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Sharing Pizza Among a Two Power Number of Friends

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Abstract: We extend the work of Hirschhorn, Hirschhorn, Hirschhorn, Hirschhorn and Hirschhorn on the problem of sharing pizza among several people. More specifically, we present a class of Coxeter arrangements in an even dimension n such that if one cuts a ball containing the origin with one of these arrangements, the slices can be partitioned into $2^{n/2}$ blocks such that each block has the same volume.

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1. Introduction

The original pizza problem was formulated by Upton as a problem in Mathematics Magazine [8]. Given a circular disc, pick an arbitrary point in the disc. Cut the disc with four lines through this point such that adjacent cuts have an angle of $\pi/4$ radians. When alternating the slices between two friends, they will receive the same amount of pizza. Goldberg [5] provided the solution for $2k$ lines where $k \geq 2$. Recently, this problem was generalized to higher dimensions independently by Brailov [2] and Ehrenborg–Morel–Readdy [3]. Returning to two dimensions Hirschhorn, Hirschhorn, Hirschhorn, Hirschhorn and Hirschhorn showed that if you cut the pizza with the dihedral arrangement with $2p$ lines, p people can share the pizza fairly [6]. In four dimensions Ehrenborg–Morel–Readdy showed that 4 people can share a 4-dimensional ball fairly if they use a Coxeter arrangement of type F_4; see [3, Section 9]. Other than this result, the problem of sharing pizza among more than two people in dimensions greater than two has remained open.

In this paper, we show that for n even, there is a class of Coxeter arrangements in n dimensions such that $2^{n/2}$ people can fairly share a pizza. Our technique is to use one of the pizza results of Ehrenborg–Morel–Readdy (see Theorem 2.2) to a class of subarrangements of our arrangement. This yields a linear equation system in the sought-after quantities. The coefficient matrix of this system is a classical Hadamard matrix, and hence it is straightforward to see that all the quantities have the same value, proving the result.

We end this note with open questions.

2. Preliminaries

Let V be a finite-dimensional real inner product space. Let $\mathcal{H}$ be a central hyperplane arrangement in the space V. The connected components of the complement $V - \bigcup_{H \in \mathcal{H}} H$ are called chambers. Select one chamber T_0 to be the base chamber. Every chamber has a sign $(-1)^T$ given by $(-1)^r$ where r is the number hyperplanes of $\mathcal{H}$ that separates the chamber T from the base chamber T_0. For measurable set K the *pizza quantity* is defined to be the alternating sum

$$P(\mathcal{H}, K) = \sum (-1)^T \cdot \mathrm{Vol}(K \cap T)$$

where T ranges over all chambers of the arrangement $\mathcal{H}$.

A Coxeter arrangements $\mathcal{H}$ is a hyperplane arrangement such that the group W generated by the orthogonal reflections in the hyperplanes of $\mathcal{H}$ is finite and the arrangement is closed under all such reflections. Given two hyperplane arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ in the spaces V_1, respectively, V_2, the product arrangement $\mathcal{H}_1 \times \mathcal{H}_2$ in the Cartesian product $V_1 \times V_2$ is described by

$$\mathcal{H}_1 \times \mathcal{H}_2 = \{H \times V_2 : H \in \mathcal{H}_1\} \cup \{V_1 \times H : H \in \mathcal{H}_2\}.$$

Note that Coxeter arrangements are closed under this product. A Coxeter arrangement that cannot factor under this product is called *irreducible*. It is well-known that the irreducible Coxeter arrangements have been classified; see for instance [1].

The only irreducible arrangement needed for this note is the dihedral arrangement. It is the arrangement in the plane $\mathbb{R}^2$ consisting of k lines through the origin, where adjacent lines meet at an angle of π/k radians. In the Coxeter classification, the dihedral arrangement has type $I_2(k)$.

