Enumerative Combinatorics and Applications
ecajournal.haifa.ac.il

ECA **5:3** (2025) Article #S2R22
https://doi.org/10.54550/ECA2025V5S3R22

Partially Restricted Stacks as Functions

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Received: April 5, 2025, **Accepted**: July 29, 2025, **Published**: August 15, 2025

Abstract: We further generalize Berlow's stack sorting map s_T to $s_{(T,k)}$, where instead of avoiding all permutations in T, the stack contains at most k permutations from a restricted set T. We introduce the (T, k)-machine as well, defined as the composition of West's stack sorting map s and $s_{(T,k)}$, as a parallel to Cerbai, Claesson, and Ferrari's $\sigma-$machine. We show that for every $s_{(T,k)}$, there exists an equivalent $s_{T'}$ and give an explicit construction of T'. We then characterize the permutations in the preimage of id_n in $s_{(\{12,21\},1)}$ and prove that the size of the preimage is the $(n-1)^{\text{th}}$ Catalan number. We also demonstrate that the $s_{(\{12,21\},1)}$ map and the $(\{12, 21\}, 1)$-machine sort all permutations after a finite number of repeated applications.

Keywords: Pattern avoidance; Stack sort; Stack sorting machine
2020 Mathematics Subject Classification: 05A05; 05A15; 37E15

1. Introduction

The stack sorting map s was first introduced by West [5] in 1990. The map sends permutations through a stack that always avoids the permutation 21 when read from top to bottom. Knuth [4] first showed that a permutation π is sorted by s if and only if π avoids the permutation 231, thus also showing that the number of permutations of length n that are sorted by s is $\frac{1}{n+1}\binom{2n}{n}$, or the nth Catalan number [4].

More recently, the map s was generalized by Cerbai, Claesson, and Ferrari [2] in 2020 to the map s_σ. Whereas the stack in s avoids the permutation 21, the stack in s_σ avoids a permutation σ. Cerbai, Claesson, and Ferrari also introduced the σ-machine, defined as the composition of s and s_σ. With stacks that avoid a single permutation, the natural next step was to avoid multiple permutations. Shortly after, Berlow [1] extended s_σ to s_T, where T is a set of permutations and the stack in s_T must avoid all permutations in T.

We further generalize s to $s_{(T,k)}$. We define the maps $s_{(T,k)}$ that avoid containing more than k distinct permutations from T in the stack at once. More specifically, the map sorts a permutation π via the following stack sorting algorithm: If adding the leftmost element of the input to the stack keeps the stack (T, k)-avoiding, push that element into the stack. Otherwise, pop the top element off the stack and append it to the output.

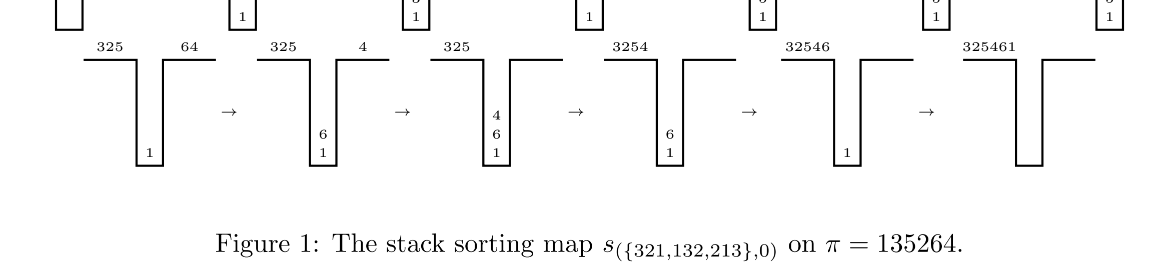

Figure 1: The stack sorting map $s_{(\{321,132,213\},0)}$ on $\pi = 135264$.

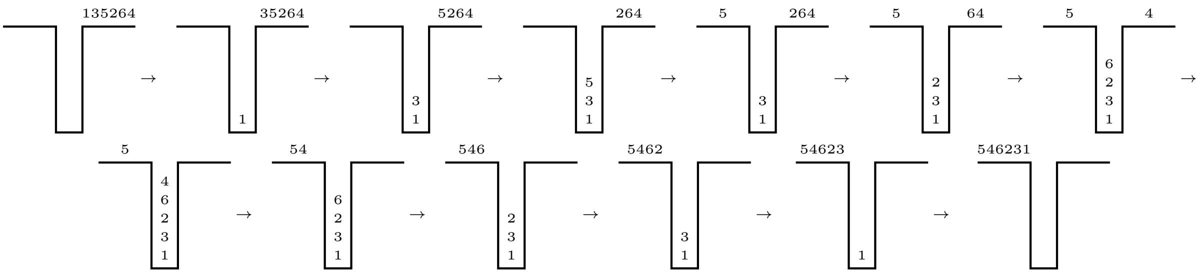

Figure 2: The stack sorting map $s_{(\{321,132,213\},1)}$ on $\pi = 135264$.

