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Enumeration of Multiplex Juggling Card Sequences Using Generalized q-Derivatives

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Abstract: In 2019, Butler, Choi, Kim, and Seo introduced a new type of juggling card that represents multiplex juggling patterns in a natural bijective way. They conjectured a formula for the generating function for the number of multiplex juggling cards with capacity 2. In this paper, we prove their conjecture. More generally, we find an explicit formula for the generating function with any capacity. We also find an expression for the generating function for multiplex juggling card sequences by introducing a generalization of the q-derivative operator. As a consequence, we show that this generating function is a rational function.

Keywords: juggling sequences; rational generating functions; q-derivatives
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1. Introduction

Juggling is an act of throwing and catching balls. Since the 1980s, juggling has been studied mathematically by many researchers; for example, see [1–4, 6–10, 12, 16, 18] and references therein. We refer the reader to [4, 15] for the history of mathematics of juggling.

Juggling can be divided into two categories: *simple juggling* and *multiplex juggling*. In simple juggling, at most one ball is caught and thrown at every beat. Multiplex juggling is a generalization of simple juggling, where at most k balls are caught and thrown at every beat. The number k is called the *(hand) capacity*.

A *simple juggling pattern* is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i) \geq i$ for all $i \in \mathbb{Z}$. This can be understood as the situation that a juggler catches a ball at beat i and throws it immediately so that it lands at beat $f(i)$. (If $f(i) = i$, then the juggler does not catch or throw any ball at beat i.) A simple juggling pattern is *periodic* if the function $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(i) = f(i) - i$ is periodic.

Buhler, Eisenbud, Graham, and Wright [4] showed that the number of simple juggling patterns with b balls and period p is equal to $(b + 1)^p - b^p$. Ehrenborg and Readdy [10] found a simple proof of this result by introducing juggling cards.

There are some models of multiplex juggling cards introduced by Ehrenborg–Readdy [10] and Butler–Chung–Cummings–Graham [6]. However, they do not represent multiplex juggling patterns in a bijective way. In 2019, Butler, Choi, Kim, and Seo [5] introduced a new type of card that represents multiplex juggling patterns in a natural bijective way. We refer the reader to [5] for more details.

In this paper, we study enumerative properties of the number of multiplex juggling card sequences. We consider three natural parameters of multiplex juggling card sequences: the number b of balls, the capacity k, and the length ℓ of a sequence of cards. Let $J(b, k, \ell)$ denote the number of multiplex juggling card sequences with given parameters k, b, and ℓ. See Section 2 for the precise definition.

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Now we review some known results on the number $J(b,k,1)$ of multiplex juggling cards with b balls and capacity k. If $k=1$, it is immediate from the definition that

$$\sum_{b\geq 0} J(b,1,1)x^b = \sum_{b\geq 0}(b+1)x^b = \frac{1}{(1-x)^2}.$$

For the other extreme case $k=\infty$, Butler et al. [5, Theorem 4] showed that the sequence $\{J(b,\infty,1)\}_{b\geq 0}$ satisfies the following simple linear recurrence: $J(0,\infty,1)=1$, $J(1,\infty,1)=2$, $J(2,\infty,1)=7$, and for $b\geq 3$,

$$J(b,\infty,1) = 4J(b-1,\infty,1) - 2J(b-2,\infty,1). \tag{1}$$

By the standard method for linear recurrences, see [17, Theorem 4.1.1], one can easily check that the recursion (1) is equivalent to

$$\sum_{b\geq 0} J(b,\infty,1)x^b = \frac{1-2x+x^2}{1-4x+2x^2}. \tag{2}$$

For the case $k=2$, Butler et al. [5, Conjecture 13] conjectured the following generating function formula:

$$\sum_{b\geq 0} J(b,2,1)x^b = \frac{1-x+x^2+x^3}{(1-x-x^2)^3}. \tag{3}$$

In this paper, we prove this conjecture; see Example 3.7. More generally, we find an explicit formula for the generating function $\sum_{b\geq 0} J(b,k,1)x^b$ for any capacity k; see Corollary 3.9. We also find an expression for

$$\sum_{b\geq 0} J(b,k,\ell)x^b \tag{4}$$

for any capacity k and length ℓ by introducing a generalization of the q-derivative operator; see Theorem 4.5. As a consequence, we show that the generating function (4) is a rational function in x.

The rest of this paper is organized as follows. In Section 2 we provide necessary definitions. In Section 3 we study the generating function (4) for the case $\ell=1$. In Section 4 we study the generating function (4) for a general ℓ. In Section 5 we show that (4) is a rational function in x. In Section 6 we summarize our results and propose some open problems.

2. Preliminaries

In this section, we give the necessary definitions. Throughout this paper we will use the notation $[n]=\{1,\dots,n\}$ for a positive integer n.

Definition 2.1. A *composition* is a sequence $\alpha=(\alpha_1,\dots,\alpha_r)$ of positive integers. Each α_i is called a *part* of α. The *size* $|\alpha|$ of α is defined by $|\alpha|=\alpha_1+\cdots+\alpha_r$. If $|\alpha|=n$, we say that α is a composition of n. The *length* $\ell(\alpha)$ of α is defined to be the number of parts in α. We denote by $\mathrm{Comp}(n)$ the set of compositions of n. We also denote by $\mathrm{Comp}(n,k)$ the set of compositions of n with k parts.

Definition 2.2. A *(multiplex juggling) card* is a triple (α,β,f) such that $\alpha=(\alpha_1,\dots,\alpha_r)$ and $\beta=(\beta_1,\dots,\beta_s)$ are compositions with $|\alpha|=|\beta|$, and $f:[r]\to\{0\}\cup[s]$ is a strictly increasing function satisfying $\alpha_i\leq\beta_{f(i)}$ for all $i\in[r]$ with $f(i)\neq 0$. We call α and β the *arrival composition* and the *departure composition* of the card, respectively. If every part of α and β is at most k, then we say that the card has *capacity* k. We also say that the card has $|\alpha|$ *balls*.