Let $\mathbb{B}(a, R)$ denote the ball centered at the point a of radius R. Goldberg's result [5] can now be stated as:

Theorem 2.1 (Goldberg)**.** *Let $\mathcal{H}$ be the dihedral arrangement $I_2(2k)$ in $\mathbb{R}^2$ for $k \geq 2$. For every point $a \in \mathbb{R}^2$ such that $0 \in \mathbb{B}(a, R)$, the pizza quantity for the disc $\mathbb{B}(a, R)$ vanishes, that is, $P(\mathcal{H}, \mathbb{B}(a, R)) = 0$.*

Ehrenborg–Morel–Readdy generalized this result to the following statement in higher dimensions [3, Theorem 1.1(i)].

Theorem 2.2 (Ehrenborg–Morel–Readdy)**.** *Let $\mathcal{H}$ be a Coxeter arrangement in a finite-dimensional inner product space V of dimension n. Assume that the number of hyperplanes of $\mathcal{H}$ is strictly greater than the dimension n and has the same parity as n. If the ball $\mathbb{B}(a, R)$ contains the origin then the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ vanishes, that is, $P(\mathcal{H}, \mathbb{B}(a, R)) = 0$.*

3. The construction

Let V_j be an inner product space of dimension n_j for $1 \leq j \leq k$ and let $\mathcal{H}_j$ be a Coxeter arrangement in V_j. The product arrangement $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_k$ is the arrangement in the Cartesian product $V = V_1 \times V_2 \times \cdots \times V_k$ consisting of the hyperplanes

$$\mathcal{H} = \{V_1 \times \cdots \times V_{j-1} \times H \times V_{j+1} \times \cdots \times V_k : H \in \mathcal{H}_j\}.$$

For a vector $\vec{\imath} = (i_1, i_2, \ldots, i_k) \in \{0, 1\}^k$ define the subarrangement $\mathcal{H}(\vec{\imath})$ of $\mathcal{H}$ to be the collection of the hyperplanes

$$\mathcal{H}(\vec{\imath}) = \{V_1 \times \cdots \times V_{j-1} \times H \times V_{j+1} \times \cdots \times V_k : i_j = 1, H \in \mathcal{H}_j\},$$

that is, for the zero index vector $\vec{0} = (0, 0, \ldots, 0)$ the subarrangement $\mathcal{H}(\vec{0})$ is the empty arrangement. For the index vector $(1, 1, \ldots, 1)$, the subarrangement $\mathcal{H}((1, 1, \ldots, 1))$ is the product arrangement $\mathcal{H}$. The arrangement $\mathcal{H}(\vec{\imath})$ is also a product arrangement, that is, it factors as follows:

$$\mathcal{H}(\vec{\imath}) = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_k,$$

where $\mathcal{F}_j$ is the arrangement $\mathcal{H}_j$ if $i_j = 1$ and is the empty arrangement on V_j if $i_j = 0$. Since each factor is a Coxeter arrangement, we conclude that $\mathcal{H}(\vec{\imath})$ is a Coxeter arrangement.

Let T_0 be a base chamber for the product arrangement $\mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_k$. For each chamber T of the product arrangement $\mathcal{H}$ define the vector $\vec{r}(T) = (r_1, r_2, \ldots, r_k) \in \{0, 1\}^k$ by letting r_j be the parity of the number hyperplanes inherited from $\mathcal{H}_j$ that separate the chamber T from the base chamber T_0.

Let $T_0(\vec{\imath})$ be the chamber of the arrangement $\mathcal{H}(\vec{\imath})$ that contains the chamber T_0. We view $T_0(\vec{\imath})$ as the base chamber of $\mathcal{H}(\vec{\imath})$. Let T be a chamber of the product arrangement $\mathcal{H}$. The chamber T is contained in a unique chamber T' of the subarrangement $\mathcal{H}(\vec{\imath})$. We claim that the sign of the chamber T' is given by $(-1)^{\vec{\imath} \cdot \vec{r}(T)}$ where $\vec{\imath} \cdot \vec{r}(T)$ denotes the inner product of the two vectors $\vec{\imath}$ and $\vec{r}(T)$. The reason is that we only count hyperplanes whose index j satisfies $i_j = 1$.

We can now state our main result.

