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Interview with Sheila Sundaram

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Photo by Priyanka Altman

Sheila Sundaram completed her Ph.D. at the Massachusetts Institute of Technology (MIT) in 1986 under the supervision of Richard Stanley. She held postdoctoral positions at the University of Michigan with Phil Hanlon, at the Institute for Mathematics and its Applications (Minneapolis, USA) and at Université du Québec à Montréal in Canada. She left the University of Miami as a tenured Associate Professor in 1995. After her children were grown, she taught at a small school for eleven years, and gradually returned to mathematical research. She is currently a Visiting Professor at the University of Minnesota in Minneapolis.

Mansour: Professor Sundaram, first of all, we would like to thank you for accepting this interview. Would you tell us broadly what combinatorics is?

Sundaram: Well, it is often defined as the "Art of Counting"[1]. What is surprising is how much of mathematics ultimately comes down to counting something. Unifying all the diverse counting techniques into a solid theoretical framework, and extracting underlying structures, is what has made Combinatorics the field that it is today.

Mansour: What do you think about the development of the relations between combinatorics and the rest of mathematics?

Sundaram: Combinatorics has been around for a very long time, and is embedded in all of mathematics. It has finally earned its place as a field in its own right, thanks to the pioneering work of Rota, Stanley, and others. However, as Richard Stanley has repeatedly pointed out, the connections to other areas of mathematics are mind-boggling. One has only to skim through EC1[2] and EC2[3] to see this. I have never forgotten this quote that I came across as an undergraduate, from an old algebra book[4]: "Never underestimate the power of a theorem that counts something".

Mansour: What have been some of the main goals of your research?

Sundaram: I would say that I am most interested in understanding the properties of a particular group action on a vector space or some other structure.

Mansour: We would like to ask you about your formative years. What were your early experiences with mathematics? Did that happen under the influence of your family or some other people?

Sundaram: I was actually a rather indifferent mathematics student as a child. I remember having trouble subtracting with carries, and with what used to be called "Interest and Annuities" problems (a peculiar predilection of the time in British Commonwealth schools). When I was ten, my father (who always loved mathematics, but had little exposure to it) taught me some algebra, and quite magically the fog cleared. The formalism really appealed to me. Suddenly arithmetic was not just a

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[1]B. Sagan, *Combinatorics: The Art of Counting*, Grad. Stud. Math. 210, Amer. Math. Soc., Providence, RI, 2020.
[2]R. P. Stanley, *Enumerative Combinatorics*, Volume 1, Cambridge University Press, Cambridge, 1997.
[3]R. P. Stanley, *Enumerative Combinatorics*, Volume 2, Cambridge University Press, Cambridge, 1999.
[4]J. Fraleigh, *A First Course in Abstract Algebra*, 7th Edition, Addison-Wesley, Boston, 1982.

motley assortment of arbitrary rules, but actually had a structure. Coordinate geometry seemed so much easier than those difficult Euclidean proofs. When I finally got to abstract algebra, it felt like I had been waiting for this all my life. I was utterly charmed by Herstein's *Topics in Algebra*[5].

Mansour: Were there specific problems that made you first interested in combinatorics?

Sundaram: I did fall in love with Ian Macdonald's book, *Symmetric Functions and Hall Polynomials*[6], which Richard Stanley[7] gave me to read, and a wonderful course (18.318) taught by Stanley when I was a graduate student. I remember very much enjoying the problem sets. There was also another 18.318 taught by Phil Hanlon on the construction of Specht modules (based on Gordon James' book[8]), and together they connected symmetric functions, the symmetric group, and the general linear group, via the representation theory. I still look at the problems in Chapter 7 of Stanley's EC2, as well as the Examples in Macdonald's book. Together they form a treasure trove of sources for inspiration, not to mention information.

Mansour: What was the reason you chose MIT for your Ph.D. and your advisor Richard Stanley?

Sundaram: My family moved around a lot, so my schooling was rather erratic and unorthodox. At some point, I must have heard about MIT, and it became something of a dream to go there.

I was very much influenced by Herstein's textbook, my first real mathematics book. I remember how important he thought counting arguments were. He makes several references to this in the very first chapter. Much later I would discover how ubiquitous they are in representation theory.