We can visualize a card (α,β,f) as follows. Suppose $\alpha=(\alpha_1,\dots,\alpha_r)$ and $\beta=(\beta_1,\dots,\beta_s)$. Consider a rectangle with r vertices on the left side labeled $\alpha_1,\dots,\alpha_r$ from bottom to top, s vertices on the right side labeled $\beta_1,\dots,\beta_s$ from bottom to top, and a vertex on the bottom side called the *ground vertex*. If $f(1)\neq 0$, then it follows from Definition 2.2 that we must have $r=s$ and $f(i)=i$ for all $i\in[r]$. In this case, draw a curve from vertex α_i to vertex β_i for all $i\in[r]$. If $f(1)=0$, then draw a curve from vertex α_1 to the ground vertex, a curve from vertex α_i to vertex $\beta_{f(i)}$ for each $i\in\{2,3,\dots,r\}$, and a curve from the ground vertex to each vertex β_j such that either $j=f(i)$ for some $i\in[r]$ with $\alpha_i<\beta_j$ or j is not in the image of f.

Such a visualization can be understood as α_i balls entering the card on level i and β_j balls leaving the card on level j. Note that if $f(1)\neq 0$ then all balls stay in the air, and if $f(1)=0$ then α_1 balls are caught and thrown again, so that these balls are redistributed.

Example 2.3. Let (α,β,f) be the card such that $\alpha=(4,2,3)$, $\beta=(4,2,3)$, and $f:\{1,2,3\}\to\{0,1,2,3\}$ is the function given by $f(1)=1$, $f(2)=2$, and $f(3)=3$. Then the card can be visualized as the left diagram in Figure 1.

Example 2.4. Let (α, β, f) be the triple such that $\alpha = (6, 1, 2, 2)$, $\beta = (3, 1, 2, 3, 2)$, and $f : \{1, 2, 3, 4\} \to \{0, 1, 2, 3, 4, 5\}$ is the function given by $f(1) = 0$, $f(2) = 1$, $f(3) = 3$, and $f(4) = 4$. Then the card can be visualized as the right diagram in Figure 1.

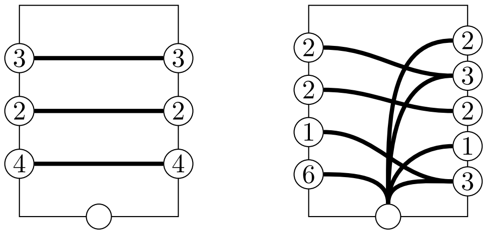

Figure 1: The left diagram is a visualization of the card in Example 2.3. The right diagram is a visualization of the card in Example 2.4.

One multiplex juggling card represents a situation of multiplex juggling at a given moment, say at beat i. We say that two juggling cards (α, β, f) and (α', β', f') are *compatible* if $\beta = \alpha'$. By listing ℓ compatible cards, we can represent a situation of multiplex juggling from beats 1 to ℓ. To be more precise, we introduce the following definition.

Definition 2.5. An *ℓ-card sequence* with b *balls* and *capacity* k is a sequence $(C_1, \dots, C_\ell)$ of ℓ cards with b balls and capacity k such that the departure composition of C_i is equal to the arrival composition of C_{i+1} for all $i \in [\ell - 1]$. We denote by $\mathcal{J}(b, k, \ell)$ the set of ℓ-card sequences with b balls and capacity k. We also define $J(b, k, \ell) = |\mathcal{J}(b, k, \ell)|$.

For example, see Figure 2.

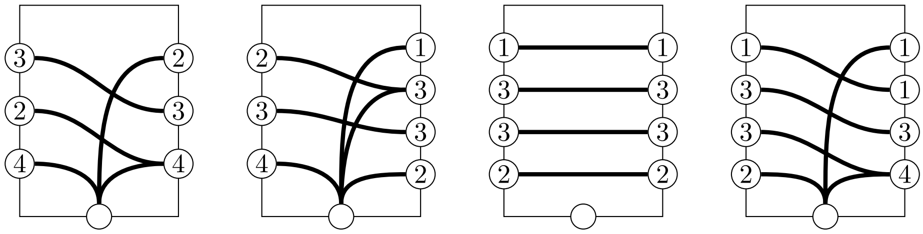

Figure 2: An example of a 4-card sequence.

3. Juggling cards with fixed capacity

In this section, we provide three expressions for the generating function

$$\sum_{b \geq 0} J(b, k, 1) x^b \tag{5}$$

with fixed capacity k. To this end, we give another description of a card using embeddings introduced in [5].

Recall that $\mathcal{J}(b, k, 1)$ is the set of all cards with b balls and capacity k. We use the notation r^s to denote the word consisting of s r's.

Definition 3.1. A (b, k)-*embedding* is a sequence $\gamma = (\gamma_1, \dots, \gamma_s)$ satisfying the following conditions:

- Each γ_i is a word of the form $\gamma_i = 0^u 1^v$ for some integers $u, v \geq 0$ with $1 \leq u + v \leq k$.
- The total number of 1's in all of $\gamma_1, \dots, \gamma_s$ is at most k.
- The sum of the lengths of γ_i for all $i \in [s]$ is equal to b.

Let $\mathcal{E}(b, k)$ be the set of (b, k)-embeddings.

