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A Descent-Excedance Correspondence in Colored Permutation Groups

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Abstract: It is well known that descents and excedances are equidistributed in the symmetric group. We show that the descent and excedance enumerators, summed over permutations with a fixed first letter, are identical when we perform a simple change of the first letter. We generalize this to type B and colored permutation groups. We find q-analogues of generating functions for descent enumerating polynomials with respect to the major index when summed over permutations that start with a fixed first letter. Using the flag major index, we also generalise to obtain type B results. In the univariate case, we show that the aforementioned polynomials are real-rooted but need not be palindromic. Finally, we show that a symmetrised descent enumerating polynomial over the hyperoctahedral group with fixed first letter is palindromic and gamma-positive.

Keywords: Carlitz identity; Colored permutation group; Eulerian statistic; Gamma-positivity
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1. Introduction

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathfrak{S}_n$ denote the set of permutations of the set $[n]$. For $\pi \in \mathfrak{S}_n$ with $\pi = \pi_1, \pi_2, \ldots, \pi_n$, let $\mathrm{DES}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\}$ be its set of descents and let $\mathrm{des}(\pi) = |\mathrm{DES}(\pi)|$ denote its number of descents. For $\pi \in \mathfrak{S}_n$ let $\mathrm{EXC}(\pi) = \{i \in [n] : \pi_i > i\}$ be its set of excedances and let $\mathrm{exc}(\pi) = |\mathrm{EXC}(\pi)|$ denote its number of excedances. The polynomial $A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}$ is defined as the n-th Eulerian polynomial and is well studied, see, for example, the book [13] by Petersen. If we define $\mathrm{AExc}_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{exc}(\pi)}$, then, for all non-negative integers n, it is well known (see [13]) that $A_n(t) = \mathrm{AExc}_n(t)$.

For a positive integer n and for $i \in [n]$, let $\mathfrak{S}_{n,i} = \{\pi \in \mathfrak{S}_n : \pi_1 = i\}$ be the subset of $\mathfrak{S}_n$ consisting of all π that start with the letter i. While working on riffle shuffles in decks with repeated cards, Conger [6] enumerated descents over the set $\mathfrak{S}_{n,i}$. Let $A_{n,d,i}$ denote the number of permutations in $\mathfrak{S}_{n,i}$ that have d descents. Define

$$A_{n,i}(t) := \sum_{\pi \in \mathfrak{S}_{n,i}} t^{\mathrm{des}(\pi)} = \sum_{d=0}^{n-1} A_{n,d,i} t^d. \tag{1}$$

The following identity and generating function were shown by Conger [6, Theorem 1 and Equation 25].

Theorem 1.1 (Conger)**.** *With the above notation, when $1 \leq i \leq n$, we have*

$$\begin{aligned} A_{n,d,i} &= \sum_{j \geq 0} (-1)^{d-j} \binom{n}{d-j} j^{i-1} (j+1)^{n-i}, \\ \frac{A_{n,i}(t)}{(1-t)^n} &= \sum_{j \geq 1} j^{i-1} (j+1)^{n-i} t^j. \end{aligned}$$

Various q-analogues of identities for descent polynomials over Coxeter groups are known, see, for example, Pan and Zeng [14] and Dey, Shankar, and Sivasubramanian [8]. In this work, we obtain the following q-analogue of Theorem 1.1. Let $\mathrm{maj}(\pi) := \sum_{i\in \mathrm{DES}(\pi)} i$. Define

$$A_{n,i}(t,q) := \sum_{\pi\in\mathfrak{S}_{n,i}} t^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} \text{ and } A_{n,d,i}(q) = \sum_{\pi\in\mathfrak{S}_{n,i}:\mathrm{des}(\pi)=d} q^{\mathrm{maj}(\pi)}.$$

Our result is the following.

Theorem 1.2. *Let $A_{n,i}(t,q)$ be defined as above. When $1 \le i \le n$, we have*

$$\begin{aligned} A_{n,d,i}(q) &= \sum_{j\ge 0} (-1)^{d-j} \binom{n}{d-j}_q q^{\binom{d-j+1}{2}+j} [j]_q^{i-1} [j+1]_q^{n-i}, \\ \frac{A_{n,i}(t,q)}{(tq;q)_n} &= \sum_{j\ge 0} q^j [j]_q^{i-1} [j+1]_q^{n-i} t^j. \end{aligned}$$

It is easy to see that Theorem 1.2 clearly generalizes Theorem 1.1. As can be expected, type B analogues of this result exist. Recall that $\mathfrak{B}_n$ is the group of signed permutations over $[\pm n]$. We give a Carlitz-type identity for the polynomials $B_{n,i}(t)$. The type B descent set is defined as follows $\mathrm{DES}_B(\pi) := \{i \in [0, n-1] | \pi_i > \pi_{i+1}\}$ where $\pi_0 := 0$. The type B descent of a permutation is defined as $\mathrm{des}_B(\pi) := |\mathrm{DES}_B(\pi)|$. For a signed permutation, define the $\mathrm{DES}_A(\pi) = \{r \in [n-1] : \pi_r > \pi_{r+1}\}$ and $\mathrm{des}_A(\pi) = |\mathrm{DES}_A(\pi)|$. The type A major index is defined by $\mathrm{maj}_A(\pi) = \sum_{r\in \mathrm{DES}_A(\pi)} r$. To obtain the result that generalises these results known over $\mathfrak{B}_n$, we use the fmajor index statistic introduced by Adin and Roichman in [1]. Also, see the paper by Gessel and Chow [4]. For $\pi \in \mathfrak{B}_n$, define its flag major index (fmaj, henceforth) as $\mathrm{fmaj}(\pi) = 2\mathrm{maj}_A(\pi) + N(\pi)$, where $N(\pi)$ is the number of negative entries in π. Define

$$B_{n,i}(t,q) := \sum_{\pi\in\mathfrak{B}_{n,i}} t^{\mathrm{des}_B(\pi)} q^{\mathrm{fmaj}(\pi)} \text{ and } \quad B_{n,d,i}(q) = \sum_{\pi\in\mathfrak{B}_{n,i}:\mathrm{des}_B(\pi)=d} q^{\mathrm{fmaj}(\pi)}. \tag{2}$$

Our result refines the Carlitz-type identity of Brenti [3, Theorem 3.4(iii)] for the type B Coxeter group and depends on the sign of the first letter. Our result is proved in Section 5 and is the following.