Becoming Richard Stanley's student was a happy accident. I believe my graduate adviser, Steve Kleiman, an algebraic geometer, was the one who suggested it.

Mansour: What would guide you in your research? A general theoretical question or a specific problem?

Sundaram: Usually this is a combination of both. Thinking about a specific problem or example often leads to theoretical questions. I do like to compute things by hand. I find it gives me insight that I would not otherwise get. Of course, not all problems lend themselves to this.

Mansour: When you are working on a problem, do you feel that something is true even before you have the proof?

Sundaram: This depends on the problem. As an example, when I was working on the partition lattice[9], I did feel very strongly that there had to be a systematic way to get all the rank-selected homologies, because the partition lattice is a special example of what Stanley calls a uniform poset, with a highly suggestive recursive structure. A more recent example is the theorem about global conjugacy classes in S_n[10]. These have a surprisingly nice characterization. I was sure the result that all the irreducibles appear in certain conjugacy classes was true, with only a little data to back me up (at the time I did not have access to the software I needed to generate the data).

Mansour: What three results do you consider the most influential in combinatorics during the last thirty years?

Sundaram: Would you allow me to go back a bit farther? Even with my limited knowledge, it is really difficult to choose three specific results; so many papers have spawned a vast literature of exciting discoveries and opened up new directions of research. Here are my three: Ian Macdonald's "A new class of symmetric functions", introducing what would come to be known as the Macdonald polynomials[11]; Ira Gessel's[12] "Multipartite P-partitions and inner products of skew-Schur functions", introducing the quasisymmetric function[13]; and Rodica

[5]I. N. Herstein, *Topics in Algebra*, Xerox College Pub., 1975.
[6]I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Second Edition, Oxford University Press (1995).
[7]*Interview with Richard P. Stanley*, Enumer. Combin. Appl. 1:1 (2021), Interview S3I1.
[8]G. D. James, *The Representation Theory of the Symmetric Groups*, Springer Berlin, Heidelberg, 1978.
[9]S. Sundaram, *The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice*, Adv. in Math. 104(2) (1994), 225–296.
[10]S. Sundaram, *On conjugacy classes of S_n containing all irreducibles*, Isr. J. Math. 225 (2018), 321–342.
[11]I. G. Macdonald, *A new class of symmetric functions*, Sém. Loth. Combin. 20 (1988), Article B20a.
[12]*Interview with Ira Gessel*, Enumer. Combin. Appl. 2:2 (2022), Interview S3I8.
[13]I. M. Gessel, *Multipartite P-partitions and inner products of skew Schur functions*, Combinatorics and algebra (Boulder, Colo., 1983), 289–317, Contemp. Math. 34, Amer. Math. Soc., Providence, RI, 1984.

Simion and Frank Schmidt's paper "Restricted permutations"[14], which sparked the huge area of permutation patterns, now an entire field unto itself.

Mansour: What are the top three open questions in your list?

Sundaram: At the risk of sounding somewhat predictable, they are: Plethysm coefficients[6], Kronecker coefficients[15], and Thrall's problem[16], which asks for a combinatorial rule for the irreducible multiplicities in the higher Lie modules.

Mansour: What kind of mathematics would you like to see in the next ten to twenty years as the continuation of your work?

Sundaram: Personally I think the character table of the symmetric group has lots of hidden surprises. There is very lovely work of Alex Miller[17] on this already, as well as papers of Kannan Soundararajan and Sarah Peluse[18].

Mansour: Do you think that there are core or mainstream areas in mathematics? Are some topics more important than others?

Sundaram: I am not sure I am qualified to comment on this. History indicates that it is very difficult to predict the importance or future impact of a topic.

Mansour: What do you think about the distinction between pure and applied mathematics that some people focus on? Is it meaningful at all in your case? How do you see the relationship between so-called "pure" and "applied" mathematics?

Sundaram: I'm afraid I am rather ignorant about applied mathematics. Some people find their inspiration in applied ("real-world") problems. I recently learned a little bit about PDEs and boundary-value problems, and I was surprised by how interesting the questions are. I would say that the big distinction is in the practical applications, but then again, pure number-theoretic and algebraic geometry results have found important applications in cryptography and error-correcting codes for ages.

