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Extremal Boolean Functions With Long Prime Implicants

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Abstract: We combine Boolean arguments with language-theoretic tools to obtain a short, constructive, and transparent proof (a book proof) of the result on the largest induced subgraphs of the n-cube that contain no 4-cycles.

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1. Introduction

We call a Boolean function F of n variables an ℓpi-function if all prime implicants of F have length at least $n-1$ (ℓpi stands for *long prime implicants*, see [2] for more general classes of Boolean functions). In this paper, we show that an ℓpi-function of n variables has at most $\lceil \frac{2^{n+1}}{3} \rceil$ true points and that for each n there is a unique (up to renaming and negating variables) function attaining this bound. Taking into account the bijection between implicants of length $n-k$ and subcubes of dimension k in the n-dimensional cube, we observe that a Boolean function is an ℓpi-function if and only if the set of its true points induces in the n-cube a subgraph containing no 4-cycles. Therefore, our paper presents an innovative proof, which is short, constructive and transparent, for the result discovered independently in [3] and [4].

Let $\mathcal{B} = \{0,1\}$. We refer to the set $\mathcal{B}^n$ as a *cube of dimension* n, and to the elements of $\mathcal{B}^n$ as *Boolean points*. We say that two Boolean functions F and G are *congruent*, denoted $F \cong G$, if they are identical up to renaming and negating variables. Let F be an n-variable Boolean function. A *partition of* F *over a variable* x is a pair (F_0, F_1) of functions of $n-1$ variables such that $F = \overline{x}F_0 \vee xF_1$, in which case F_0 and F_1 are said to be the *restrictions* of F to $x=0$ and $x=1$, respectively, written as $F_0 = F_{|x=0}$ and $F_1 = F_{|x=1}$. Let $(X^1, \ldots, X^{2^n})$ be the Boolean points of $\mathcal{B}^n$ ordered lexicographically, then the word $f = F(X^1)\ldots F(X^{2^n})$ is the *vector of values of* F. We denote the reverse of a word f by f^* and the number of true points of F by $|F|$. For Boolean terminology not defined here, we refer the reader to [1].

2. Extremal ℓpi-functions

We define inductively a sequence of Boolean functions A_n $(n = 1, 2, \ldots)$ on $\mathcal{B}^n$. The definition is different for odd and even values of n and involves an auxiliary function B_n. We define A_n and B_n via their vectors of values denoted by a_n and b_n, respectively:

$n = 1$: $a_1 = 11$ and $b_1 = 10$.
$n > 1$ even: $a_n = a_{n-1}b_{n-1}$ and $b_n = b_{n-1}b^*_{n-1}$.
$n > 1$ odd: $a_n = a^*_{n-1}a_{n-1}$ and $b_n = a_{n-1}b_{n-1}$.

Lemma 2.1. *A_n and B_n are ℓpi-functions for all $n \geq 1$.*

Proof. For any function $F : \mathcal{B}^n \to \mathcal{B}$ with an implicant of length at most $n-2$, there is a restriction with an implicant of length at most $n-3$. If $F \cong A_n$ or $F \cong B_n$, then *any* restriction of F is congruent to A_{n-1} or B_{n-1} by definition of A_n and B_n and due to the notion of congruency. Therefore, the result follows by induction on n, with small values of n being easily verifiable. □

Lemma 2.2. *The functions A_n and B_n satisfy the following properties:* (a) $|B_n| = |A_n| - 1$ *for all values of n,* (b) $|A_n| = \frac{2^{n+1}+1}{3}$ *for even n and* $|A_n| = \frac{2^{n+1}+2}{3}$ *for odd n.*

Proof. From the definition of A_n and B_n, we know that $|A_1| = 2$, $|B_1| = 1$, $|A_n| = |A_{n-1}| + |B_{n-1}|$, $|B_n| = 2|B_{n-1}|$ for even n, and $|A_n| = 2|A_{n-1}|$, $|B_n| = |A_{n-1}| + |B_{n-1}|$ for odd n. This immediately implies (a) by induction on n. Also, using these equalities and (a), we derive by induction on n that $|A_n| = |A_{n-1}| + |B_{n-1}| = \frac{2^n+2}{3} + \frac{2^n+2}{3} - 1 = \frac{2^{n+1}+1}{3}$ for an even n, and $|A_n| = 2|A_{n-1}| = 2\frac{2^n+1}{3} = \frac{2^{n+1}+2}{3}$ for an odd n, as required. □

Theorem 2.1. *For each value of $n \geq 1$, A_n is the unique (up to renaming and negating variables) ℓpi-function on $\mathcal{B}^n$ with the maximum number of true points.*

Proof. We prove the theorem by induction on n. For $n \leq 2$, the result can be checked by direct inspection. From now on, we assume that $n \geq 3$ and that the theorem is valid for values smaller than n. This implies that for any ℓpi-function F on $\mathcal{B}^n$ with $|F| = |A_n|$,

(o) if n is odd, then any partition of F consists of two functions congruent to A_{n-1},
(e) if n is even, then in any partition of F, one of the restrictions is congruent to A_{n-1}.