Theorem 3.1. *Assume that for each index i the dimension n_i of V_i is even. Furthermore, assume that the number of hyperplanes of the arrangement $\mathcal{H}_i$ is even and is strictly greater than the sum $n = n_1 + n_2 + \cdots + n_k$. Let $\mathbb{B}(a, R)$ be a ball that contains the origin. Then for all index vectors $\vec{\jmath} \in \{0, 1\}^k$ the following equality holds:*

$$\frac{\mathrm{Vol}(\mathbb{B}(a, R))}{2^k} = \sum_{T \,:\, \vec{r}(T) = \vec{\jmath}} \mathrm{Vol}(T \cap \mathbb{B}(a, R)). \tag{1}$$

Proof. For $\vec{\jmath} \in \{0,1\}^k$ let $x_{\vec{\jmath}}$ be the sum on the right-hand side of equation (1). Directly, we have the equality

$$\mathrm{Vol}(\mathbb{B}(a,R)) = \sum_{\vec{\jmath} \in \{0,1\}^k} x_{\vec{\jmath}}. \tag{2}$$

Let $\vec{\imath}$ be a non-zero index vector in $\{0,1\}^k$. The subarrangement $\mathcal{H}(\vec{\imath})$ has more hyperplanes than the dimension n. Furthermore, the number of hyperplanes is even, that is, the same parity as the dimension n. Hence Theorem 2.2 applies and we have

$$0 = \sum_{T} (-1)^{\vec{\imath}\cdot\vec{r}(T)} \cdot \mathrm{Vol}(T \cap \mathbb{B}(a,R)) = \sum_{\vec{\jmath} \in \{0,1\}^k} \sum_{T \,:\, \vec{r}(T)=\vec{\jmath}} (-1)^{\vec{\imath}\cdot\vec{\jmath}} \cdot \mathrm{Vol}(T \cap \mathbb{B}(a,R)) = \sum_{\vec{\jmath} \in \{0,1\}^k} (-1)^{\vec{\imath}\cdot\vec{\jmath}} \cdot x_{\vec{\jmath}}. \tag{3}$$

Observe that equations (2) and (3) form a linear equation system in the unknown variables $x_{\vec{\jmath}}$. The coefficient matrix has rows and columns indexed by the set $\{0,1\}^k$ and is given by

$$M = \left((-1)^{\vec{\imath}\cdot\vec{\jmath}}\right)_{\vec{\imath},\vec{\jmath} \in \{0,1\}^k}.$$

This is a Hadamard matrix, in fact, first considered by Sylvester [7]. Hence, the matrix M is non-singular and the linear system has a unique solution. It is now straightforward to verify that the solution is given by $x_{\vec{\jmath}} = \mathrm{Vol}(\mathbb{B}(a,R))/2^k$ for all index vectors $\vec{\jmath}$. □

Using the dihedral arrangement, we obtain the following result.

Corollary 3.1. *For even dimension n, the Coxeter arrangement of the product type*

$$I_2(k_1) \times I_2(k_2) \times \cdots \times I_2(k_{n/2}), \tag{4}$$

where $k_1, k_2, \ldots, k_{n/2}$ are all even and greater than or equal to $n+2$, allows one to share pizza fairly between $2^{n/2}$ people.

Note that the number of hyperplanes in the arrangement in equation (4) is $k_1 + k_2 + \cdots + k_{n/2} \geq n/2 \cdot (n+2)$. The smallest such arrangement is the following.

Corollary 3.2. *For even dimension n, the Coxeter arrangement of type $I_2(n+2)^{n/2}$ allows us to share pizza fairly between $2^{n/2}$ people.*

It might be a bit impractical since we would have a total number of $(2n+4)^{n/2}$ slices and each person receives $(n+2)^{n/2}$ slices. Also note that for $n = 4$ our construction uses 12 hyperplanes whereas, the Ehrenborg–Morel–Readdy construction [3, Section 9] based on the Coxeter arrangement of type F_4 uses 24 hyperplanes.

4. Open questions

To our knowledge, it is still an open question on how to share pizza in odd dimensions among several people. Are there ways to share a pizza among not a two power number of people? Finally, is there a dissection proof of Theorem 3.1 in the spirit of the results in [4]?

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