Theorem 1.3. *Let $B_{n,i}(t,q)$ be defined as in (2).*

1. *We have the following identities. When $i > 0$,*

$$B_{n,d,i}(q) = \sum_{j=0}^{n-1} (-1)^{d-j} \binom{n}{d-j}_{q^2} q^{2\binom{d-j+1}{2}+2j} [2j]_q^{i-1} [2j+1]_q^{n-i}.$$

When $i < 0$,

$$B_{n,d,i}(q) = \sum_{j=0}^{n-1} (-1)^{d-j} \binom{n}{d-j}_{q^2} q^{2\binom{d-j+1}{2}+2j-1} [2j]_q^{i-1} [2j-1]_q^{n-i}.$$

2. *We have the following generating function for these polynomials. For $0 < i \le n$,*

$$\frac{B_{n,i}(t,q)}{(tq^2;q^2)_n} = \sum_{j\ge 0} q^{2j} [2j]^{i-1} [2j+1]^{n-i} t^j.$$

For $0 < i \le n$,

$$\frac{B_{n,i}(t,q)}{(tq^2;q^2)_n} = \sum_{j\ge 0} q^{2j-1} [2j]^{i-1} [2j-1]^{n-i} t^j.$$

Using the identity $[2j]^n - [2j-1]^n = ([2j] - [2j-1]) \sum_{r\ge 0} [2j]^r [2j-1]^{n-r}$ and summing over all $i, \bar{i}$ gives us Brenti's result [3, Theorem 3.4(iii)]. A Carlitz identity for the wreath product $\mathfrak{S}_n \wr \mathbb{Z}_r$ was given by Chow and Mansour [5]. It would be interesting to see if their identity can be refined along these lines by fixing the first letter and the color. As mentioned earlier, it is well known that $A_n(t) = \mathrm{AExc}_n(t)$. We ask if a similar result holds when we consider the set $\mathfrak{S}_{n,i}$. Towards answering this, for a positive integer n and for $1 \le j \le n$, recall $A_{n,i}(t)$ from (1) and define $\mathrm{AExc}_{n,i}(t) = \sum_{\pi\in\mathfrak{S}_{n,i}} t^{\mathrm{exc}(\pi)}$ to be the polynomials enumerating descents and excedances respectively in $\mathfrak{S}_{n,i}$. When $n = 6$, we get the following table giving $A_{6,i}(t)$ and $\mathrm{AExc}_{6,i}(t)$ (seen better on a colour monitor).

i	$A_{6,i}(t)$ wrt descents	$\mathsf{AExc}_{6,i}(t)$ wrt excedances
1	$1 + 26t + 66t^2 + 26t^3 + t^4$	$1 + 26t + 66t^2 + 26t^3 + t^4$
2	$16t + 66t^2 + 36t^3 + 2t^4$	$t + 26t^2 + 66t^3 + 26t^4 + t^5$
3	$8t + 60t^2 + 48t^3 + 4t^4$	$2t + 36t^2 + 66t^3 + 16t^4$
4	$4t + 48t^2 + 60t^3 + 8t^4$	$4t + 48t^2 + 60t^3 + 8t^4$
5	$2t + 36t^2 + 66t^3 + 16t^4$	$8t + 60t^2 + 48t^3 + 4t^4$
6	$t + 26t^2 + 66t^3 + 26t^4 + t^5$	$16t + 66t^2 + 36t^3 + 2t^4$

Our main result is Theorem 1.4 (proved in Section 3), which holds for colored permutation groups. As a corollary, when the number of colors is 1, we get the following.

Corollary 1.1. *When $n \geq 2$, we have $A_{n,1}(t) = \mathrm{AExc}_{n,1}(t) = A_{n-1}(t)$. When $n \geq 2$ and $2 \leq i \leq n$, we have $A_{n,i}(t) = \mathrm{AExc}_{n,n+2-i}(t)$. Thus, for a fixed n, the two families of polynomials $A_{n,i}(t)$ and $\mathrm{AExc}_{n,i}(t)$ are the same up to a permutation of the second index i in the subscript.*

For the proof of Theorem 1.4, we need a modification of Foata's first fundamental transformation. To the best of our knowledge, we have not seen this modified version, see Remark 3.1 for the exact relation. Another application of Foata's first fundamental transformation mapping descents of a special kind to *pure excedances* in $\mathfrak{S}_n$ was recently given by Baril and Kirgizov in [2]. They defined pure excedance of $\pi \in \mathfrak{S}_n$ to be the number of positions i such that $i < \pi_i$ and there is no $j < i$ with $i \leq \pi_j < \pi_i$.

For a positive integer r, let $[r]_0 = \{0, 1, \ldots, r-1\}$ be a set of size r. The colored permutation groups are denoted by $\mathfrak{S}_n \wr \mathbb{Z}_r$ (see, for example Steingrímsson [15]). To define descents and excedances over $\mathfrak{S}_n \wr \mathbb{Z}_r$, we need a linear order L on $[n] \times [r]_0$. With respect to such a linear order L, for a permutation π, we denote descents as $\mathrm{ldes}(\pi)$ and excedances as $\mathrm{lexc}(\pi)$ (see Section 2 for our definition). We denote $\mathfrak{S}_n \wr \mathbb{Z}_r$ alternatively as $\mathfrak{S}_n^r$ and call this the colored permutation group.

In Section 2, we define the min-one linear order on elements of colored permutations. It will be clear when there is a single color that the min-one order reduces to the natural order on $\mathbb{N}$ and that ldes and lexc with respect to these orders give the usual definition of des and exc in $\mathfrak{S}_n$ respectively. Let $S_{(i,j)}$ be the set of colored permutations in $\mathfrak{S}_n^r$ that start with the letter i that further have colour j for the letter i. With respect to the min-one order, we show that enumerating ldes over $S_{(i,j)}$ gives the same polynomial as enumerating lexc over $S_{(\sigma(i),\mu(j))}$ for some permutations σ and μ. Our result proved in Section 3 is the following.

Theorem 1.4. *With respect to the min-one order, for all positive integers n, r, there exists a bijection $\Gamma : \mathfrak{S}_n^r \to \mathfrak{S}_n^r$ such that $\mathrm{ldes}(p) = \mathrm{lexc}(\Gamma(p))$ for all $p \in \mathfrak{S}_n^r$ which additionally satisfies $\Gamma(S_{(i,j)}) = S_{(n+2-i,r+1-j)}$, when $i \geq 2$ and $j \in [n]$. When $i = 1$ and $j \in [n]$, we have $\Gamma(S_{(1,j)}) = S_{(1,r+1-j)}$.*

With respect to the min-one order, for positive integers n, r, the polynomials enumerating ldes in $\mathfrak{S}_n^r$ have degree $n-1$. Recall that $\mathfrak{B}_n$ is the group of signed permutations over $[\pm n]$ and enumeration of the type B descent, des_B, over $\mathfrak{B}_n$ gives a degree n polynomial. Thus, if we use the min-one order and have two colors, we will not get a generalization of known results for $\mathfrak{B}_n$.