Mansour: Would you tell us about your interests besides mathematics?

Sundaram: I have always been interested in languages, their origin, and their evolution. I was fortunate to be brought up among people who spoke many languages fluently, and my mother taught me to read, write, and speak in Hindi and Tamil (two Indian languages with completely different scripts). I grew up reading English and French literature. I haven't kept up with more modern work, though.

Mansour: One of your most cited papers is *The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice*[19]. Could you explain the significance of these results in a broader context of your work?

Sundaram: This paper was inspired by Stanley's *Some aspects of groups acting on finite posets*[20], which he had given me to read in graduate school. For me, it was one of the most influential papers in Combinatorics. In that paper, Stanley determines the rank-selected homology for the boolean lattice and the face lattice of the cross-polytope. But little was known about the partition lattice; the most important result was a consequence of a now-famous character computation of Phil Hanlon[21].

A result that caught my eye one day was Anders Björner's description of Baclawski's Whitney homology[22]. I saw then that instead of computing character values, one could use Whitney homology at the level of modules. I remember being very excited when I realized the importance of the plethysm operation and its connection to the recursive structure of the poset.

The culmination was a general plethystic recurrence for the rank-selected homology (and later a wider class of subposets); it led to

[14] R. Simion and F. W. Schmidt, *Restricted Permutations*, European J. Combin. 6:4 (1985), 383–406.

[15] G. Panova, *Complexity and asymptotics of structure constants, Open problems in algebraic combinatorics*, 61–85. Proc. Sympos. Pure Math. 110. Amer. Math. Soc., Providence, RI, 2024.

[16] R. M. Thrall, *On symmetrized Kronecker powers and the structure of the free Lie ring*, Amer. J. Math. 64 (1942), 371–388.

[17] A. Miller, *The characters of symmetric groups that depend only on length*, Math. Z. 304:9 (2023), 1–14.

[18] S. Peluse and K. Soundararajan, *Almost all entries in the character table of the symmetric group are multiples of any given prime*, Journal für die reine und angewandte Mathematik (Crelles Journal), vol. 2022, no. 786 (2022), 45–53.

[19] S. Sundaram, *The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice*, Adv. in Math. 104: 2 (1994), 225–296.

[20] R. P Stanley, *Some aspects of groups acting on finite posets*, J. Combin. Theory, Ser. A 32:2 (1982), 132–161.

[21] P. Hanlon, *The fixed-point partition lattices*, Pacific J. Math. 96:2 (1981), 319–341.

[22] A. Björner, *On the homology of geometric lattices*, Algebra Universalis 14 (1982), 107–128.

[23] S. Sundaram, *The homology of partitions with an even number of blocks*, J. Algebraic Combin. 4:1 (1995), 69–92.

many other discoveries. For instance, the subposet where the *number* of blocks is even[23], has a very interesting homology representation that is supported entirely on involutions. The plethystic recurrence strongly suggested it was in fact a permutation module. Further manipulation showed that the proof hinged on a purely enumerative fact: the nonnegativity of a sequence of integers defined by a quite ghastly recurrence. I was very happy when, a few years later, an academic sibling, Benjamin Joseph, in his Ph.D. thesis from MIT, was able to take that recurrence and construct an intricate sign-reversing involution to establish the positivity.

This paper was also the impetus for the discovery of the simsun permutations that Rodica Simion and I defined[9]. The Whitney homology technique has since been useful to me for many other posets.

The many enumerative (and still open) problems resulting from this work are collected in this paper[24]. For instance, Stanley had shown that the multiplicity of the trivial representation in the homology of the rank-selected subposets refines the Euler numbers by subsets. Phil Hanlon[25] proved a vanishing conjecture of Stanley for these multiplicities, and I was able to determine a few more cases, as well as formulate more conjectures, some of which Hanlon and Patricia Hersh later proved using quite sophisticated methods[26].

This project motivated me to learn to use John Stembridge's then-new symmetric functions package SF in Maple[27], which became an indispensable tool. I was by no means alone. In addition to his mathematics, Stembridge's Maple packages have had an enormous impact on the Algebraic Combinatorics community.