Let $(\alpha, \beta, f) \in \mathcal{J}(b, k, 1)$, where $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_s)$. Let $\gamma = (\gamma_1, \dots, \gamma_s)$ be the (b, k)-embedding defined by

$$\gamma_j = \begin{cases} 1^{\beta_j} & \text{if } j \text{ is not in the image of } f, \\ 0^{\alpha_i} 1^{\beta_j - \alpha_i} & \text{if } j = f(i). \end{cases}$$

This can also be understood using the visualization of the card (α, β, f) as follows. For each vertex β_j, if it is connected to vertex α_i for some $i \in [r]$, then $\gamma_j = 0^{\alpha_i} 1^{\beta_j - \alpha_i}$ and otherwise $\gamma_j = 1^{\beta_j}$. The 0's encode the balls passing over the ground vertex, whereas the 1's encode the balls that were thrown up from this vertex.

Example 3.2. If (α, β, f) is the card in Example 2.3, then $\gamma = (0000, 00, 000)$. If (α, β, f) is the card in Example 2.4, then $\gamma = (011, 1, 00, 001, 11)$. See Figure 3.

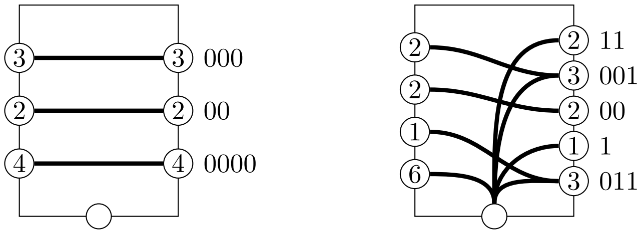

Figure 3: The left diagram shows the card in Example 2.3 and its corresponding embedding $(0000, 00, 000)$. The right diagram shows the card in Example 2.4 and its corresponding embedding $(011, 1, 00, 001, 11)$.

It is easy to see that the map $(\alpha, \beta, f) \mapsto \gamma$ is a bijection from $\mathcal{J}(b, k, 1)$ to $\mathcal{E}(b, k)$. Therefore, we can identify a card with b balls and capacity k with a (b, k)-embedding.

Now we are ready to find an expression for the generating function in (5). For a formal power series $F(z)$ in z, the notation $[z^k]F(z)$ denotes the coefficient of z^k in $F(z)$.

Proposition 3.3. *For a positive integer k, we have*

$$\sum_{b\geq 0} J(b,k,1)x^b = [z^k]\left(\frac{1}{1-z}\cdot\frac{1}{1-\sum_{i=1}^k x^i \sum_{j=0}^i z^j}\right).$$

Proof. Since $J(0, k, 1) = 1$ and $J(b, k, 1) = |\mathcal{J}(b, k, 1)| = |\mathcal{E}(b, k)|$ for $b \geq 1$, we can instead consider the (b, k)-embeddings for all $b \geq 1$. Recall that if $\gamma = (\gamma_1, \ldots, \gamma_s)$ is a (b, k)-embedding, then $\gamma_i = 0^{u_i} 1^{v_i}$ for some integers $u_i, v_i \geq 0$ with $1 \leq u_i + v_i \leq k$ such that $0 \leq v_1 + \cdots + v_s \leq k$ and $u_1 + \cdots + u_s + v_1 + \cdots + v_s = b$.

Let W be the set of words $0^u 1^v$ such that $u, v \geq 0$ and $1 \leq u + v \leq k$. For a word $w = 0^u 1^v \in W$, let $\ell(w) = u + v$ and $\ell_1(w) = v$. By definition, we have

$$\sum_{w\in W} x^{\ell(w)} z^{\ell_1(w)} = \sum_{i=1}^k x^i \sum_{j=0}^i z^j.$$

Thus we obtain

$$\frac{1}{1-\sum_{i=1}^k x^i \sum_{j=0}^i z^j} = 1 + \sum_{t\geq 1} \sum_{(w_1,\ldots,w_t)\in W^t} x^{\ell(w_1)+\cdots+\ell(w_t)} z^{\ell_1(w_1)+\cdots+\ell_1(w_t)}. \tag{6}$$

On the other hand, for $b \geq 1$, $\mathcal{E}(b, k)$ is the set of $\gamma = (\gamma_1, \ldots, \gamma_t) \in W^t$, $t \geq 1$, such that $\ell(\gamma_1)+\cdots+\ell(\gamma_t) = b$ and $0 \leq \ell_1(\gamma_1) + \cdots + \ell_1(\gamma_t) \leq k$. Thus

$$\begin{aligned}
\sum_{b\geq 0} J(b,k,1)x^b &= 1 + \sum_{b\geq 1} |\mathcal{E}(b,k)|x^b \\
&= 1 + \sum_{t\geq 1} \sum_{\substack{(w_1,\ldots,w_t)\in W^t \\ 0\leq \ell_1(\gamma_1)+\cdots+\ell_1(\gamma_t)\leq k}} x^{\ell(w_1)+\cdots+\ell(w_t)} \\
&= \sum_{p=0}^k [z^p]\left(1 + \sum_{t\geq 1} \sum_{(w_1,\ldots,w_t)\in W^t} x^{\ell(w_1)+\cdots+\ell(w_t)} z^{\ell_1(w_1)+\cdots+\ell_1(w_t)}\right).
\end{aligned}$$

By equation (6), this expression is equal to

$$\sum_{p=0}^k [z^p]\left(\frac{1}{1-\sum_{i=1}^k x^i \sum_{j=0}^i z^j}\right) = [z^k]\left(\frac{1}{1-z}\cdot\frac{1}{1-\sum_{i=1}^k x^i \sum_{j=0}^i z^j}\right), \tag{7}$$

as desired. □

Remark 3.4. Proposition 3.3 can be used to prove equation (2), which is equivalent to the result of Butler et al. [5, Theorem 4]. To see this, note that, by equation (7), Proposition 3.3 can be rewritten as

$$\sum_{b\geq 0} J(b,k,1)x^b = \sum_{p=0}^{k}[z^p]\left(\frac{1}{1-\sum_{i=1}^{k} x^i \sum_{j=0}^{i} z^j}\right).$$