Thus, to obtain a generalisation over $\mathfrak{B}_n$, we consider colored permutations with an even number of colors and append a zero to the front of all such permutations. We denote this as $\overline{\mathfrak{S}_n^{2r}}$. The element zero is also given a place in the linear order. We thus define our second linear order called the symmetric order with respect to which, enumeration of ldes gives a degree n polynomial for all n, r. Further, this linear order gives the standard linear order on $\mathbb{Z}$ when we have two colors, and further, in this case, ldes reduces to des_B.

We define a statistic bexc that generalises Brenti's type B excedance statistic to colored permutations with an even number of colors. Our second main result, proved in Section 6, is the following.

Theorem 1.5. *With respect to the symmetric order, for even colored permutation groups $\overline{\mathfrak{S}_n^{2r}}$, we have a bijection $\Gamma : \overline{\mathfrak{S}_n^{2r}} \mapsto \overline{\mathfrak{S}_n^{2r}}$ satisfying $\mathrm{ldes}(p) = \mathrm{bexc}(\Gamma(p))$. Further, our bijection Γ satisfies $\Gamma(S_{(i,j)}) = S_{(i,\bar{j})}$ and $\Gamma(S_{(i,\bar{j})}) = S_{(i,j)}$ when $1 \leq i \leq n$ and $1 \leq j \leq r$.*

Define the restricted type B Eulerian numbers that count descents and excedances when the starting letter and its colour are fixed. Let $\mathfrak{B}_{n,i} := \{\pi \in \mathfrak{B}_n : \pi_1 = i\}$. When $i \in [-n, n] \backslash \{0\}$, define the polynomials

$$B_{n,i}(t) := \sum_{\pi \in \mathfrak{B}_{n,i}} t^{\mathrm{des}_B(\pi)}, \qquad \mathrm{BE}_{n,i}(t) := \sum_{\pi \in \mathfrak{B}_{n,i}} t^{\mathrm{exc}_B(\pi)}.$$

When $d = 2$, we obtain the following corollary.

Corollary 1.2. *For positive integers n, k with $1 \leq i \leq n$, the following are equidistributed.*

1. *When $i = 1$, we have $B_{n,i}(t) = \mathrm{BE}_{n,i}(t)$ and $B_{n,\bar{i}}(t) = \mathrm{BE}_{n,\bar{1}}(t)$.*
2. *When $2 \leq i \leq n$, we have $B_{n,i}(t) = \mathrm{BE}_{n,\bar{i}}(t)$ and $B_{n,\bar{i}}(t) = \mathrm{BE}_{n,i}(t)$.*

A degree n polynomial $f(t) = \sum_{r=0}^{n} f_r t^r$ with $f_n \neq 0$ is said to be palindromic if $f_r = f_{n-r}$ when $0 \leq r \leq n$. A degree n palindromic polynomial $f(t)$ is said to be gamma positive if it can be written as $f(t) = \sum_{r=0}^{\lfloor n/2 \rfloor} \gamma_r t^r (1+t)^{n-2r}$ with $\gamma_r \geq 0$ for all r. Nevo, Petersen and Tenner [11] showed gamma positivity of the polynomial $A_{n,i}(t) + A_{n,n+1-i}(t)$.

Theorem 1.6 (Nevo, Petersen and Tenner). *For a positive integer n and for $1 \leq i \leq n$, the sum $A_{n,i}(t) + A_{n,n+1-i}(t)$ is palindromic and gamma positive.*

In Subsection 6.2, we show the following type B analogue of Theorem 1.6. For $1 \leq i \leq n$, define the restricted and symmetrized type B Eulerian polynomials, as

$$\overline{B}_{n,i}(t) = B_{n,i}(t) + B_{n,\bar{i}}(t) \quad \text{and} \quad \widetilde{B}_{n,i}(t) = tB_{n,i}(t) + B_{n,\bar{i}}(t).$$

Theorem 1.7. *For $1 \leq i \leq n$, both $\overline{B}_{n,i}(t)$ and $\widetilde{B}_{n,i}(t)$ are gamma-positive with the centers of symmetry $n/2$ and $(n+1)/2$ respectively.*

A polynomial is said to be real-rooted if all its roots are real. Dey [7] showed that the polynomials $A_{n,i}(t)$ are real-rooted when $n \geq 2$ and $1 \leq i \leq n$. Our next result (proved in Section 6) is a type B analogue of this result.

Theorem 1.8. *For positive integers n, i with $1 \leq i \leq n$, the polynomials $B_{n,i}(t)$ and $B_{n,\bar{i}}(t)$ are real-rooted.*

2. Preliminaries

A colored permutation is an element of the group $\mathfrak{S}_n^r = \mathbb{Z}_r \wr \mathfrak{S}_n$. We represent a colored permutation as a product $\pi \times c$ of a permutation word $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$ and an n-tuple $c = (c_1, \ldots, c_n)$ with each $c_i \in \mathbb{Z}_r$.

We also think of an element of $\pi \times c \in \mathfrak{S}_n^r$, where $\pi = \pi_1 \ldots \pi_n \in \mathfrak{S}_n$ and $c = (c_1, \ldots, c_n)$ with $c_i \in \mathbb{Z}_r$, as an n-tuple of pairs $((\pi_1, c_1), \ldots, (\pi_n, c_n))$. We will also somet replace the pair (π_i, c_i) by the letter π_i with the subscript c_i. This will let us write the element $\pi \times c$ as $(\pi_1)_{c_1} \ldots (\pi_n)_{c_n}$.

Given a linear ordering $>_L$ on the elements of the set $[n] \times [r]_0$, we will define three statistics on the set of colored permutations $\mathfrak{S}_n^r$ using this linear order.

Definition 2.1. *The L-*descent set *of a colored permutation $p = \pi \times c$ with respect to the linear ordering $>_L$ is the set,* $\mathrm{DescSet}_L(p) := \{i \in [n-1] \ : \ (\pi_i, c_i) >_L (\pi_{i+1}, c_{i+1})\}$. *The cardinality of* $\mathrm{DescSet}_L(p)$ *is denoted by* $\mathrm{ldes}(p)$.

Definition 2.2. *The L-*ascent set *of a colored permutation $p = \pi \times c$ with respect to the linear ordering $>_L$ is the set,* $\mathrm{AscSet}_L(p) := \{i \in [n-1] \ : \ (\pi_{i+1}, c_{i+1}) >_L (\pi_i, c_i)\}$. *The cardinality of* $\mathrm{AscSet}_L(p)$ *is denoted by* $\mathrm{lasc}(p)$.