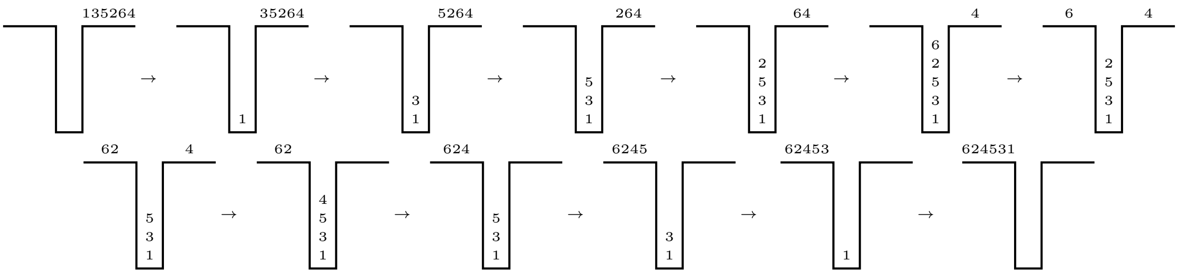

Figure 3: The stack sorting map $s_{(\{321,132,213\},2)}$ on $\pi = 135264$.

Figures 1, 2, and 3 show the sorting process for $\pi = 135264$, with $T = \{321, 132, 213\}$ and k set to 0, 1, and 2, respectively, with each value of k producing a different result. Note that when $k = 0$, the map is identical to Berlow's map by definition.

We prove a number of analogues of classical results for West's stack sorting map for $s_{(T,k)}$. In Section 2, we establish preliminaries. In Section 3, we prove the existence of an equivalence between the $s_{(T,k)}$ maps and Berlow's [1] s_T maps (Lemma 3.1). In Section 4, we characterize the one-stack-sortable permutations (Theorem 4.1) and prove that all permutations are sortable by repeated sorts (Theorem 4.2) for $T = \{12, 21\}$ and $k = 1$. We study the map $s_{(\{12,21\},1)}$ specifically because it is the closest parallel to the original 21-avoiding stack sorting map s. In Section 5, we establish (T, k)-machines, defined as $(s \circ s_{(T,k)})$, as a natural analogue to Cerbai, Claesson, and Ferrari's [2] machine. We also prove that the $(\{12, 21\}, 1)-$machine eventually sorts all permutations (Theorem 5.1).

2. Preliminaries

For a permutation $\pi = \pi_1\pi_2 \dots \pi_n \in S_n$, we let $\mathrm{id}_n = 12 \dots n$ and $\mathrm{inv}_n = n(n-1)\dots 1$. We also use $\pi \pm k$ to denote $(\pi_1 \pm k)(\pi_2 \pm k) \cdots (\pi_n \pm k)$, use $\mathrm{rev}(\pi)$ to denote $\pi_n\pi_{n-1} \dots \pi_1$, and use $\pi_{[a:b]}$ to denote $\pi_a\pi_{a+1} \cdots \pi_{b-1}\pi_b$ for integers $1 \le a < b \le n$. We let $|\pi| = n$, and refer to this quantity as the *length* of π. We define the *lead length* of a permutation, denoted by $\mathrm{lead}(\pi)$, as the greatest integer i such that $\pi_{[1:i]} = 12 \dots i$ if $\pi_1 = 1$, and 0 otherwise. For a permutation π and a set of indices $A = \{a_1, a_2, \dots, a_i\}$ sorted in ascending order, we define $\pi_{[A]}$ as $\pi_{a_1}\pi_{a_2} \dots \pi_{a_i}$. In addition, for a set of permutations T and $a < |T|$, let $\mathrm{long}_a(T)$ be the sum of the lengths of the $a + 1$ longest permutations in T. For two stack sorting maps q and r, we say that they are *equivalent* if for all permutations π the sorting process for $q(\pi)$ and $r(\pi)$ are the same; that is, at every point in the process, $q(\pi)$ and $r(\pi)$ undergo identical push and pop operations.

Given two permutations π and σ, we say π *contains* σ if there exist $a_1, a_2, \dots, a_k$ such that the subpermutation $\pi_{a_1}\pi_{a_2} \dots \pi_{a_k}$ is order-isomorphic to σ. Otherwise, we say π avoids σ. The set of all permutations that avoid σ is denoted by $\mathrm{Av}(\sigma)$; the set of all permutations of length n that avoid σ is denoted by $\mathrm{Av}_n(\sigma)$. Furthermore, for a set T of permutations and a nonnegative integer k, we say π is (T, k)-*avoiding* if π contains at most k elements of T. The set of all permutations that are (T, k)-avoiding is denoted by $\mathrm{Av}^k(T)$, while the set of permutations of length n that are (T, k)-avoiding is denoted by $\mathrm{Av}^k_n(T)$. Finally, a set of permutations $\mathcal{C}$ is called a *permutation class* if for all $\pi \in \mathcal{C}$, if π contains σ then $\sigma \in \mathcal{C}$.

Throughout the paper, we use t to denote $s_{(\{12,21\},1)}$. We also make frequent use of a well-known result from Knuth [4].

Lemma 2.1 (Knuth [4])**.** *A permutation $\pi \in S_n$ satisfies $s(\pi) = \mathrm{id}_n$ if and only if $\pi \in \mathrm{Av}(231)$. Furthermore, the number of permutations $\pi \in S_n$ such that $s(\pi) = \mathrm{id}_n$ is $\frac{1}{n+1}\binom{2n}{n}$.*

3. Equivalence to s_T

In this section, we show that every $s_{(T,k)}$ map is equivalent to a $s_{T'}$ map for some T'.