Taking the limit as k tends to infinity, we have

$$\sum_{b\geq 0} J(b,\infty,1)x^b = \sum_{p=0}^{\infty}[z^p]\left(\frac{1}{1-\sum_{i=1}^{\infty} x^i \sum_{j=0}^{i} z^j}\right).$$

Adding the coefficient of z^p for all nonnegative p is equivalent to substituting $z=1$. Hence we obtain

$$\sum_{b\geq 0} J(b,\infty,1)x^b = \frac{1}{1-\sum_{i=1}^{\infty} x^i(i+1)} = \frac{1}{1-\frac{d}{dx}\left(\frac{1}{1-x}-1-x\right)} = \frac{1-2x+x^2}{1-4x+2x^2}.$$

The sequence $\{J(b,\infty,1)\}_{b\geq 0}$ is A003480 in the On-Line Encyclopedia of Integer Sequences (OEIS) [14]:

$$1, 2, 7, 24, 82, 280, 956, 3264, 11144, 38048, 129904, 443520, 1514272, \dots.$$

Now we give another expression for the generating function in (5). For a composition $\alpha \in \mathrm{Comp}(n)$, we define $\ell_2(\alpha)$ to be the number of parts of α at least 2.

Theorem 3.5. *For a positive integer k, we have*

$$\sum_{b\geq 0} J(b,k,1)x^b = \sum_{\alpha\in \mathrm{Comp}(k)} \frac{(-1)^{\ell_2(\alpha)} x^{k-\ell(\alpha)}}{(1-x-\cdots-x^k)^{1+\ell(\alpha)}}.$$

Proof. We will modify the right-hand side of Proposition 3.3. We have

$$\begin{aligned}
\frac{1}{1-z}\cdot\frac{1}{1-\sum_{i=1}^{k} x^i \sum_{j=0}^{i} z^j} &= \frac{1}{1-z-\sum_{i=1}^{k} x^i \sum_{j=0}^{i} z^j + \sum_{i=1}^{k} x^i \sum_{j=0}^{i} z^{j+1}}\\
&= \frac{1}{1-z+\sum_{i=1}^{k} x^i(z^{i+1}-1)}\\
&= \frac{1}{1-\sum_{i=1}^{k} x^i - z\left(1-\sum_{i=1}^{k} x^i z^i\right)}\\
&= \frac{1}{1-\sum_{i=1}^{k} x^i}\cdot\frac{1}{1-z\left(1-\sum_{i=1}^{k} x^i z^i\right)\left(1-\sum_{i=1}^{k} x^i\right)^{-1}}\\
&= \sum_{r\geq 0}\frac{z^r\left(1-\sum_{i=1}^{k} x^i z^i\right)^r}{\left(1-\sum_{i=1}^{k} x^i\right)^{1+r}}.
\end{aligned}$$

Therefore by Proposition 3.3, we obtain

$$\sum_{b\geq 0} J(b,k,1)x^b = [z^k]\left(\sum_{r\geq 0}\frac{(z-xz^2-\cdots-x^k z^{k+1})^r}{(1-x-\cdots-x^k)^{1+r}}\right). \tag{8}$$

Observe that

$$(z-xz^2-\cdots-x^k z^{k+1})^r = \sum_{\alpha_1,\dots,\alpha_r\in[k+1]} z^{\alpha_1+\cdots+\alpha_r}(-1)^{\ell_2(\alpha_1,\dots,\alpha_r)} x^{\alpha_1+\cdots+\alpha_r-r}.$$

Hence, we can rewrite (8) to obtain the result of the theorem. □

For small values of k, one can easily find explicit formulas for the generating function in (5) using Theorem 3.5 as follows.

Example 3.6. If $k=1$ then $\mathrm{Comp}(k)=\{(1)\}$. Thus we have

$$\sum_{b\geq 0} J(b,1,1)x^b = \frac{1}{(1-x)^2}.$$

Example 3.7. If $k = 2$ then $\mathrm{Comp}(k) = \{(2), (1, 1)\}$. Thus we have

$$\sum_{b\geq 0} J(b, 2, 1)x^b = \frac{-x}{(1 - x - x^2)^2} + \frac{1}{(1 - x - x^2)^3} = \frac{1 - x + x^2 + x^3}{(1 - x - x^2)^3}.$$

This proves the identity (3) conjectured by Butler et al. [5, Conjecture 13]. The sequence $\{J(b, 2, 1)\}_{b\geq 0}$ is A370304 in OEIS [14]:

$$1, 2, 7, 17, 41, 91, 195, 403, 812, 1601, 3102, 5922, 11165, 20824, 38477, \dots.$$

Example 3.8. If $k = 3$ then $\mathrm{Comp}(k) = \{(3), (2, 1), (1, 2), (1, 1, 1)\}$. Thus we have

$$\begin{aligned}\sum_{b\geq 0} J(b, 3, 1)x^b &= \frac{-x^2}{(1 - x - x^2 - x^3)^2} + \frac{-2x}{(1 - x - x^2 - x^3)^3} + \frac{1}{(1 - x - x^2 - x^3)^4} \\ &= \frac{1 - 2x + x^2 + 4x^3 + 3x^4 - 3x^6 - 2x^7 - x^8}{(1 - x - x^2 - x^3)^4}.\end{aligned}$$

The sequence $\{J(b, 3, 1)\}_{b\geq 0}$ is A370306 in OEIS [14]:

$$1, 2, 7, 24, 70, 198, 532, 1370, 3418, 8296, 19677, 45770, 104687, 235972, \dots.$$

In fact, using Theorem 3.5, we can find an explicit formula for the generating function in (5) for a general capacity k.