For a coloured word $w = w_1 \ldots w_n$, define its reverse word as $\mathrm{REV}(w) = w_n \ldots w_1$. We note that both the value along with its color are reversed by this operation. It is clear that $\mathrm{REV} : \mathfrak{S}_n^r \mapsto \mathfrak{S}_n^r$ is a bijection. Also, it is easy to check that $\mathrm{lasc}(p) = \mathrm{ldes}(\mathrm{REV}(p))$.

Definition 2.3. *The L-*excedance set *of a colored permutation $p = \pi \times c$ with respect to the linear ordering $>_L$ is the set,* $\mathrm{ExcSet}_L(p) := \{i \in [n] \ : \ (\pi_{\pi_i}, c_{\pi_i}) >_L (\pi_i, c_i)\}$. *The cardinality of* $\mathrm{ExcSet}_L(p)$ *is denoted by* $\mathrm{lexc}(p)$.

Example 2.1. *Let $p = (241563, 013302) \in \mathfrak{S}_6^4$. Since $r = 4$, the order L is:*

$$1_0 <_L \cdots <_L 6_0 <_L 1_1 <_L \cdots <_L 6_1 <_L 1_2 <_L \cdots <_L 6_2 <_L 1_3 <_L \cdots <_L 6_3.$$

Here, $5_3 >_L 6_0$ and therefore, $\mathrm{ldes}(p) = 1$. *Further, $4_1 >_L 2_0$, $5_3 >_L 4_1$, $3_2 >_L 6_0$, $1_3 >_L 3_2$ and so,* $\mathrm{lexc}(p) = 4$.

The colored permutation $p = (\pi, c)$ written in word notation is the word $(\pi_1, c_1) \ldots (\pi_n, c_n)$ or $(\pi_1)_{c_i} \ldots (\pi_n)_{c_i}$ in the alphabet $[n] \times [r]_0$. We introduce our cycle notation for colored permutations as follows. Let $p = (\pi, c)$ and $\pi = \mathrm{Cyc}_1 \ldots \mathrm{Cyc}_k$ be the cycle decomposition of π. The cycle decomposition of p is obtained by replacing the each element π_i in the cycle decomposition of π by the pair (π_i, c_i) or $(\pi_i)_{c_i}$. This will be called the cycle decomposition of the colored permutation.

Example 2.2. *Let $p = (3241, 0110) \in \mathfrak{S}_4^2$. The cycle decomposition of p is $(1_0, 3_1, 4_0)(2_1)$ which we think is better notation than the equivalent $((1,0),(3,1),(4,0))((2,1))$.*

We will look at the following linear ordering on $[n] \times [r]_0$, which will give us the standard linear order on $\mathbb{N}$ when we set $r = 1$.

Definition 2.4. *The* min-one order $>_{\text{mo}}$ *on* $[n]\times[r]_0$ *is defined to be*

$$1_0 <_{\text{mo}} 1_1 <_{\text{mo}} \cdots <_{\text{mo}} 1_{r-1} <_{\text{mo}} 2_0 <_{\text{mo}} 3_0 <_{\text{mo}} \cdots <_{\text{mo}} n_0$$
$$n_0 <_{\text{mo}} 2_1 <_{\text{mo}} \cdots <_{\text{mo}} n_1 <_{\text{mo}} \cdots <_{\text{mo}} 2_{r-1} <_{\text{mo}} \cdots <_{\text{mo}} n_{r-1}$$

Furthermore, for the order, when $d=1$, the statistics $\text{ldes}(\pi)$, $\text{des}(\pi)$, $\text{exc}(\pi)$ and $\text{lexc}(\pi)$ coincide. For $(i,j)\in[n]\times[r]_0$, let $S_{(i,j)}$ be set of all colored permutations $\pi\times c\in\mathfrak{S}_n^r$ with $\pi_1=i$ and $c_1=j$.

3. Proof of Theorem 1.4

We move to proving one of the main results of this work. Before we see our proof, we need a definition. Define a map $s:[n]\times[r]_0\mapsto[n]\times[r]_0$ that takes $s(i,j)=(n+2-i,r+1-j)$ if $i\neq 1$ and when $i=1$, for all $j\in[r]_0$, define $s(1,j)=(1,r+1-j)$. It is easy to see that the map s is a bijection.

Proof of Theorem 1.4. In cycle notation, let $p=(\pi,c)\in\mathfrak{S}_n^r$. Let $p=\text{Cyc}_1\ldots\text{Cyc}_k$ be the cycle decomposition of p. First, replace each pair (i,j) by the pair $s(i,j)$. Within cycles, we order the cycles of p such that the last element of each cycle is the smallest pair (π_i,c_i) in the linear order L in that cycle. Further, we order the cycles such that the last elements are ascending in the order L. Finally, we remove the parentheses. The colored permutation obtained is the required permutation. Whenever there is an i such that neither π_i nor π_{π_i} is 1 and

$$(\pi_{\pi_i},c_{\pi_i}) >_{\text{mo}} (\pi_i,c_i),$$

then we necessarily have

$$(n+2-\pi_{\pi_i},n+1-c_{\pi_i}) <_{\text{mo}} (n+2-\pi_i,n+1-c_i).$$

For these indices, an L-excedance gets turned into an L-descent. If $\pi_i=1$, then

$$(\pi_1,c_1) >_{\text{mo}} (1,c_i)$$

is always true. Here, $(1,c_i)$ is the last element in the first cycle and (π_1,c_1) is the first element of that cycle according to our arrangement. This does not generate an L-descent under our bijection. However, we have the inequality

$$(\pi^{-1}(1),c_{\pi^{-1}(1)}) >_{\text{mo}} (1,c_i)$$

Under our bijection, this produces an L-descent as $(n+2-\pi^{-1}(1),n+1-c_{\pi^{-1}(1)}) >_{\text{mo}} (1,n+1-c_i)$. Therefore, every L-excedance of p is taken to an L-descent of $\Gamma(p)$, which was to be shown. What is left to be shown is that there are no L-descents occurring between cycles when parentheses are being removed. This is taken care of by the fact that the last element in the cycle is always smaller in the 'mo' order than the first element of the succeeding cycle, by our arrangement of the cycles. ☐

Example 3.1. *An illuminating example is when the number of colors is one and the order is just the order of* $\mathbb{Z}$. *Take a permutation written in one-line notation, say* $\pi=891624375$ *with* $\text{exc}(\pi)=3$. *Replace* i *by* $n+2-i$ *for* $i\neq 1$. *We get* $\pi'=321597846$. *Write this in cycle form as specified, that is,* $\pi'=(31)(2)(596784)$. *Remove the parenthesis to get* $\Phi(\pi)=312596784$ *which has* $\text{des}(\Phi(\pi))=3$.