Mansour: Together with Michelle Wachs, in the paper *The homology representations of the k-equal partition lattice*[28], you determined the character of the action of the symmetric group on the homology of the induced subposet of the lattice of partitions of the set $\{1, 2, \ldots, n\}$ obtained by restricting block sizes to the set $\{1, k, k+1, \ldots\}$. Would you say more about this work?

Sundaram: This is probably the first example of a non-pure poset[29]. It very quickly inspired the two seminal papers of Anders Björner and Michelle Wachs[30,31] on non-pure shellability. Its homotopy type turns out to be a wedge of spheres in varying dimensions, and we were able to give a plethystic formula for the action of the symmetric group on the homology modules. A very pleasing aspect of this work is the strong interplay between the combinatorial topology and the representation theory. Michelle Wachs[32,33] had constructed remarkable bases for the homology and cohomology of the partition lattice. Somewhat miraculously, in the case of the k-equal partition lattice, the analogous basis perfectly reflected the plethystic formula for the homology representation. That formula is a key ingredient in subsequent work with Volkmar Welker[34] on the equivariant Goresky-Macpherson formula for the cohomology ring of configuration spaces.

It was a privilege to be able to work with some of the giants in the field!

Mansour: Some of your works include some applications of combinatorics in physics, such as *Vector partition functions and Kronecker coefficients*[35], coauthored by Mishna and Rosas. Would you please expand on this?

Sundaram: I was happy to be a part of this project since I have been interested in the Kronecker product for a long time. Mercedes

[24] S. Sundaram, *Some problems arising from partition poset homology*, in "The Mathematical Legacy of Richard P. Stanley", Amer. Math. Soc., Providence, RI, 2016, 335–352.

[25] P. Hanlon, *A proof of a conjecture of Stanley concerning partitions of a set*, European J. Combin. 4:2 (1983), 137–141.

[26] P. Hanlon and P. Hersh, *Multiplicity of the trivial representation in rank-selected homology of the partition lattice*, J. Algebra 266:2 (2003), 521–538.

[27] J. Stembridge, *A Maple package for symmetric functions*, J. Symb. Comput. 30 (1995), 755–768.

[28] S. Sundaram and M. L. Wachs, *The homology representations of the k-equal partition lattice*, Trans. Amer. Math. Soc. 349:3 (1997), 935–954.

[29] A. Björner, L. Lovász, and A. Yao, *Linear decision trees: volume estimates and topological bounds*, In: Proc. 24th ACM Symp. on Theory of Computing, 170–177. New York: ACM Press 1992.

[30] A. Björner and M. L. Wachs, *Shellable nonpure complexes and posets. I*, Trans. Amer. Math. Soc. 348 (1996), 1299–1327.

[31] A. Björner and M. L. Wachs, *Shellable nonpure complexes and posets. II*, Trans. Amer. Math. Soc. 349 (1996), 3945–3975.

[32] M. L. Wachs, *A basis for the homology of the d-divisible partition lattice*, Adv. in Math. 117 (1996), 294–318.

[33] M. L. Wachs, *On the (co)homology of the partition lattice and the free Lie algebra*, Discrete Math. 193 (1998), 287–319.

[34] S. Sundaram and V. Welker, *Group actions on arrangements of linear subspaces and applications to configuration spaces*, Trans. Amer. Math. Soc. 349(4) (1997), 1389–1420.

[35] M. Mishna, M. Rosas, and S. Sundaram, *Vector partition functions and Kronecker coefficients*, J. Physics A: Math. and Theor. 20 (2021), Article 205204.

Rosas, an expert on this topic, led the project with her idea of using vector partition functions to study Kronecker products in a new way, starting from the formula for the Schur function as a quotient of alternants. Marni Mishna is an expert on techniques from Analytic Combinatorics (in the style of Flajolet), which we use here. One surprise was yet another formula for the Kronecker coefficients for two-rowed partitions. Another is a curious digraph that captures bounds on the coefficients. This digraph has been extended to the next case (three-rowed shapes) by Stefan Trandafir, a student of Mishna, in his Ph.D. thesis.