Corollary 3.9. *For a positive integer k, we have*

$$\sum_{b\geq 0} J(b, k, 1)x^b = \sum_{r=1}^{k}\sum_{s=0}^{r} \frac{(-1)^{r-s}\binom{r}{s}\binom{k-r-1}{r-s-1}x^{k-r}}{(1 - x - \cdots - x^k)^{1+r}},$$

where we extend the binomial coefficient by $\binom{-1}{n} = \binom{n}{-1} = 1$ if $n = -1$ and $\binom{-1}{n} = \binom{n}{-1} = 0$ otherwise.

Proof. We can rewrite Theorem 3.5 as

$$\sum_{b\geq 0} J(b, k, 1)x^b = \sum_{r=1}^{k} \frac{x^{k-r}}{(1 - x - \cdots - x^k)^{1+r}} \sum_{\alpha\in\mathrm{Comp}(k,r)} (-1)^{\ell_2(\alpha)}. \tag{9}$$

It is easy to see that the number of $\alpha \in \mathrm{Comp}(k, r)$ with exactly s parts equal to 1 is $\binom{r}{s}\binom{k-r-1}{r-s-1}$. Hence,

$$\sum_{\alpha\in\mathrm{Comp}(k,r)} (-1)^{\ell_2(\alpha)} = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}\binom{k-r-1}{r-s-1}. \tag{10}$$

By combining equations (9) and (10) we obtain the desired formula. □

Remark 3.10. Note that the right-hand side of (10), say

$$C(k, r) = \sum_{s=0}^{r} (-1)^{r-s}\binom{r}{s}\binom{k-r-1}{r-s-1}$$

can be defined for any nonnegative integers k and r. Since $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$, we have, for $k < r$,

$$C(k, r) = -\sum_{s=0}^{r}\binom{r}{s}\binom{2r-k-s-1}{r-s-1} = -\sum_{s=1}^{r}\binom{r}{s}\binom{r-k-1}{s-1}.$$

This shows that, for $1 \leq k < r$, we have $C(k, r) = -A(r, r - k + 1)$, where $A(n, k)$ is the sequence A050143 in OEIS [14]. We note that the sequence $A(n, k)$ has an interpretation using certain lattice paths; see [14].

4. Juggling card sequences with fixed capacity

In this section, we find an expression for the generating function for $J(b,k,\ell)$ with fixed capacity k and length ℓ. To this end we introduce (b,k,ℓ)-embeddings, which generalize the notion of (b,k)-embeddings of cards to ℓ-card sequences. In order to analyze (b,k,ℓ)-embeddings, we then introduce an operator which generalizes the q-derivative operator.

Observe that the (b,k)-embedding of a card keeps track of when balls are thrown. In what follows, we extend this notion to ℓ-card sequences.

Let $(C_1,\dots,C_\ell) \in J(b,k,\ell)$. For $i = 0,1,\dots,\ell$, we will define a sequence $\alpha^{(i)}$ of words of the form $0^{c_1}1^{c_1}\cdots i^{c_i}$ as follows.

- First, we define $\alpha^{(0)} = (0^{a_1},\dots,0^{a_s})$, where $(a_1,\dots,a_s)$ is the departure composition of C_1.
- For $i \in [\ell]$, suppose $(\beta_1,\dots,\beta_t)$ is the (b,k)-embedding of C_i. Then each β_j is a word of the form 0^u1^v.

  Case 1 There are no 1's in $\beta_1,\dots,\beta_t$. In this case, we define $\alpha^{(i)} = \alpha^{(i-1)}$.

  Case 2 There is at least one 1 in $\beta_1,\dots,\beta_t$. Then, first, we replace each 1 by i in $\beta_1,\dots,\beta_t$. Let $\beta_{d_1},\dots,\beta_{d_m}$ be the words among $\beta_1,\dots,\beta_t$ containing at least one 0, where $d_1 < \cdots < d_m$. Then $\alpha^{(i-1)}$ must have $m+1$ words. Let $\alpha^{(i-1)} = (\alpha_1^{(i-1)},\dots,\alpha_{m+1}^{(i-1)})$. For each $j \in [m]$, we replace the subword of β_{d_j} consisting of zeros by $\alpha_{j+1}^{(i-1)}$.

Example 4.1. Let $(C_1,\dots,C_4)$ be the 4-card sequence in Figure 2. Then we have $\alpha^{(0)} = (0000,00,000)$, $\alpha^{(1)} = (0011,000,11)$, $\alpha^{(2)} = (22,000,112,2)$, $\alpha^{(3)} = (22,000,112,2)$, and $\alpha^{(4)} = (0004,112,2,4)$. See Figure 4.

Note that a 0 in $\alpha^{(i)}$ means that the associated ball came from the original set of balls whereas a j in $\alpha^{(i)}$ means that the associated ball was thrown up by card C_j.

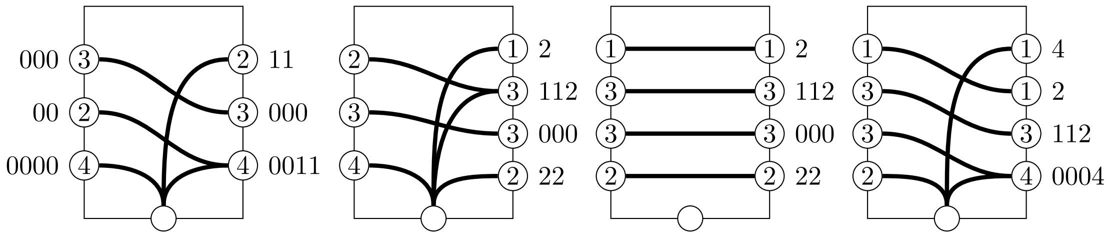

Figure 4: A 4-card sequence (C_1,C_2,C_3,C_4) with the data $(\alpha^{(0)},\alpha^{(1)},\alpha^{(2)},\alpha^{(3)},\alpha^{(4)})$ of when balls are thrown. Here $\alpha^{(i-1)}$ and $\alpha^{(i)}$ are shown on the left and on the right, respectively, of each card C_i.