Theorem 3.1. *The maps $s_{(T,k)}$ and $s_{T'}$ are equivalent, where $T' = \bigcup_{i=1}^{\mathrm{long}_k(T)} S_i \setminus \mathrm{Av}_i^k(T)$.*

Proof. It suffices to show that a permutation π contains a permutation in T' if and only if it contains at least $k+1$ permutations in T. If π contains at least $k+1$ permutations in T, let $k+1$ of those permutations be $t_1, t_2, \ldots, t_{k+1}$ and their corresponding order-isomorphic contained permutations in π be $\pi_{[A_1]}, \pi_{[A_2]}, \ldots, \pi_{[A_{k+1}]}$. Then, π contains the permutation $\sigma = \pi_{[\mathbb{A}]}$, where $\mathbb{A} = A_1 \cup A_2 \cup \ldots \cup A_{k+1}$. Furthermore, we have that $\sigma \notin \mathrm{Av}^k(T)$, as σ contains $t_1, t_2, \ldots, t_{k+1}$. However, since $|\mathbb{A}| \leq |A_1| + |A_2| + \cdots + |A_{k+1}| \leq \mathrm{long}_k(T)$, it follows that $|\sigma| \leq \mathrm{long}_k(T)$; it follows that $\sigma \in T'$. For the converse, if π contains a permutation $\sigma \in T'$, we have that σ must contain at least $k+1$ permutations in T by definition. Thus, π contains at least $k+1$ permutations in T because π contains σ. Therefore, the (T,k)-avoiding stack is equivalent to the T'-avoiding stack. □

While T' is a finite set of permutations, it is not necessarily the smallest possible set, nor are the permutations in it necessarily the ones of least length. As a specific example, take $s_{(\{12,21\},1)}$, where $T = \{12, 21\}$, $k = 1$, and $\mathrm{long}_k(T) = \mathrm{long}_1(\{12,21\}) = |12| + |21| = 4$; then, the corresponding T' contains 26 permutations:

$$\begin{aligned}
T' &= \bigcup_{i=1}^{4} S_i \setminus \mathrm{Av}_i^1(\{12,21\}) \\
&= (S_1 \setminus \{1\}) \cup (S_2 \setminus \{12,21\}) \cup (S_3 \setminus \{123,321\}) \cup (S_4 \setminus \{1234,4321\}) \\
&= (\varnothing \cup \varnothing \cup S_3 \cup S_4) \setminus \{123,321,1234,4321\} \\
&= (S_3 \cup S_4) \setminus \{123,321,1234,4321\} \\
&= \{132, 213, 231, 312, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, \\
&\quad 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312\}
\end{aligned}$$

However, a much smaller restricted set exists that still generates an equivalent map.

Lemma 3.1. *The maps $s_{(\{12,21\},1)}$ and $s_{\{132,213,231,312\}}$ are equivalent.*

Proof. For all permutations π, we have that $\pi \in \mathrm{Av}(132, 213, 231, 312)$ if and only if $\pi \in \mathrm{Av}^1(12, 21)$, because a permutation that contains at least one of $\{132, 213, 231, 312\}$ must contain both 12 and 21 and vice versa. Thus, the stack sorting process is the same for both maps, so they are equivalent. □

4. The Properties of $s_{(\{12,21\},1)}$

We begin by proving supporting lemmas that will build up to a full characterization of the preimage of id_n under t. Given a permutation $\pi \in S_n$, we note that $t(\pi)_n = \pi_1$, because π_1 is pushed into the stack first and is popped out last.

Proposition 4.1. *For all $\pi \in S_n$, we have that*

$$t(\pi)_n = \pi_1.$$

Next, we prove that for the permutations π with $\pi_1 = |\pi|$, the maps t and s are equivalent.

Lemma 4.1. *For all $\pi \in S_n$ such that $\pi_1 = n$, we have that*

$$t(\pi) = s(\pi).$$

Proof. We have that $\pi_1 = n$ is always at the bottom of the stack. Then the stack must contain the permutation 12 when it has more than 1 element. Therefore, the stack always avoids 21, so t is equivalent to s by definition. □

Now, we can prove Theorem 4.1.

Theorem 4.1. *For $\pi \in S_n$, we have $t(\pi) = \mathrm{id}_n$ if and only if $\pi_{[2:n]} \in \mathrm{Av}_{n-1}(231)$ and $\pi_1 = n$.*

Proof. Let $\pi \in S_n$ satisfy $t(\pi) = \mathrm{id}_n$. Then $t(\pi)_n = n$, so $\pi_1 = n$ by Proposition 4.1. Next, by Lemma 4.1, we have that $t(\pi) = s(\pi)$, so $s(\pi_{[2:n]}) = \mathrm{id}_{n-1}$. It follows that $\pi_{[2:n]} \in \mathrm{Av}(231)$ by Lemma 2.1. Conversely, let $\pi \in S_n$ be of the form described in the theorem statement. Because $\pi_1 = n$ is the largest element in π and $\pi_{[2:n]} \in \mathrm{Av}_{n-1}(231)$, we have that $\pi \in \mathrm{Av}_n(231)$. Then $t(\pi) = s(\pi) = \mathrm{id}_n$ by Lemmas 2.1 and 4.1. □

A direct corollary is the enumeration of the one-stack-sortable permutations under t.