Mathematical Physics journals have a long history of publishing papers on various computational aspects of representation theory; we were pleased that it appealed to them.

Mansour: In one of your papers *On a curious variant of the S_n-module Lie_n*[36], you introduced a variant of the much-studied Lie representation of the symmetric group S_n. Why is this variant curious and what makes it important?

Sundaram: I do like this one. A well-known result of Cadogan[37] from the 1970s tells us that the sign-twisted Lie characters yield the compositional inverse of the sum of the trivial characters of the symmetric group. In terms of symmetric functions, this is the plethystic inverse of the sum of the complete homogeneous symmetric functions. It turns out that this sign-twisted variant gives the plethystic inverse of the sum of the elementary symmetric functions. The remarkable thing about this variant is that it offers a counterpart for nearly every interesting property enjoyed by the Lie character. I am still trying to understand this.

Mansour: In a very recent publication *0-Hecke modules for row-strict dual immaculate functions*[38], coauthored with E. Niese, S. van Willigenburg, J. Vega, and Sh. Wang, you introduced a new basis of quasisymmetric functions, namely the row-strict dual immaculate functions. Would you tell us more about this work?

Sundaram: This was a collaboration led by Elizabeth Niese, whose idea it was to consider the row-strict version of the dual immaculate quasisymmetric functions. We then constructed 0-Hecke modules whose quasisymmetric characteristic (the analogue of the Frobenius characteristic) is the row-strict dual immaculate function. By defining a poset that we call the immaculate Hecke poset, we were able to find many other 0-Hecke modules with combinatorially interesting quasisymmetric characteristics, including some that had been previously discovered by Berg-Bergeron-Saliola-Serrano-Zabrocki[39] and by Dominic Searles[40], but also several new ones. This is currently a very active area of research; many research groups (including Bardwell and Searles, Seung-Il Choi, Young-Hun Kim, Young-Tak Oh, and coauthors) have been prolific in their discoveries.

Mansour: You gave talks at numerous conferences, workshops, and seminars. What do you think about the importance of such activities for researchers?

Sundaram: Early-career mathematicians today are very fortunate in that there is an abundance of mentored workshops and supportive collaborative experiences available to them, for example, the Graduate Research Workshop in Combinatorics (GRWC), and the Mathematical Research Communities series. These are in my opinion invaluable resources.

The most important thing is that such activities foster a sense of community among researchers. We are all connected by this common passion, and driven by a common curiosity. Mathematics can be a very isolating endeavour. Watching others work, and hearing them speak about their struggles, their ideas and especially what prompted them to pursue a particular line of thought can be helpful, motivating, and inspiring. I am continually amazed by how a chance remark by one person can turn into a valuable exploration for another. We are individually inspired by our community, and these interactions are not to be discounted. It also shows how much mathematics is built on the work of those who went

[36] S. Sundaram, *On a curious variant of the S_n-module Lie_n*, Alg. Combin. 3:4 (2020), 985–1009.

[37] C. C. Cadogan, *The Möbius function and connected graphs*, J. Combin. Theory Ser. B 11 (1971), 193–200.

[38] E. Niese, S. Sundaram, S. van Willigenburg, J. Vega, and S. Wang, *0-Hecke modules for row-strict dual immaculate functions*, Trans. Amer. Math. Soc. 377 (2024), 2525–2582.

[39] C. Berg, N. Bergeron, F. Saliola, L. Serrano, and M. Zabrocki, *Indecomposable modules for the dual immaculate basis of quasi-symmetric functions*, Proc. Amer. Math. Soc. 143:3 (2015), 991–1000.

[40] D. Searles, *Indecomposable 0-Hecke modules for extended Schur functions*, Proc. Amer. Math. Soc. 148:5 (2020), 1933–1943.

before us, the debt we owe them, and the fascinating interconnections.

Mansour: Professor Sundaram: *What are total cut complexes of graphs?*[41]

Sundaram: First there were cut complexes, which are simplicial complexes defined on the vertex set of a graph. The k-cut complex has facets which are the complements of vertex subsets of size k that induce disconnected subgraphs. It is the brainchild of a very talented young coauthor, Mark Denker, on this project, who in turn was inspired by a famous theorem of Ralf Fröberg in commutative algebra, characterising certain Stanley-Reisner ideals. For example, for the graph on n vertices with no edges, the k-cut complex is the $(n-k-1)$-skeleton of the $(n-1)$-simplex, and the face lattice is just a truncated boolean lattice.