The ℓ-card sequence $(C_1,\dots,C_\ell)$ can be recovered from $(\alpha^{(0)},\dots,\alpha^{(\ell)})$ because the (b,k)-embedding of C_i is obtained from $\alpha^{(i)}$ by replacing every integer less than i by 0 and every i by 1. Moreover, $\alpha^{(i-1)}$ is also determined by $\alpha^{(i)}$ and the history of the balls thrown at beat i. More precisely, let δ_i be the first word in $\alpha^{(i-1)}$ if there is at least one i in $\alpha^{(i)}$, and let $\delta_i = \emptyset$ otherwise. Then $\alpha^{(i-1)}$ is determined by $\alpha^{(i)}$ and δ_i as follows. If $\delta_i = \emptyset$, then $\alpha^{(i-1)} = \alpha^{(i)}$. Otherwise, $\alpha^{(i-1)}$ is obtained from $\alpha^{(i)}$ by deleting all i's, discarding all empty words if there are any, and adding δ_i at the beginning.

Applying this process iteratively, the whole sequence $(\alpha^{(0)},\dots,\alpha^{(\ell)})$ is determined by the pair (γ,δ), where $\gamma = \alpha^{(\ell)}$ and $\delta = (\delta_1,\dots,\delta_\ell)$. Observe that the length of the word δ_i is equal to the total number of i's in $\alpha^{(i)}$, which is also equal to the total number of i's in $\delta_{i+1},\dots,\delta_\ell$, and $\alpha^{(\ell)}$.

Example 4.2. The 4-card sequence in Figure 4 corresponds to the pair (γ,δ), where $\gamma = (0004,112,2,4)$, and $\delta = (0000,0011,\emptyset,22)$.

The above observations imply that the ℓ-card sequence $(C_1,\dots,C_\ell)$ can be identified with the pair (γ,δ). This leads us to the following definition.

Definition 4.3. A (b,k,ℓ)-*embedding* is a pair (γ,δ) of two sequences $\gamma = (\gamma_1,\dots,\gamma_r)$ and $\delta = (\delta_1,\dots,\delta_\ell)$ satisfying the following conditions:

- Each γ_i is a word of the form $0^{c_0}1^{c_1}\cdots\ell^{c_\ell}$ for some integers $c_0,c_1,\dots,c_\ell \ge 0$ with $1 \le c_0+c_1+\cdots+c_\ell \le k$.
- The sum of the lengths of γ_i for all $i \in [r]$ is equal to b.
- Each δ_i is a (possibly empty) word of the form $0^{d_0}1^{d_1}\cdots(i-1)^{d_{i-1}}$ for some integers $d_0,d_1,\dots,d_{i-1} \ge 0$ such that $0 \le d_0+d_1+\cdots+d_{i-1} \le k$ and $d_0+d_1+\cdots+d_{i-1}$ is equal to the total number of i's in $\gamma_1,\dots,\gamma_r$ and $\delta_{i+1},\delta_{i+2},\dots,\delta_\ell$.

Let $\mathcal{E}(b,k,\ell)$ be the set of (b,k,ℓ)-embeddings.

By the observations above, the map $(C_1,\dots,C_\ell)\mapsto(\gamma,\delta)$ is a bijection from $\mathcal{J}(b,k,\ell)$ to $\mathcal{E}(b,k,\ell)$. In order to enumerate $\mathcal{E}(b,k,\ell)$ we need some definitions.

For a nonnegative integer n, the *complete homogeneous symmetric function* $h_n(x_1,\dots,x_m)$ is defined by

$$h_n(x_1,\dots,x_m)=\sum_{i_1+\cdots+i_m=n} x_1^{i_1}\cdots x_m^{i_m},$$

where the sum is over all m-tuples $(i_1,\dots,i_m)$ of nonnegative integers summing to n. Note that we have $h_0(x_1,\dots,x_m)=1$.

Definition 4.4. For indeterminates $z_1,\dots,z_k$, we denote by $D_{z_1,\dots,z_k}$ the linear operator on the space of formal power series in z_k defined by

$$D_{z_1,\dots,z_k} z_k^n = h_n(1,z_1,\dots,z_{k-1})z_k^n.$$

Note that this can be seen as a generalization of the q-*derivative operator* $(\frac{d}{dz})_q$, which is the linear operator on the space of formal power series in z defined by

$$\left(\frac{d}{dz}\right)_q z^n=(1+q+\cdots+q^{n-1})z^{n-1}.$$

Hence $D_{q,z}$ is equal to the operator $(\frac{d}{dz})_q z$, which multiplies z and then takes the q-derivative.

Now we are ready to find an expression for the generating function for $J(b,k,\ell)$ when k and ℓ are fixed.