Corollary 4.1. *The number of permutations $\pi \in S_n$ such that $t(\pi) = \mathrm{id}_n$ is $\frac{1}{n}\binom{2n-2}{n-1}$.*

Defant and Zheng [3] proved that the set of one-stack-sortable permutations under the map $(s \circ s_{12})$ is a permutation class. However, this is not the case for t, as the permutation 4213 is one-stack-sortable, but the contained permutation 213 is not.

Remark 4.1. *The set of one-stack-sortable permutations under t is not a permutation class.*

Now, to prove Theorem 4.2, we first characterize the behavior of $t(\pi)$ when $\pi_1 = 1$.

Lemma 4.2. *For $\pi \in S_n$, if $\mathrm{lead}(\pi) \geq 1$ then*

$$t(\pi)_{[n-\mathrm{lead}(\pi):n]} = (\mathrm{lead}(\pi)+1)\mathrm{lead}(\pi)\dots 1.$$

Proof. All of $12\dots\mathrm{lead}(\pi)$ will enter the stack, so the stack is 12-avoiding. The remaining inputs are greater than $\mathrm{lead}(\pi)$, so $12\dots\mathrm{lead}(\pi)$ will remain in the stack until the input is empty. When $\mathrm{lead}(\pi)+1$ is pushed, it will pop all elements greater than $\mathrm{lead}(\pi)+1$. Since the remaining inputs are greater than $\mathrm{lead}(\pi)+1$, the elements $(\mathrm{lead}(\pi)+1)\dots 21$ remain in the stack until they are popped at the end. ☐

Next, we characterize the behavior of $t(\pi)$ when $\pi_{|\pi|} = 1$.

Lemma 4.3. *For $\pi \in S_n$ and an integer $1 \leq i \leq n-1$, if $\pi_{[n-i+1:n]} = i(i-1)\dots 1$, there exists $1 \leq j \leq n-i$ such that $t(\pi)_{[j:j+i-1]} = 12\dots i$.*

Proof. Since $i < n$, we have $\pi_1 \neq i$, so $\pi_1 > i$. Thus, when i is pushed, the stack becomes 21-avoiding. Then, after $(i-1)\dots 21$ is pushed, the stack will read from top to bottom as

$$12\dots i\dots\pi_1.$$

Since the input is empty, the elements above will be popped in the stated order. At least 1 element (the element π_1) is popped after i is, so the lemma statement follows. ☐

Thirdly, we characterize the behavior of $t(\pi)$ when $\pi_1 \neq 1$ and $\pi_{|\pi|} \neq 1$.

Lemma 4.4. *For $\pi \in S_n$ and positive integers i, j where $2 \leq i$, if $\pi_{[i:i+j-1]} = 12\dots j$, there exists $1 \leq \ell < i$ such that $t(\pi)_{[\ell:\ell+j-1]} = 12\dots j$.*

Proof. As in Lemma 4.3, we have $\pi_1 > j \geq 1$. When $\pi_i = 1$ is pushed, the stack becomes 21-avoiding. Let a be the total number of elements in the stack that have not been popped when π_i is pushed. Since π_1 cannot be popped until the input is empty, we have $a < i-1$. Next, when π_h is pushed for $i+1 \leq h \leq i+j$, it pops only π_{h-1} and keeps the stack 21-avoiding. Thus $t(\pi)_{[a+1:a+j]} = \pi_{[i:i+j-1]} = 12\dots j$. Since $a < i-1$, we have that $\ell := a+1 < i$. ☐

We end by showing that repeatedly applying t eventually increases the lead length of π.

Lemma 4.5. *For $\pi \in S_n$, if there exists an $a \geq 0$ such that $\mathrm{lead}(t^a(\pi)) \geq 1$, there also exists a $b > a$ such that $\mathrm{lead}(t^b(\pi)) > \mathrm{lead}(t^a(\pi))$.*

Proof. By Lemmas 4.2 and 4.3, there exists $1 \leq i \leq n - \mathrm{lead}(t^a(\pi))$ such that

$$t^{a+2}(\pi)_{[i:i+\mathrm{lead}(t^a(\pi))]} = 12\dots(\mathrm{lead}(t^a(\pi))+1).$$

If $i = 1$ then we are done. Otherwise, Lemma 4.4 concludes. ☐

Now, we can prove Theorem 4.2.

Theorem 4.2. *For every $\pi \in S_n$, there exists a positive integer i such that $t^i(\pi) = \mathrm{id}_n$.*

Proof. By Lemma 4.4, there exists an $a \geq 0$ such that $t^a(\pi)_1 = 1$. Then by Lemma 4.5 there exists some $b > a$ such that $\mathrm{lead}(t^b(\pi)) = n$. By definition, we have $t^b(\pi) = \mathrm{id}_n$. ☐

As all permutations are eventually sorted by t, a natural question is what permutations $\pi \in S_n$ take the maximal number of sorts and what that maximal sort time is. Let $\mathrm{m}_t(\pi)$ be the smallest nonnegative integer j such that $t^j(\pi) = \mathrm{id}_n$.