We prove Corollary 1.1.

Proof of Corollary 1.1. When $r=1$, for all $\pi\in\mathfrak{S}_n$, we clearly have $\text{ldes}(\pi)=\text{des}(\pi)$ and $\text{lexc}(\pi)=\text{exc}(\pi)$. ☐

Remark 3.1. *When* $r=1$, *the bijection* Γ *is* $\sigma\circ\text{FFT}'\circ\sigma$ *where* $\sigma=\prod_{k\neq 1}(k,n+2-k)$ *and* FFT' *is a variant of Foata's fundamental transformation with the cycles arranged slightly differently as given in Lothaire [10, Theorem 10.2.3].*

4. Proof of Theorem 1.2

The main aim of this section is to supply a proof of Theorem 1.2.

Proposition 4.1. *For* $1\le i\le n$, *the polynomials* $A_{n,d,i}(q)$ *satisfy the following recurrence relation:*

$$A_{n+1,d,i}(q)=q^{d+1}[n-d]_qA_{n,d-1,i}(q)+[d+1]_qA_{n,d,i}(q).$$

For $k=n+1$,

$$A_{n+1,d,n+1}(q)=q^dA_{n,d-1}(q).$$

Initial conditions are $A_{2,0,1}(q)=1$, $A_{2,1,1}(q)=0$, $A_{2,1,1}(q)=0$ *and* $A_{2,1,2}(q)=q$.

Proof. We prove the recurrence using the insertion process. We insert the element $n+1$ into two different sets of permutations to get the permutations we require.

1. We insert $n+1$ into a descent position or at the end of a permutation of $\mathfrak{S}_{n,i}$ with d descents.
2. We insert $n+1$ into an ascent position of a permutation of $\mathfrak{S}_{n,i}$ with $d-1$ descents.

The descent change is straightforward. In the first case, there is no change in descents. In the second case, the number of descents increases by one. Now, we keep track of how the major index changes in the insertion process. Let $\{i_1 < i_2 < \cdots < i_d\}$ be the descent indices. In the first case, inserting in the position i_l moves each subsequent descent position by 1, thereby increasing the major index by $d-l$. Therefore, the total change is $(1+q+\cdots+q^d)A_{n,d,i}(q) = [d+1]_q A_{n,d,i}(q)$.

In the second case, indices that are ascents are precisely $[n]\backslash\{i_1,\ldots,i_{d-1}\}$. Note that inserting in a position $i_l < y < i_{l+1}$ increases the major index by $y+1+d-1-l = y+d-l$. Let σ be the permutation. The insertion process gives us

$$q^{\mathrm{maj}(\sigma)}(\sum_{l=0}^{d-1}\sum_{y=i_l+1}^{i_{l+1}-1} q^{y+d-l}) = q^{\mathrm{maj}(\sigma)}(q^{d+1}+\cdots+q^n)$$

where $i_0 = 0$, $i_d = n$. A straightforward calculation shows that when summing over all permutations, this is $(q^{d+1}+\cdots+q^n)A_{n,d-1,i}(q) = q^{d+1}[n-d]_q A_{n,d-1,i}(q)$. The $i = n+1$ case is straightforward, and its proof is omitted. This completes our proof. □

We need the following lemma on q-binomial coefficients.

Lemma 4.1. *For $r+1 \le d \le n$, we have*

$$q^{\binom{d-r+1}{2}+r}[r+1]_q\binom{n+1}{d-r}_q = q^{\binom{d-r+1}{2}+r}[d+1]_q\binom{n}{d-r}_q - q^{\binom{d-r}{2}+r+d+1}[n-d]_q\binom{n}{d-1-r}_q$$

Proof. Cancelling the terms $q^{\binom{d-r+1}{2}+r}$ from both sides simplifies it to

$$[r+1]_q\binom{n+1}{d-r}_q = [d+1]_q\binom{n}{d-r}_q - q^{r+1}[n-d]_q\binom{n}{d-1-r}_q$$

Using the q-Pascal recurrence simplifies the lemma to

$$q^{d-r}[r+1]_q\binom{n}{d-r}_q + [r+1]_q\binom{n}{d-1-r}_q = [d+1]_q\binom{n}{d-r}_q - q^{r+1}[n-d]_q\binom{n}{d-1-r}_q$$

Moving terms appropriately simplifies the lemma to

$$([r+1]_q + q^{r+1}[n-d]_q)\binom{n}{d-1-r}_q = ([d+1]_q - q^{d-r}[r+1]_q)\binom{n}{d-r}_q,$$

which can further be simplified to

$$[n-d+r+1]_q\binom{n}{d-1-r}_q = [d-r]_q\binom{n}{d-r}_q.$$

This follows by using the definition of the Gaussian binomial coefficients. □

Proof of Theorem 1.2. We show that the generating function is equivalent to the following identity:

$$\begin{aligned}\frac{A_{n,i}(t,q)}{(tq;q)_n} &= \sum_{j\ge 0} q^j[j]_q^{i-1}[j+1]_q^{n-i}t^j,\\ &= \left(\sum_{r=0}^{n}(-1)^r\binom{n}{r}_q t^r q^{\binom{r+1}{2}}\right)\times\left(\sum_{j\ge 0} q^j[j]_q^{i-1}[j+1]_q^{n-i}t^j\right),\\ &= \sum_{d\ge 0}\left(\sum_{r\ge 0}(-1)^{d-r}\binom{n}{d-r}_q q^{\binom{d-r+1}{2}+r}[r]_q^{i-1}[r+1]_q^{n-i}\right)t^d.\end{aligned}$$

The second equality follows from the q-binomial theorem. Define

$$A'_{n,d,i}(q) := \sum_{r\ge 0}(-1)^{d-r}\binom{n}{d-r}_q q^{\binom{d-r+1}{2}+r}[r]_q^{i-1}[r+1]_q^{n-i}.$$

We will verify that $A'_{n,d,i}(q)$ satisfies the recurrence in Proposition (4.1). Using Lemma 4.1 completes the proof. The initial conditions when $n=2$ are easy to verify. □

5. Proof of Theorem 1.3

Proposition 5.1. *For $0 < i \leq n$, the polynomials $B_{n,d,i}(q)$ satisfy the following recurrence relation:*

$$B_{n+1,d,i}(q) = [2d+1]_q B_{n,d,i}(q) + q^{2d+1}[2(n-d)+1]_q B_{n,d-1,i}(q).$$

For $i = n+1$,

$$B_{n+1,d,n+1} = q^{2d} B_{n,d-1}(q).$$

The initial conditions are $B_{1,0,1}(q) = 1$, $B_{1,1,1}(q) = 0$.

