}{2}}$ and $\mathrm{ps}\,e_n = \frac{q^{\binom{n}{2}}}{\prod_{j=1}^{n}(1-q^j)}$.

Proposition 6.1. *We have, for $t \geqslant 1$,*

$$\mathrm{ps}\,\mathrm{Pch}(\tilde{H}_{n-2}(B_n^{(t)})) = \mathrm{ps}\,\beta_n^{(t)} = \frac{W_n^{(t)}(q)}{\prod_{j=1}^{n}(1-q^j)^t}.$$

Proof. From (2.2), we obtain

$$\begin{bmatrix} n \\ i \end{bmatrix}_q \cdot \prod_{j=i+1}^{n} (1-q^j)^{-1} = \prod_{j=1}^{n-i} (1-q^j)^{-1} = \mathrm{ps}\,h_{n-i}.$$

Thus, dividing (2.4) throughout by $\prod_{j=1}^{n}(1-q^j)^t$, we obtain

$$\frac{W_n^{(t)}(q)}{\prod_{j=1}^{n}(1-q^j)^t} = \sum_{i=0}^{n-1} (-1)^{n-1-i} \left(\prod_{j=1}^{n-i} (1-q^j)^{-1} \right)^t \frac{W_i^{(t)}(q)}{\prod_{j=1}^{i}(1-q^j)^t},$$

showing that $\frac{W_n^{(t)}(q)}{\prod_{j=1}^{n}(1-q^j)^t}$ and $\mathrm{ps}\,\beta_n^{(t)}$ satisfy the same recurrence. They also satisfy the same initial conditions, since $\beta_0^{(t)} = 1$ implies $\mathrm{ps}\,\beta_0^{(t)} = 1 = W_0^{(t)}(q)$, and $\beta_1^{(t)} = h_1^t$ implies $\mathrm{ps}\,\beta_1^{(t)} = \frac{1}{(1-q)^t} = \frac{W_1^{(t)}(q)}{(1-q)^t}$. This completes the proof. ☐

Now we turn to rank-selection. Recall that we denote by $\beta_n^{(t)}(J)$ the product Frobenius characteristic of the top homology $\tilde{H}_{k-1}(B_n^{(t)}(J))$ of the rank-selected subposet of $B_n^{(t)}$ corresponding to the rank-set $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$. Similarly, for the t-fold Segre power of the subspace lattice $B_{n,q}$, in Section 2 we denoted by $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ its rank-selected Betti number, i.e., the dimension of the top homology module of the rank-selected subposet $B_{n,q}^{(t)}(J)$ corresponding to the rank-set S. Thus $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ is the absolute value of the Möbius number of $B_{n,q}^{(t)}(J)$. In particular, from Theorem 2.4, $W_n^{(t)}(q) = \tilde{\beta}_{B_{n,q}^{(t)}}([n-1])$. Using the recurrences in Theorem 5.5 and Equation (2.3), we can now derive the rank-selected analogue of Proposition 6.1.

Theorem 6.2. *The stable principal specialisation of $\beta_n^{(t)}(J)$ for the rank-selected homology module of the t-fold Segre power of the Boolean lattice B_n, and the rank-selected Betti number $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ of the t-fold Segre power of the subspace lattice $B_{n,q}$, are related by the equation*

$$\mathrm{ps}\,\beta_n^{(t)}(J) = \frac{\tilde{\beta}_{B_{n,q}^{(t)}}(J)}{\prod_{i=1}^{n}(1-q^i)^t}.$$

Proof. We verify that each side of the above equation satisfies the same recurrence. For a fixed subset J of the nontrivial ranks $[1, n-1]$, the rank-selected subposet $B_n^{(t)}(J)$ is defined to be the bounded poset $\{x \in B_n^{(t)} : \mathrm{rank}(x) \in J\}$ with the top and bottom elements $\hat{0}, \hat{1}$ appended. First, taking the principal specialisation in Theorem 5.5 gives us the following recurrence for the left-hand side, $\mathrm{ps}\,\beta_n^{(t)}(J)$.

$$\operatorname{ps}\beta_n^{(t)}(J) + \operatorname{ps}\beta_n^{(t)}(J \setminus \{j_r\}) = \operatorname{ps}\beta_{j_r}^{(t)}(J \setminus \{j_r\}) \cdot (\operatorname{ps} h_{n-j_r})^t. \tag{6.2}$$

For the right-hand side of the statement, use the recurrence for $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ in Equation (2.3), which may be restated in the equivalent form

$$\tilde{\beta}_{B_{n,q}^{(t)}}(J) + \tilde{\beta}_{B_{n,q}^{(t)}}(J \setminus \{j_r\}) = \tilde{\beta}_{B_{j_r}^{(t)}(q)}(J \setminus \{j_r\}) \cdot (\operatorname{ps} h_{n-j_r})^t \cdot \prod_{i=1+j_r}^{n} (1-q^i)^t. \tag{6.3}$$

Dividing (6.3) by $\prod_{i=1}^{n}(1-q^i)^t$ and comparing with the recurrence (6.2) for the principal specialisation immediately shows that the expressions $\operatorname{ps}\beta_n^{(t)}(J)$ and $\frac{\tilde{\beta}_{B_{n,q}^{(t)}}(J)}{\prod_{i=1}^{n}(1-q^i)^t}$ satisfy the same recurrence on subsets $J \subseteq [n-1]$. Note also that the theorem has been established when $J = [n-1]$.

Now consider the case when J consists of a single rank $\{r\}$. Then we have

$$\beta_n^{(t)}(J) = \prod_{j=1}^{t} h_r(X^j) \prod_{j=1}^{t} h_{n-r}(X^j) - \prod_{j=1}^{t} h_n(X^j),$$

$$\begin{aligned}
\text{and so} \qquad \operatorname{ps}\beta_n^{(t)}(J) &= \left(\prod_{i=1}^{r}(1-q^i)^{-1} \prod_{i=1}^{n-r}(1-q^i)^{-1}\right)^t - (\prod_{i=1}^{n}(1-q^i)^{-1})^t \\
&= \prod_{i=1}^{n}(1-q^i)^{-t} \left(\begin{bmatrix} n \\ r \end{bmatrix}_q^t - 1\right) \\
&= \frac{\tilde{\beta}_{B_{n,q}^{(t)}}(J)}{\prod_{i=1}^{n}(1-q^i)^t}.
\end{aligned}$$

An inductive argument on the size of J now completes the proof. □

7. Further questions

In a future paper, we examine the diagonal $\mathfrak{S}_n$-action on the (rank-selected) homology of the t-fold Segre power of B_n (Theorem 4.20) more closely, investigating the conjectures at the end of Section 4. It would also be interesting to see what more can be said about the map Φ_t.

In [14], Stanley examined the rank-selected homology of the subspace lattice. A logical next step is to carry out the program of this paper for the action of $GL_n(q)$ on the Segre powers of the subspace lattice.

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