Theorem 4.5. *For fixed positive integers k and ℓ, we have*

$$\sum_{b\ge 0} J(b,k,\ell)x^b=[z_1^k\cdots z_\ell^k]\left(\frac{1}{1-z_1}\cdots\frac{1}{1-z_\ell}D_{z_1,z_2}D_{z_1,z_2,z_3}\cdots D_{z_1,\dots,z_\ell}\frac{1}{2-h_k(1,x,xz_1,\dots,xz_\ell)}\right).$$

Proof. We proceed similarly as in the proof of Proposition 3.3. Let W be the set of words of the form $0^{c_0}1^{c_1}\cdots\ell^{c_\ell}$ for some integers $c_0,c_1,\dots,c_\ell\ge 0$ with $1\le c_0+c_1+\cdots+c_\ell\le k$. For a word $w=0^{c_0}1^{c_1}\cdots\ell^{c_\ell}\in W$, we define $\operatorname{len}(w)=c_0+\cdots+c_\ell$ and

$$\operatorname{wt}(w)=z_1^{c_1}\cdots z_\ell^{c_\ell}.$$

Then we have

$$\sum_{w\in W} x^{\operatorname{len}(w)}\operatorname{wt}(w)=\sum_{1\le c_0+\cdots+c_\ell\le k} x^{c_0+\cdots+c_\ell}z_1^{c_1}\cdots z_\ell^{c_\ell}=h_k(1,x,xz_1,\dots,xz_\ell)-1.$$

Therefore we obtain

$$A:=\frac{1}{2-h_k(1,x,xz_1,\dots,xz_\ell)}=1+\sum_{r\ge1}\sum_{(w_1,\dots,w_r)\in W^r} x^{\operatorname{len}(w_1)+\cdots+\operatorname{len}(w_r)}\operatorname{wt}(w_1)\cdots\operatorname{wt}(w_r). \tag{11}$$

First, we consider how the operator $D_{z_1,\dots,z_\ell}$ acts on a monomial in the right-hand side of (11). For $(w_1,\dots,w_r)\in W^r$, we have

$$x^{\operatorname{len}(w_1)+\cdots+\operatorname{len}(w_r)}\operatorname{wt}(w_1)\cdots\operatorname{wt}(w_r)=x^b z_1^{n_1}\cdots z_\ell^{n_\ell}$$

for some integers $b\ge1$ and $n_1,\dots,n_\ell\ge0$. Then

$$\begin{aligned}D_{z_1,\dots,z_\ell}x^b z_1^{n_1}\cdots z_\ell^{n_\ell}&=x^b z_1^{n_1}\cdots z_\ell^{n_\ell}h_{n_\ell}(1,z_1,\dots,z_{\ell-1})\\&=x^b z_1^{n_1}\cdots z_\ell^{n_\ell}\sum_{c_0+\cdots+c_{\ell-1}=n_\ell} z_1^{c_1}\cdots z_{\ell-1}^{c_{\ell-1}}.\end{aligned}$$

This implies that

$$D_{z_1,\dots,z_\ell}A=1+\sum_{(w_1,\dots,w_r,u_\ell)\in X_\ell} x^{\operatorname{len}(w_1)+\cdots+\operatorname{len}(w_r)}\operatorname{wt}(w_1)\cdots\operatorname{wt}(w_r)\operatorname{wt}(u_\ell), \tag{12}$$

where X_ℓ is the set of tuples $(w_1,\dots,w_r,u_\ell)$ such that $w_i\in W$ for $i\in[r]$ and u_ℓ is a word of the form $0^{c_0}\cdots(\ell-1)^{c_{\ell-1}}$ for some nonnegative integers $c_0,\dots,c_{\ell-1}$ with the condition that $c_0+\cdots+c_{\ell-1}$ is equal to the total number of ℓ's in $w_1,\dots,w_r$.

Next, we consider how the operator $D_{z_1,\dots,z_{\ell-1}}$ acts on a monomial in the right-hand side of (12). For $(w_1,\dots,w_r,u_\ell)\in X_\ell$, we have

$$x^{\mathrm{len}(w_1)+\cdots+\mathrm{len}(w_r)}\,\mathrm{wt}(w_1)\cdots\mathrm{wt}(w_r)\,\mathrm{wt}(u_\ell)=x^b z_1^{n_1}\cdots z_\ell^{n_\ell}$$

for some integers $b\ge 1$ and $n_1,\dots,n_\ell\ge 0$. Then we have

$$\begin{aligned}D_{z_1,\dots,z_{\ell-1}}x^b z_1^{n_1}\cdots z_\ell^{n_\ell} &= x^b z_1^{n_1}\cdots z_\ell^{n_\ell}h_{n_{\ell-1}}(1,z_1,\dots,z_{\ell-2})\\ &= x^b z_1^{n_1}\cdots z_\ell^{n_\ell}\sum_{c_0+\cdots+c_{\ell-2}=n_{\ell-1}} z_1^{c_1}\cdots z_{\ell-1}^{c_{\ell-1}}.\end{aligned}$$

This implies that

$$D_{z_1,\dots,z_{\ell-1}}D_{z_1,\dots,z_\ell}A=1+\sum_{(w_1,\dots,w_r,u_\ell,u_{\ell-1})\in X_{\ell-1}} x^{\mathrm{len}(w_1)+\cdots+\mathrm{len}(w_r)}\,\mathrm{wt}(w_1)\cdots\mathrm{wt}(w_r)\,\mathrm{wt}(u_\ell)\,\mathrm{wt}(u_{\ell-1}),$$

where $X_{\ell-1}$ is the set of tuples $(w_1,\dots,w_r,u_\ell,u_{\ell-1})$ such that $(w_1,\dots,w_r,u_\ell)\in X_\ell$ and $u_{\ell-1}$ is a word of the form $0^{c_0}\cdots(\ell-2)^{c_{\ell-2}}$ for some nonnegative integers $c_0,\dots,c_{\ell-2}$ with the condition that $c_0+\cdots+c_{\ell-2}$ is equal to the total number of $(\ell-1)$'s in $w_1,\dots,w_r,u_\ell$.