Conjecture 4.1. *Fix a positive integer $n \geq 3$. When n is odd, we conjecture that $\max_{\pi\in S_n}(\mathrm{m}_t(\pi)) = 2n-3$. When n is even, we conjecture that $\max_{\pi\in S_n}(\mathrm{m}_t(\pi)) = 2n-4$.*

In the following two lemmas, we partially resolve the conjecture by showing that there exist permutations of both odd and even length that require the conjectured maximum number of sorts. We begin with the permutations of odd length.

Lemma 4.6. *For odd $n \geq 3$ and $\pi \in S_n$, if $\pi_{[n-1:n]} = 1n$ and $t(\pi_{[1:n-2]}) = \text{inv}_{n-2} + 1$, we have that $\text{m}_t(\pi) = 2n - 3$.*

Proof. First, note that $\text{m}_t(213) = 3$, so the statement holds for $n = 3$. Next, assume the statement is true for all permutations of length $i \geq 3$ for some odd i and proceed by induction. Let $n = i + 2$ and $\pi \in S_n$ be an arbitrary permutation in the form of the lemma statement. Since $t(\pi_{[1:n-2]}) = \text{inv}_{n-2} + 1$, we have $\pi_1 = 2$ by Proposition 4.1. Thus, $\pi_{n-2} > \pi_1$, so the stack avoids 12 when $\pi_{n-1} = 1$ is pushed. Then 1 will pop out all of the stack except $\pi_1 = 2$. Hence, $\pi_{[2:n-2]}$ will sort to $\text{inv}_{n-3} + 2$. It follows that $t(\pi) = (n-1)(n-2)\cdots 431n2$. Then, we have $t^4(\pi) = 34\cdots(n-3)(n-2)12(n-1)n$. Since $1, 2$ and $n, (n-1)$ are adjacent, $t^4(\pi)$ is isomorphic to $23\cdots(n-4)(n-3)1(n-2)$, which is a permutation of length $n - 2 = i$ in the form described in the lemma statement. Thus $\text{m}_t(\pi) = \text{m}_t(t^4(\pi)) + 4 = (2(n-2) - 3) + 4 = 2n - 3$. □

From Lemma 4.6, we have the following corollary.

Corollary 4.2. *For all odd n, there are at least $\frac{1}{n-1}\binom{2n-4}{n-2}$ permutations $\pi \in S_n$ such that $\text{m}_t(\pi) = 2n - 3$.*

We now prove the existence of even-length permutations that reach the conjectured bound.

Lemma 4.7. *For even $n \geq 4$ and $\pi \in S_n$, if $\pi_1\pi_{n-1}\pi_n = n(n-1)1$ and $\pi_{[2:n-2]}$ avoids 231 then $\text{m}_t(\pi) = 2n-4$.*

Proof. First, note that $\text{m}_t(4231) = 4$, so the lemma holds for $n = 4$. Next, assume the statement is true for all permutations of length $i \geq 4$ for some even i and proceed by induction. Let $n = i + 2$ and $\pi \in S_n$ be a permutation of the form specified in the lemma statement. It follows from Lemmas 2.1 and 4.1 that $t(\pi) = 23\ldots(n-2)1(n-1)n$, as $\pi_{[2:n-2]}$ is sorted to $23\ldots(n-2)$; then, when $n-1$ is pushed in, it pops out everything in the stack except for n, followed by 1 being pushed in and the entire stack being popped out.

The next application of t is as follows: first, $23\cdots(n-2)$ is pushed into the stack. Second, 1 pops $(n-2)\cdots 43$ to the output before being pushed into the stack. Third, $n-1$ pops 1 and $(n-1)n$ is pushed into the stack. Fourth, $n(n-1)2$ is popped to the output. Therefore, $t^2(\pi) = (n-2)(n-3)\cdots 31n(n-1)2$.

Then, for the third application of t, the elements $(n-2)(n-3)\cdots 31$ are first pushed into the stack. Second, n pops $(n-3)(n-4)\cdots 31$ to the output before being pushed into the stack. Third, $n-1$ pops n and is pushed into the stack. Fourth, 2 pops $n-1$ and is pushed into the stack. Fifth, $2(n-2)$ is popped to the output. Thus, $t^3(\pi) = 134\cdots(n-3)n(n-1)2(n-2)$.

Lastly, the fourth application of t is as follows: first, $134\cdots(n-3)n$ is pushed into the stack. Second, $n-1$ is pushed into the stack, popping n in the process. Third, 2 pops $(n-1)(n-3)(n-2)\cdots 3$ to the output and is pushed into the stack. Fourth, $(n-2)$ pops 2 and is pushed into the stack. Fifth, 21 is popped to the output.

As a result, we have that $t^4(\pi) = n(n-1)(n-3)(n-4)\cdots 3(n-2)21$. However, because n and $(n-1)$ are adjacent and 2 and 1 are adjacent, we have that $t^4(\pi)$ is equivalent to $(n-2)(n-4)(n-5)\cdots 2(n-3)1$, which is a length $n-2 = i$ in the form in the lemma statement. Thus, $\text{m}_t(\pi) = \text{m}_t(t^4(\pi))+4 = (2(n-2)-4)+4 = 2n-4$. □

From Lemma 4.7, we have the following corollary.