Total cut complexes are a variation invented by another talented coauthor, Lei Xue, in the course of our collaboration. Here the facets are the complements of *independent* vertex subsets of size k in the graph.

In many cases the resulting simplicial complexes end up having well-behaved topology, very often a wedge of equidimensional spheres, and then the symmetry groups yielded interesting representations on the homology. This was a GRWC project, and a great example of the interactions between different fields: combinatorics, topology, and algebra. I learned a lot from it, and from my collaborators.

Mansour: In your work, you have extensively used combinatorial reasoning to address important problems. How do enumerative techniques engage in your research?

Sundaram: Especially in the representation theory of the symmetric group, combinatorics plays a huge role, so it is almost inescapable. Many combinatorial identities have an underlying representation-theoretic meaning. So most often that is the direction I take: when I see a particularly interesting combinatorial identity, I always wonder if it is an enumerative feature of some group action.

Mansour: Would you tell us about your thought process for the proof of one of your favorite results? How did you become interested in that problem? How long did it take you to figure out a proof? Did you have a "eureka moment"?

Sundaram: One favourite is about the poset of nonmodular partitions[42]. Here one removes the set partitions of $\{1, 2, \ldots, n\}$ having at most one non-singleton block. It seems like a strange thing to do, but this subposet turns out to be Cohen-Macaulay; in fact, its order complex is homotopy equivalent to that of the partition lattice one index lower, of the set $\{1, 2, \ldots, n-1\}$. The starting point was examining the plethystic formula for the Lefschetz module; it was the representation that led me to the poset and the topological results. That Lefschetz module turned out to be what is called the Whitehouse module.

I worked on the plethystic formula for the partitions with an even number of blocks for many months[23], eventually extracting from it an expression showing that the homology decomposes as a (virtual) sum of permutation modules supported entirely on involutions. This, in turn, led to many enumerative results, and Benjamin Joseph's difficult proof of my conjecture. But the initial hand computations were what led me to the original conjecture, again.

Somewhat curiously, the action on the maximal chains of the full partition lattice decomposes in a similar fashion. I think there is more here to be elucidated.

Mansour: Is there a specific problem you have been working on for many years? What progress have you made?

Sundaram: I'm very interested in plethysm and Thrall's problem, and figuring out where that curious variant of Lie_n fits into all this. I don't know if this is progress, but there seem to be more places where the variant comes up.

Mansour: In a very recent short article[43], published in the Newsletter of the European Mathematical Society, Professor Melvyn B. Nathanson, while elaborating on the ethical aspects of the question "Who Owns the Theorem?", concluded that "Mathematical truths exist and mathematicians only discover them." On the other side, there are opinions that

[41] M. Bayer, M. Denker, M. J. Milutinović, R. Rowlands, S. Sundaram, and L. Xue, *Total cut complexes of graphs*, Discrete Comput. Geom. (2024).

[42] S. Sundaram, *Homotopy of non-modular partitions and the Whitehouse module*, J. Algebraic Combin. 9:3 (1999), 251–269.

[43] M. B. Nathanson, *Who Owns the Theorem?* The best writing on Mathematics 2021, Princeton: Princeton University Press, 2022, 255–257.

"mathematical truths are invented". As a third way, some people claim that it is both invented and discovered. What do you think about this old discussion? More precisely, do you believe that you *invent* or *discover* your theorems?

Sundaram: Well, people invent new concepts and then discover that there are interesting theorems about them. What is "interesting" is of course in the eye of the beholder. There is a certain amount of skill involved if one wants to invent interesting things. So I suppose I lean more in the direction of serendipitously discovering them.

Mansour: Professor Sundaram, I would like to thank you for this very interesting interview on behalf of the journal Enumerative Combinatorics and Applications.

Sundaram: Thank you for doing me this honor, and for your thoughtful, inspiring, and mathematically stimulating questions!