For $0 < i \leq n$, the polynomials $B_{n,d,\bar{i}}(q)$ satisfy the following recurrence relation:

$$B_{n+1,d,\bar{i}}(q) = [2d-1]_q B_{n,d,\bar{i}}(q) + q^{2d-1}[2(n-d+1)+1]_q B_{n,d-1,\bar{i}}(q).$$

For $i = \overline{n+1}$,

$$B_{n+1,d,\overline{n+1}} = q^{2d-1} B_{n,d}(q).$$

The initial conditions are $B_{1,0,\bar{1}}(q) = 0, B_{1,1,\bar{1}}(q) = q$.

Proof. The process is carried out using the insertion technique. We insert $n+1, \overline{n+1}$ into two different sets of permutations to get the permutations we require.

1. We insert $n+1, \overline{n+1}$ into a descent position or $n+1$ at the end of a permutation of $\mathfrak{B}_{n,i}$ with d descents.
2. We insert $n+1, \overline{n+1}$ into an ascent position or $\overline{n+1}$ at the end of a permutation in $\mathfrak{B}_{n,i}$ with $d-1$ descents.

In the first case, there is no change in the number of descents. In the second case, the number of descents goes up by one. Now, we keep track of the fmajor index increase in the insertion process. Let $\{i_1 < \cdots < i_d\}$ be the descent indices. In the first case, inserting $n+1, \overline{n+1}$ in the position i_l moves each subsequent descent position by 1, thereby increasing the fmajor index by $2(d-l)$. The placement of $\overline{n+1}$ keeps the descent positions $i_1, \ldots, i_l$ intact, adding 1 to the number of negatives, but $n+1$ insertion moves i_l to i_l+1, adding 2 to the fmajor. Finally, adding $n+1$ at the end doesn't change the fmajor index.

In the second case, we insert in all indices that are ascents. Note that inserting in a position that is between $i_l < y < i_{l+1}$ increases the fmajor index by $2(y+d-l)$ for $n+1$ and $2(y+d-l-1)+1$ for $\overline{n+1}$. If the starting letter was negative, then inserting in a position that is between $i_l < y < i_{l+1}$ increases the fmajor index by $2(y+d-1-l)$ for $n+1$ and $2(y+d-2-l)+1$ for $\overline{n+1}$. The calculation for the sum contribution of all such insertions is straightforward and is omitted. This concludes the proof. □

Proof of Theorem 1.3. The generating function is equivalent to the identity as follows.

$$\begin{aligned} B_{n,i}(t,q) &= \left(\sum_{r=0}^{n}(-1)^r \binom{n}{r}_{q^2} t^r q^{2\binom{r+1}{2}}\right) \times \left(\sum_{j\geq 0} q^{2j}[2j]_q^{i-1}[2j+1]_q^{n-i} t^j\right) \\ &= \sum_{d\geq 0}\left(\sum_{r=0}^{n-1}(-1)^{d-r}\binom{n}{d-r}_{q^2} q^{2\binom{d-r+1}{2}+2r}[2r]_q^{i-1}[2r+1]_q^{n-i}\right)t^d \end{aligned}$$

Define

$$B'_{n,d,i}(q) := \sum_{d=0}^{n-1}(-1)^{d-r}\binom{n}{d-r}_{q^2} q^{2\binom{d-r+1}{2}+2r}[2r]_q^{i-1}[2r+1]_q^{n-i}.$$

We will show that $B'_{n,d,i}(q)$ satisfies the recurrence in Proposition 5.1, thereby proving our theorem. Showing that $B'_{n,d,i}(q)$ satisfies the recurrence in Proposition 5.1 is equivalent to the following claim.

Claim 1. *We have*

$$[2r+1]_q \binom{n+1}{d-r}_{q^2} = [2d+1]_q \binom{n}{d-r}_{q^2} - q^{2r+1}[2(n-d)+1]_q \binom{n}{d-1-r}_{q^2}.$$

The proof of the claim is very similar to the type A case and is omitted.

The second part of the theorem follows from setting

$$B''_{n,d,i}(q) := \sum_{r=0}^{n-1}(-1)^{d-r}\binom{n}{d-r}_{q^2} q^{2\binom{d-r+1}{2}+2r-1}[2r]_q^{i-1}[2r-1]_q^{n-i}.$$

and showing that $B''_{n,d,i}(q)$ satisfies the second recurrence in Proposition 5.1. The initial condition verification is straightforward and omitted. This finishes our proof. □

6. Colored permutations with an even number of colors

An important point to note is that the maximum number of type B descents for a signed permutation in the hyperoctahedral group is n as opposed to $n-1$ for elements of the symmetric group. This additional descent is caused by the introduction of the element $\pi_0 = 0$. However, in the group $\mathfrak{S}_n^2$, the maximum number of ldes or lexc is $n-1$.

Definition 6.1. *Define $I := [-r, r]\backslash\{0\}$. Define the set $\overline{\mathfrak{S}_n^r}$ to be the set of colored permutations in $\mathfrak{S}_n^r$ with the pair $(0,0)$ or 0_0 (which we will write as just 0), prefixed to each colored permutation.*

We introduce a linear order on the set $\{(0,0)\} \cup [n] \times I$. We define a linear order that generalises the standard linear order of $\mathbb{Z}$ when we have $d = 2$.

Definition 6.2. *The symmetric order $>_{\mathrm{S}}$ is the linear order on $\{(0,0)\} \cup [n] \times I$ defined as follows:*

$$n_{\overline{r}} <_{\mathrm{S}} \cdots <_{\mathrm{S}} n_{\overline{1}} <_{\mathrm{S}} (n-1)_{\overline{r}} <_{\mathrm{S}} \cdots <_{\mathrm{S}} (n-1)_{\overline{1}} <_{\mathrm{S}} \cdots <_{\mathrm{S}} 1_{\overline{r}} <_{\mathrm{S}} \cdots <_{\mathrm{S}} 1_{\overline{1}} <_{\mathrm{S}} 0$$

$$0 <_{\mathrm{S}} 1_1 <_{\mathrm{S}} \cdots <_{\mathrm{S}} 1_r <_{\mathrm{S}} 2_1 <_{\mathrm{S}} \cdots <_{\mathrm{S}} 2_r <_{\mathrm{S}} \cdots <_{\mathrm{S}} n_1 <_{\mathrm{S}} \cdots <_{\mathrm{S}} n_r$$

The L-descent and L-ascent of a colored permutation in $\overline{\mathfrak{S}_n^r}$ are defined in the same way as for $\mathfrak{S}_n^r$. We define a new statistic on the set $\overline{\mathfrak{S}_n^{2r}}$.