Corollary 4.3. *For all even n, there are at least $\frac{1}{n-2}\binom{2n-6}{n-3}$ permutations $\pi \in S_n$ such that $\text{m}_t(\pi) = 2n - 4$.*

Note that there exist permutations of even length that take the maximal number of sorts that are not characterized by Lemma 4.7; for example, for $n = 4$, the permutation $\pi = 2314$ does not have $\pi_1\pi_3\pi_4 = 431$, but still takes 4 sorts to reach the identity, as

$$t^4(2314) = t^3(3142) = t^2(1423) = t(4321) = 1234.$$

We also establish a necessary condition for the permutations that are minimally sorted.

Lemma 4.8. *If $\pi \in S_n$ satisfies $\text{m}_t(\pi) \geq \text{m}_t(\pi')$ for all $\pi' \in S_n$ then $t^{\text{m}_t(\pi)-1}(\pi) = \text{inv}_\text{n}$.*

Proof. Let $\sigma \in S_n$ satisfy $\sigma_i = \pi_{(n-i+1)}$ for $1 \leq i \leq n$. Then $t^{\text{m}_t(\pi)}(\sigma) = \text{inv}_n$ by symmetry, so $t^{\text{m}_t(\pi)+1}(\sigma) = \text{id}_n$. If $t^{\text{m}_t(\pi)-1}(\sigma) = \text{id}_n$ also, then $t^{\text{m}_t(\pi)-1}(\pi) = \text{inv}_\text{n}$, and we are done. Otherwise, we have $\text{m}_t(\sigma) = \text{m}_t(\pi) + 1$. Then $\text{m}_t(\pi) < \text{m}_t(\sigma)$, a contradiction. □

Additionally, we partially characterize the two-stack-sortable permutations under t.

Lemma 4.9. *Given $\pi \in S_n$ such that $\pi_i = n$ and $i \neq 1$, if $t^2(\pi) = \text{id}_n$ then*

$$\pi_1 < \pi_2 < \cdots < \pi_{i-1} < \pi_i = n.$$

Proof. Assume otherwise. Then, there must exist a smallest $j < i$ such that $\pi_j > \pi_{j+1}$. If $j = 1$, then the stack is 21-avoiding after π_2 is pushed. Next, consider whether the stack is 21-avoiding or 12-avoiding after the first $i-1$ elements have been pushed. If the stack is 21-avoiding, then $\pi_i = n$ will pop π_{i-1} and $t(\pi)_1 \neq n$. If the stack is 12-avoiding, then π_2 must have been popped before π_i and so $t(\pi)_1 \neq n$. If $j > 1$, then during the sorting process, when π_{j+1} enters the stack, π_j is first popped. Therefore, $t(\pi)_1 \neq n$, so by Proposition 4.1, we have that $t^2(\pi) \neq \mathrm{id}_n$, a contradiction. □

This leads to a method for generating length n two-stack-sortable permutations from length $n-1$ two-stack-sortable permutations.

Lemma 4.10. *Given $\pi \in S_n$ such that $\pi_i = n$ and $i \neq 1$, let $\sigma = \pi_{[1\ldots i-1]}\pi_{[i+1\ldots n]}$. If $t^2(\sigma) = \mathrm{id}_{n-1}$ then $t^2(\pi) = \mathrm{id}_n$.*

Proof. By Lemma 4.9, when t is applied to π, nothing is popped before n is pushed onto the stack, and n does not pop anything from the stack. Then, n is immediately popped. Therefore, $t(\pi) = nt(\sigma)$ where $nt(\sigma)$ is the permutation obtained by placing n at the beginning of $t(\sigma)$. However, Proposition 4.1 means that $t(\sigma)_1 = n-1$. Thus, when applying t to $nt(\sigma)$, we have that $n-1$ is pushed into the stack right after n is and the stack is 21-avoiding throughout the entire sort. As a result, $t(\sigma)$ is sorted as if n was not there, so

$$t^2(\pi) = t(nt(\sigma)) = t^2(\sigma)n = \mathrm{id}_{n-1}n = \mathrm{id}_n.$$ □

5. The Properties of the $(\{12, 21\}, 1)-$ Machine

We prove that the machine $(s \circ t)$ eventually sorts all permutations through a series of supporting lemmas. We first characterize the output of the map $(s \circ t)$ when $\pi_1 = 1$.

Lemma 5.1. *Given $\pi \in S_n$, if $\mathrm{lead}(\pi) \geq 1$, some integer $1 \leq i \leq n - \mathrm{lead}(\pi)$ satisfies*

$$(s \circ t)(\pi)_{[i:i+\mathrm{lead}(\pi)]} = 12\ldots(\mathrm{lead}(\pi) + 1).$$

Proof. From Lemma 4.2, we have that $t(\pi) = \sigma(\mathrm{lead}(\pi) + 1)\mathrm{lead}(\pi)\ldots 1$ where σ is some (possibly empty) permutation. Then, in the stack sorting process of s, when $\mathrm{lead}(\pi) + 1$ is added to the stack, the rest of $\mathrm{lead}(\pi)\ldots 1$ will be added to the stack immediately after; the lemma statement follows. □

Next, we characterize the output of $(s \circ t)$ when $\pi_1 \neq 1$ and $\pi_n \neq 1$.