Now we prove a series of other claims for an ℓpi-function F on $\mathcal{B}^n$.

(1) *If n is odd and F is congruent to A_n, then the two functions congruent to A_{n-1} in the partition of F, written as words, are reverse of each other.* This is because a_n is palindromic, i.e. A_n, and hence F, takes equal values on opposite (complement) Boolean points.

(2) *If n is odd and $|F| = |A_n|$, then $F \cong A_n$.* To see this, partition F over two variables and write it as a word $\gamma_{00}\gamma_{01}\gamma_{10}\gamma_{11}$. By (o) and (e), two "opposite" subwords, say γ_{01} and γ_{10}, represent functions congruent to A_{n-2}, and each of the four words $\gamma_{00}\gamma_{01}$, $\gamma_{00}\gamma_{10}$, $\gamma_{01}\gamma_{11}$, $\gamma_{10}\gamma_{11}$ represents a function congruent to A_{n-1}. We assume that $\gamma_{01} = a_{n-2}$ (by renaming and negating variables), which implies $\gamma_{11} = b_{n-2}$ and $\gamma_{10} = a_{n-2}$. For the word $\gamma_{00}\gamma_{01}$ to represent a function congruent to A_{n-1}, we must read it from right to left and conclude that $\gamma_{00} = b^*_{n-2}$ (remember that a_{n-2} is palindromic). Hence, F can be written as $b^*_{n-2}a_{n-2}a_{n-2}b_{n-2} = a_n$, i.e. $F \cong A_n$.

(3) *If n is even and $|F| = |A_n|$, then $F \cong A_n$.* To prove this, we apply (e) to each of the n variables $x_1, \ldots x_n$ to obtain n restrictions $F_{|x_i=\alpha_i}$ of F congruent to A_{n-1}. By renaming and negating variables, we may assume that $\alpha_1 = 0$ and $F_{|x_1=0} = A_{n-1} = A_{n|x_1=0}$. For all $i > 1$, since $n-1$ is odd, we conclude that the two functions $F_{|x_1=0,x_i=\overline{\alpha}_i}$ and $F_{|x_1=1,x_i=\alpha_i}$ coincide, as each of them, written as a word, is the reverse of $F_{|x_1=0,x_i=\alpha_i}$ by (1). Thus, F coincides with A_n on all points but one, and since $|F| = |A_n|$, they must coincide on the remaining point too.

(4) *If n is even, then $|F| < 2|A_{n-1}|$.* We partition F over two variables, write it as a word $\gamma_{00}\gamma_{01}\gamma_{10}\gamma_{11}$, and assume the contrary: $|F| \geq 2|A_{n-1}| = 4|A_{n-2}|$. Then each subword γ_{ij} represents a function congruent to A_{n-2} (by induction) and $\gamma_{00} = \gamma^*_{01} = \gamma_{11} = \gamma^*_{10}$ (by (1)). Therefore, among any four Boolean points $(0,0,\alpha)$, $(0,1,\alpha)$, $(1,0,\alpha)$, $(1,1,\alpha)$ ($\alpha \in \mathcal{B}^{n-2}$), an even number of the points are true, and this number is different from 4, since otherwise a 4-cycle arises. Thus, $|F| \leq 2 \cdot 2^{n-2} < 2\frac{2^n+2}{3} = 2|A_{n-1}| \leq |F|$, which is a contradiction.

To finish the proof of the theorem, we consider an ℓpi-function F on $\mathcal{B}^n$ with the maximum number of true points. By induction we know that for any partition (F_1, F_2) of F we have $|F_1| \leq |A_{n-1}|$ and $|F_2| \leq |A_{n-1}|$. If n is odd and $|F| = 2|A_{n-1}| = |A_n|$, then $F \cong A_n$ by (2). If n is even, then $|F| < 2|A_{n-1}|$ by (4), and if $|F| = 2|A_{n-1}| - 1 = |A_n|$, then $F \cong A_n$ by (3). □

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