Definition 6.3. *A B-excedance of a colored permutation $p = (\pi, c) \in \overline{\mathfrak{S}_n^{2r}}$, denoted by $\mathrm{bexc}(p)$, is the cardinality of the set $\{i \in [n] : (\pi_{\pi_i}, c_\pi) >_{\mathrm{S}} (\pi_i, c_i)\} \cup \{i \in [n] : \pi_i = i \text{ and } c_i < 0\}$.*

Define a map $t : [n] \times I \mapsto [n] \times I$ that sends (i, j) to $(i, \overline{j})$ with the convention that $\overline{\overline{j}} = j$. With this, we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Each colored permutation p in $\overline{\mathfrak{S}_n^{2r}}$ has the form $0, p'$ where $p' \in \mathfrak{S}_n^{2r}$. Consider the cycle decomposition of p'. For every pair in the colored permutation such that $\pi_i \neq i$, change the pair (i, j) to the pair $t(i, j)$. Arrange the pairs in each cycle such that the last pair has the smallest first component in that cycle. Arrange the cycles such that the first component of the last elements is increasing. Remove the parentheses to get a colored permutation in word form. This is a bijection Γ as all the steps are clearly reversible. We claim that this is the required colored permutation by arguing that $\mathrm{ldes}(\Gamma(p)) = \mathrm{bexc}(p)$.

Inside each cycle, assuming π_i is not the first component of the last element of the cycle, if $(\pi_{\pi_i}, c_{\pi_i}) >_{\mathrm{S}} (\pi_i, c_i)$, then $t(\pi_i, c_i) >_{\mathrm{S}} t(\pi_{\pi_i}, c_{\pi_i})$ and this turns into an L-descent when we remove the parentheses. If π_i is the first component of the last element and suppose we had $(\pi_{\pi_i}, c_{\pi_i}) >_{\mathrm{S}} (\pi_i, c_i)$, then since $\pi_{\pi_i} > \pi_i$ by construction, c_{π_i} is strictly positive. Therefore, $t(\pi_{\pi_i}, c_{\pi_i})$ is negative in the second component. Since this is the first element of a cycle and the last element of the previous cycle is smaller in the first component, this will create an L-descent between this cycle and the previous cycle.

Similarly, for an instance where $\pi_i = i$ and $c_i < 0$, by the way we have arranged the cycles, there will be an L-descent between cycles as the last element of the previous cycle will be smaller in the first component. Therefore, the number of B-excedances in p is the number of L-descents in $\Gamma(p)$. □

6.1 Real-rootedness of the type B restricted Eulerian polynomials

Define the restricted type B Eulerian polynomials, for $i \in [-n, n]$ to be $B_{n,i}(t) = \sum_{\pi \in \mathfrak{B}_{n,i}} t^{\mathrm{des}_B(\pi)}$. Using Proposition 5.1, we next prove recurrences for the polynomials $B_{n,i}(t)$. Let D be the operator $\frac{d}{dt}$.

Proposition 6.1. *For positive integers n, i with $1 \leq i \leq n-1$, the polynomials $B_{n,i}(t)$ and $B_{n,\overline{i}}(t)$ satisfy the following recurrences:*

$$B_{n,i}(t) = [1 + (2n-3)t]B_{n-1,i}(t) + 2t(1-t)DB_{n-1,i}(t), \quad (3)$$

$$B_{n,\overline{i}}(t) = [(2n-1)t - 1]B_{n-1,\overline{i}}(t) + 2t(1-t)DB_{n-1,\overline{i}}(t) \quad (4)$$

where $B_{0,0}(t) = 1, B_{1,1}(t) = 1, B_{1,\overline{1}}(t) = t$. Moreover, for $n \geq 2$, we have $B_{n,n}(t) = B_{n,\overline{n}}(t) = 2^{n-1}tA_{n-1}(t)$.

Proof. We first prove (3). Using Proposition 5.1, we have

$$B_{n,i}(t) = \sum_{d=0}^{n} B_{n,d,i}t^d = \sum_{d=0}^{n} \big((2d+1)B_{n-1,d,i} + (2(n-d-1)+1)B_{n-1,d-1,i}\big)t^d$$

$$\begin{aligned} &= \sum_{d=0}^{n} B_{n-1,d,i}t^d + 2t\sum_{d=0}^{n} dB_{n-1,d,i}t^{d-1} + (2n-3)t\sum_{d=0}^{n} B_{n-1,d-1,i}t^{d-1} \\ &\quad -2t^2\sum_{d=0}^{n}(d-1)B_{n-1,d-1,i}t^{d-2} \\ &= [1+(2n-3)t]B_{n-1,i}(t) + 2t(1-t)DB_{n-1,i}(t) \end{aligned}$$

This completes the proof of (3). As the proof of (4) is identical, we omit its proof. Finally, if $\pi_1 = n$, then we can drop the letter n and treat $\pi_2, \ldots, \pi_n$ as a permutation in $\mathfrak{S}_{n-1}$ and this gives $B_{n,n}(t) = 2^{n-1}tA_{n-1}(t)$. The proof is complete. □

Definition 6.4. *Let f be a real-rooted polynomial with real roots $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\deg(f)}$ and g be a real-rooted polynomial with real roots $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{\deg(g)}$. We say that f interlaces g if*

$$\ldots \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1.$$

Note that in this case we must have $\deg(f) \leq \deg(g) \leq \deg(f)+1$. If f interlaces g or g interlaces f, then we also say that f and g have interlacing roots.

We need the following result of Obreschkoff [12]. This can also be found in [9, Theorem 4.3].

Theorem 6.1 (Obreschkoff)**.** *Let $f, g \in \mathbb{R}[t]$ with $\deg(f) \leq \deg(g) \leq \deg(f)+1$. Then, f interlaces g if and only if $c_1 f + c_2 g$ has only real roots for all $c_1, c_2 \in \mathbb{R}$.*

We now prove Theorem 1.8 using Theorem 6.1.

Proof of Theorem 1.8. As $B_{n,n}(t) = B_{n,\overline{n}}(t) = 2^{n-1}tA_n(t)$, the real-rootedness of the polynomials $B_{n,n}(t)$ and $B_{n,\overline{n}}(t)$ follow from the real-rootedness of $A_n(t)$.

We next prove that for $1 \leq i \leq n-1$, the polynomials $B_{n,i}(t)$ are real-rooted. We proceed by induction on n. When $n = 1$, the assertion trivially holds. We assume the statement to be true for $n-1$ and prove that $B_{n,i}$ is real-rooted for $1 \leq i \leq n-1$.