Lemma 5.2. *Given $\pi \in S_n$ and integers i and j such that $2 \leq i < i + j - 1 \leq n$, if $\pi_{[i:i+j-1]} = 12\ldots j$, there exists an integer $1 \leq \ell < i$ such that*

$$(s \circ t)(\pi)_{[\ell:\ell+j-1]} = 12\ldots j.$$

Proof. From Lemma 4.4, we have that $t(\pi)_{[\ell:\ell+j-1]} = 12\ldots j$ for some $1 \leq \ell < i$. Then, in the stack sorting process of s, all entries $1 \leq i \leq j-1$ are popped in order by $i+1$, respectively, until j enters the stack. If j is not the last entry of π, then the entry immediately after will pop j into the output. If j is the last entry of π, then j will be added to the output by definition. The lemma statement follows. □

We end by showing repeated applications of T eventually increase the lead length of π.

Lemma 5.3. *Given $\pi \in S_n$, if for some positive integer a we have that $\mathrm{lead}((s \circ t)^a(\pi)) < n$, there exists an integer $b > a$ such that*

$$\mathrm{lead}((s \circ t)^b(\pi)) > \mathrm{lead}((s \circ t)^a(\pi)).$$

Proof. Let $\ell = \mathrm{lead}((s \circ t)^a(\pi))$. By Lemma 5.1, some integer $1 \leq i \leq n - k$ satisfies $(s \circ t)^{a+1}(\pi)_{[i:i+\ell]} = 12\ldots(\ell + 1)$. If $i = 1$ we are done. If $i \neq 1$, Lemma 5.2 concludes. □

Now, we can prove that the $(\{12, 21\}, 1)-$machine eventually sorts all permutations.

Theorem 5.1. *For every positive integer n and every permutation $\pi \in S_n$, there exists a nonnegative integer j such that $(s \circ t)^j(\pi) = \mathrm{id}_n$*

Proof. If $\pi_n = 1$, then clearly $t(\pi)_n \neq 1$ and thus $(s \circ t)(\pi)_n \neq 1$ as 1 is popped by the subsequent entry in $t(\pi)$ during the stack sorting process of s. Then, from Lemma 5.2, there exists a such that $(s \circ t)^a(\pi)_1 = 1$. If $\pi_n \neq 1$, the same result follows immediately from Lemma 5.2. Then from Lemma 5.3 there exists some $b > a$ satisfying $\mathrm{lead}((s \circ t)^b(\pi)) = n$. We thus have that $(s \circ t)^b(\pi) = \mathrm{id}_n$ by definition. □

We again conjecture on what permutations take the most number of sorts and what that number is. Let $\mathrm{m}_{(s \circ t)}(\pi)$ be the smallest nonnegative integer j such that $(s \circ t)^j(\pi) = \mathrm{id}_{|\pi|}$.

Conjecture 5.1. *For all $\pi \in S_n$, the greatest value of* $\mathrm{m}_{(s\circ t)}(\pi)$ *is* $2n-5$.

To partially resolve the conjecture by showing that there are permutations π such that $\mathrm{m}_{(s\circ t)}(\pi) = 2n-5$, we first prove a supporting lemma.

Lemma 5.4. *For any $n \geq 1$ and some $3 \leq i < n-1$, we have that*

$$(s\circ t)^2(i(\mathrm{id}_{n-i-1}+i)(\mathrm{id}_{i-1})n) = (\mathrm{id}_{n-i-1}+i)(\mathrm{id}_{i-1})in.$$

Proof. First, t is applied. When $i(\mathrm{id}_{n-i-1}+i)$ is pushed into the stack, the stack is effectively 12−avoiding. Next, 1 pops $(\mathrm{id}_{n-i-1}+i)$ (in reverse order), turning the stack to 21−avoiding. Then, each entry of (id_{i-1}) pops the entry prior to itself. Finally, n pops $i-1$, and ni is popped to the output from the stack, resulting in the output $(\mathrm{rev}(\mathrm{id}_{n-i-1}+i))(\mathrm{id}_{i-1}ni)$.

Then, when s is applied, all of $\mathrm{rev}(\mathrm{id}_{n-i-1}+i)$ is pushed into the stack. Next, (id_{i-1}) is pushed into the stack, with each entry of (id_{i-1}) popping the entry prior to itself. Afterwards, n pops the entire stack, adding $(i-1)(\mathrm{id}_{n-i-1}+i)$ to the output. Finally, i is pushed into the stack and in is popped to the output, resulting in $(\mathrm{id}_{i-1})(\mathrm{id}_{n-i-1}+i)in$.

When t is first applied in the next application of $(s\circ t)$, all of $(\mathrm{id}_{i-1})(\mathrm{id}_{n-i-1}+i)$ is pushed into the effectively 12−avoiding stack. Then, i pops all of $(\mathrm{id}_{n-i-1}+i)$ and n gets pushed into the stack. Finally, $ni(\mathrm{rev}(\mathrm{id}_{i-1}))$ is popped and added to the output, resulting in $(\mathrm{rev}(\mathrm{id}_{n-i-1}+i))ni(\mathrm{rev}(\mathrm{id}_{i-1}))$.