Let $F_{n,i}(t) = tB_{n,i}(t)$. Using (3), it is easy to see that the polynomials $F_{n,i}(t)$ satisfy the following recurrence for $1 \leq i \leq n-1$:

$$F_{n,i}(t) = [(2n-1)t-1]F_{n-1,i}(t) + 2t(1-t)DF_{n,i}(t).$$

By induction, the polynomial $F_{n-1,i}(t)$ is real-rooted. Moreover, the polynomial $DF_{n-1,i}(t)$ clearly interlaces $F_{n-1,i}(t)$. Therefore, $[(2n-1)t-1]F_{n-1,i}(t)$ interlaces $2t(1-t)DF_{n-1,i}(t)$. By Theorem 6.1, the polynomial $F_{n,i}(t)$ is real-rooted. Hence, $B_{n,i}(t)$ is real-rooted. In an identical manner, one can show that the polynomials $B_{n,\bar{i}}(t)$ are real-rooted for $1 \leq i \leq n-1$. This completes the proof. □

6.2 Gamma positivity of symmetrized type B Eulerian polynomials

The polynomial $B_{n,i}(t)$, in general, is not palindromic. However, in this subsection, we show that if we add two such polynomials, then we get a gamma-positive polynomial. Our result is thus similar to the result [11, Lemma 4.5] of Nevo, Petersen, and Tenner. Recall for $1 \leq i \leq n$, we had defined $\overline{B}_{n,i}(t) = B_{n,i}(t) + B_{n,\bar{i}}(t)$ and $\widetilde{B}_{n,i}(t) = tB_{n,i}(t) + B_{n,\bar{i}}(t)$. We begin by showing palindromicity of both these polynomials.

Proposition 6.2. *For $1 \leq i \leq n$, the polynomials $\overline{B}_{n,i}(t)$ and $\widetilde{B}_{n,i}(t)$ are palindromic.*

Proof. Let $i > 0$. Define $f : \mathfrak{B}_{n,i} \cup \mathfrak{B}_{n,\bar{i}} \to \mathfrak{B}_{n,i} \cup \mathfrak{B}_{n,\bar{i}}$ by

$$f(\pi_1, \pi_2, \ldots, \pi_n) = \overline{\pi_1}, \overline{\pi_2}, \ldots, \overline{\pi_n}.$$

We note that $\mathrm{des}(\pi) = n - \mathrm{des}(f(\pi))$. This proves that $\overline{B}_{n,i}(t)$ is palindromic. Let $B_{n,i}(t) = a_0 + \cdots + a_n t^n$. Then, $B_{n,\bar{i}}(t) = a_n + \cdots + a_0 t^n$. Then,

$$\widetilde{B}_{n,i}(t) = a_n + (a_0 + a_{n-1})t + \cdots + (a_{n-1} + a_0)t^n + a_n t^{n+1}.$$

Clearly, $\widetilde{B}_{n,i}(t)$ is palindromic, completing the proof. □

Lemma 6.1. *For positive integers n, i with $1 \leq i \leq n$, we have*

$$B_{n+1,i}(t) = \sum_{j=\overline{n}}^{\overline{1}} B_{n,j}(t) + t\sum_{j=1}^{i-1} B_{n,j}(t) + \sum_{j=i}^{n} B_{n,j}(t), \quad (5)$$

$$B_{n+1,\bar{i}}(t) = t\sum_{j=\overline{n}}^{\bar{i}} B_{n,j}(t) + \sum_{j=\overline{i-1}}^{1} B_{n,j}(t) + t\sum_{j=1}^{n} B_{n,j}(t). \quad (6)$$

Proof. We consider (5) first. Let $\pi \in \mathfrak{B}_{n+1,i}$ and we observe the effect of dropping i from the first position of permutations in $\mathfrak{B}_{n+1}$. Let π' denote the permutation in $\mathfrak{B}_{\{1,2,\ldots,i-1,i+1,\ldots,n+1\}}$ which we get from π after dropping i.

If $\overline{n+1} \leq \pi_2 \leq \overline{1}$, then dropping $\pi_1 = i$ does not change the number of descents of π. If $1 \leq \pi_2 \leq i-1$, then dropping $\pi_1 = i$ decreases the number of descents of π by 1. Again, if $i+1 \leq \pi_2 \leq n+1$, then dropping $\pi_1 = i$ does not cause any change in the number of descents of π. This completes the proof. As the proof of (6) follows by a similar argument, we omit it. □

We are now in a position to prove Theorem 1.7.

Proof of Theorem 1.7. We prove this by induction on n. It is easy to verify that $\overline{B}_{2,1}(t) = (1+t)^2, \widetilde{B}_{2,1}(t) = 2t(1+t), \overline{B}_{2,2}(t) = 4t$, and $\widetilde{B}_{2,2}(t) = 2t(1+t)$. Thus, the statement is true for $n=2$. We assume the statement to be true for n and prove this for $n+1$. By applying Lemma 6.1, we have

$$\begin{aligned}
\overline{B}_{n+1,i}(t) &= B_{n+1,i}(t) + B_{n+1,\overline{i}}(t) \\
&= (1+t)\sum_{j=\overline{n}}^{\overline{i}} B_{n,j}(t) + 2\sum_{j=\overline{i-1}}^{\overline{1}} B_{n,j}(t) + 2t\sum_{j=1}^{i-1} B_{n,j}(t) + (1+t)\sum_{j=i}^{n} B_{n,j}(t) \\
&= (1+t)\sum_{j=i}^{n} \overline{B}_{n,j}(t) + 2\sum_{j=1}^{i-1} \widetilde{B}_{n,j}(t).
\end{aligned}$$

By induction, $\overline{B}_{n,j}(t)$ is gamma positive with center of symmetry $n/2$ and $\widetilde{B}_{n,j}(t)$ is gamma positive with center of symmetry $(n+1)/2$. Therefore, $\overline{B}_{n+1,i}(t)$ is gamma positive with center of symmetry $(n+1)/2$. In a similar way, one can show that

$$\widetilde{B}_{n+1,i}(t) = 2t\sum_{j=i}^{n} \overline{B}_{n,j}(t) + (1+t)\sum_{j=1}^{i-1} \widetilde{B}_{n,j}(t).$$

Thus, $\widetilde{B}_{n+1,i}(t)$ is gamma positive with center of symmetry $(n+2)/2$. Finally, we observe that $B_{n+1,n+1}(t) = B_{n+1,\overline{n+1}}(t) = \sum_{j=1}^{n} \widetilde{B}_{n,j}(t)$. Hence, both the polynomials $\overline{B}_{n+1,n+1}(t)$ and $\widetilde{B}_{n+1,n+1}(t)$ are gamma positive and have respective centers of symmetry $(n+1)/2$ and $(n+2)/2$. This completes the proof of the theorem. □

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