When s is applied again, all of $\mathrm{rev}(\mathrm{id}_{n-i-1}+i)$ are pushed into the stack. Then, n pops all of $\mathrm{rev}(\mathrm{id}_{n-i-1}+i)$ out and all of $i(\mathrm{rev}(\mathrm{id}_{i-1}))$ are pushed into the stack. Finally, all of $(\mathrm{id}_{i-1})in$ are popped to the output, resulting in $(\mathrm{id}_{n-i-1}+i)(\mathrm{id}_{i-1})in$. ☐

Now, we prove the existence of permutations π such that $\mathrm{m}_{(s\circ t)}(\pi) = 2n-5$.

Lemma 5.5. *For $n \geq 3$ and $\pi \in S_n$, if $\pi = 2\sigma 1n$ where $\sigma \in Av_{n-3}(213)$ then* $\mathrm{m}_{(s\circ t)}(\pi) = 2n-5$.

Proof. When all such permutations are sorted by t, 2 enters the stack first. As all elements of σ are greater than 2, the stack is effectively 12−avoiding. Consider the first element $\sigma_i \in \sigma$ that is popped and assume $\sigma_i \neq n-1$. If $i = n-3$ then $\sigma_i = n-1$, a contradiction. Otherwise, $\sigma_{i+1} < \sigma_i$. Because $\sigma \in \mathrm{Av}_{n-3}(213)$, there must be no entries larger than σ_i after σ_i. σ_i would therefore pop $n-1$ from the stack when it first enters, a contradiction. Thus, $n-1$ is the first entry that is popped from σ, and analogously $n-k$ is the kth entry that is popped. Thus, $t(\pi) = \mathrm{rev}(\mathrm{id}_{n-3}+2)1n2$, and subsequently $(s\circ t)(\pi) = 1(\mathrm{id}_{n-3}+2)2n$.

In the next sort of t, all of $1(\mathrm{id}_{n-3}+2)$ enter the stack; then, 2 entering pops out $\mathrm{id}_{n-3}+2$, resulting in the output $\mathrm{rev}(\mathrm{id}_{n-3}+2)n21$. In the next sort of s, all of $\mathrm{rev}(\mathrm{id}_{n-3}+2)$ enter the stack and n immediately pops them out. It follows that $(s\circ t)^2(\pi) = (\mathrm{id}_{n-3}+2)12n$. Then, by Lemma 5.4, we have that $t^{2n-8}(\mathrm{id}_{n-3}+2)12n) = (n-1)\mathrm{id}_{n-2}n$. Lastly, we have that $t((n-1)\mathrm{id}_{n-2}n) = \mathrm{id}_{n-2}n(n-1)$ and $s(\mathrm{id}_{n-2}n(n-1)) = \mathrm{id}_n$, for a total of $2n-5$ sorts. ☐

We also conjecture that the permutations in Lemma 5.5 are the only minimally sorted permutations with respect to the map $(s\circ t)$.

Conjecture 5.2. *The permutation π takes $2n-5$ sorts to be sorted to the identity by $(s\circ t)$ if and only if $\pi = 2\sigma 1n$ and $\sigma \in Av_{n-3}(213)$.*

Finally, we show that two-stack-sortable permutations under t can generate one-stack-sortable permutations under $(s\circ t)$ and vice versa. First, we start with a two-stack-sortable permutation under t.

Lemma 5.6. *Given $\pi \in S_{n+1}$ such that $\pi_i = n+1$ and $i \neq 1$, let $\sigma = \pi_{[1\ldots i-1]}\pi_{[i+1\ldots n]}$. If $t^2(\pi) = \mathrm{id}_{n+1}$ then $(s\circ t)(\sigma) = \mathrm{id}_n$.*

Proof. As in Lemma 4.10, we have that $t(\pi) = (n+1)t(\sigma)$. Moreover, $t(\pi)_{[2:n+1]} = t(\sigma)$ is 231-avoiding because $t^2(\pi) = \mathrm{id}_{n+1}$. Thus, from Lemma 2.1, $(s\circ t)(\sigma) = \mathrm{id}_n$. ☐

Next, we start with a one-stack-sortable permutation under $(s\circ t)$.

Lemma 5.7. *Given $\sigma \in S_n$ and $1 \leq i \leq n$ such that $\sigma_1 < \sigma_2 < \cdots < \sigma_i$, let $\pi = \sigma_{[1\ldots i]}(n+1)\sigma_{[i+1\ldots n]}$. If $(s\circ t)(\sigma) = \mathrm{id}_n$ then $t^2(\pi) = \mathrm{id}_{n+1}$.*

Proof. Since $\pi_1 < \pi_2 < \cdots < \pi_{i+1} = n+1$, we have that $n+1$ is the first element popped to the output, so $t(\pi) = (n+1)t(\sigma)$. Then, because $(s\circ t)(\sigma) = \mathrm{id}_n$, it follows that $t(\sigma) = t(\pi)_{[2:n+1]}$ is 231-avoiding. Therefore, from Lemma 2.1, $t^2(\pi) = \mathrm{id}_{n+1}$. ☐

Acknowledgements

We thank the anonymous referee for detailed comments on the presentation of the paper.

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