Applying the above argument iteratively, we obtain that

$$D_{z_1,z_2}\cdots D_{z_1,\dots,z_\ell}A=1+\sum_{(w_1,\dots,w_r,u_\ell,\dots,u_1)\in X_1} x^{\mathrm{len}(w_1)+\cdots+\mathrm{len}(w_r)}\,\mathrm{wt}(w_1)\cdots\mathrm{wt}(w_r)\,\mathrm{wt}(u_\ell)\cdots\mathrm{wt}(u_1), \tag{13}$$

where X_1 is the set of tuples $(w_1,\dots,w_r,u_\ell,\dots,u_1)$ such that for each $i\in[r]$, $w_i\in W$ and for each $j\in[\ell]$, u_j is a word of the form $0^{c_0}\cdots(j-1)^{c_{j-1}}$ for some nonnegative integers $c_0,\dots,c_{j-1}$ with the condition that $c_0+\cdots+c_{j-1}$ is equal to the total number of j's in $w_1,\dots,w_r,u_\ell,\dots,u_{j+1}$. By equation (13), we have

$$[z_1^k\cdots z_\ell^k]\left(\frac{1}{1-z_1}\cdots\frac{1}{1-z_\ell}D_{z_1,z_2}D_{z_1,z_2,z_3}\cdots D_{z_1,\dots,z_\ell}A\right)=1+\sum_{b\ge1}|Y_b|x^b,$$

where Y_b is the set of tuples $(w_1,\dots,w_r,u_\ell,\dots,u_1)\in X_1$ such that

$$x^{\mathrm{len}(w_1)+\cdots+\mathrm{len}(w_r)}\,\mathrm{wt}(w_1)\cdots\mathrm{wt}(w_r)\,\mathrm{wt}(u_\ell)\cdots\mathrm{wt}(u_1)=x^b z_1^{n_1}\cdots z_\ell^{n_\ell}$$

for some integers $0\le n_1,\dots,n_\ell\le k$.

It is immediate from the definitions of Y_b and $\mathcal{E}(b,k,\ell)$ that the map

$$(w_1,\dots,w_r,u_\ell,\dots,u_1)\mapsto(\gamma,\delta),$$

where $\gamma=(w_1,\dots,w_r)$ and $\delta=(u_1,\dots,u_\ell)$, is a bijection from Y_b to $\mathcal{E}(b,k,\ell)$. Therefore we conclude

$$1+\sum_{b\ge1}|Y_b|x^b=1+\sum_{b\ge1}|\mathcal{E}(b,k,\ell)|x^b=\sum_{b\ge0}J(b,k,\ell)x^b,$$

which completes the proof. □

5. Rationality of the generating function

In this section, as a consequence of Theorem 4.5, we show that the generating function for $J(b,k,\ell)$ with fixed k and ℓ is a rational function. This is equivalent to the statement that the sequence $\{J(b,k,\ell)\}_{b\ge0}$ satisfies a linear recurrence relation; see [17, Theorem 4.1.1].

We first review some basic properties of derivatives and q-derivatives and extend these properties to the operator $D_{z_1,\dots,z_m}$. By the quotient rule in calculus, one can easily deduce that the derivative of a rational function is also a rational function. For the q-derivative, it is well known [13, Equation (11.4.1)] that

$$\left(\frac{d}{dx}\right)_q f(x)=\frac{f(x)-f(qx)}{x-qx}.$$

This implies that the q-derivative of a rational function is also a rational function. Note that since

$$D_{z_1,z_2}z_2^n=(1+z_1+\cdots+z_1^n)z_2^n=\frac{(1-z_1^{n+1})z_2^n}{1-z_1},$$

we have

$$D_{z_1,z_2}f(z_2) = \frac{f(z_2) - z_1 f(z_1 z_2)}{1 - z_1}. \tag{14}$$

Hence, if $f(z_2)$ is a rational function in z_2, then so is $D_{z_1,z_2}f(z_2)$.

Our strategy is to show that the operator $D_{z_1,\dots,z_m}$, for any $m \ge 2$, also preserves the rationality of a formal power series. To do this, we need the following two lemmas.

Lemma 5.1. *For integers $n \ge 0$ and $m \ge 2$, we have*

$$h_n(1, z_1, \dots, z_m) = \frac{z_{m-1} h_n(1, z_1, \dots, z_{m-1}) - z_m h_n(1, z_1, \dots, z_{m-2}, z_m)}{z_{m-1} - z_m}.$$

Proof. Let $\mathbb{R}[z_1, \dots, z_k]$ denote the space of polynomials in the variables $z_1, \dots, z_k$. Consider the linear operator $\Delta : \mathbb{R}[x] \longrightarrow \mathbb{R}[x, y]$ defined by

$$\Delta(p(x)) = \frac{x \cdot p(x) - y \cdot p(y)}{x - y}.$$

Note that $\Delta(x^n) = \sum_{i+j=n} x^i y^j$. Hence, by letting $x = z_{m-1}$ and $y = z_m$ and applying this operator to $h(z_1, \dots, z_{m-1})$, we obtain the desired identity. ☐

Lemma 5.2. *For an integer $m \ge 2$, we have*

$$D_{z_1,\dots,z_{m+1}} = \frac{z_{m-1} D_{z_1,\dots,z_{m-1},z_{m+1}} - z_m D_{z_1,\dots,z_{m-2},z_m,z_{m+1}}}{z_{m-1} - z_m}.$$

Proof. Both sides are linear operators in the space of formal power series in z_{m+1}. Therefore, it suffices to show that they act on z_{m+1}^n in the same way, that is,

$$D_{z_1,\dots,z_{m+1}} z_{m+1}^n = \frac{z_{m-1} D_{z_1,\dots,z_{m-1},z_{m+1}} - z_m D_{z_1,\dots,z_{m-2},z_m,z_{m+1}}}{z_{m-1} - z_m} z_{m+1}^n.$$

But this is equivalent to

$$h_n(1, z_1, \dots, z_m) z_{m+1}^n = \frac{z_{m-1} h_n(1, z_1, \dots, z_{m-1}) - z_m h_n(1, z_1, \dots, z_{m-2}, z_m)}{z_{m-1} - z_m} z_{m+1}^n,$$

which follows from Lemma 5.1. ☐

Now we are ready to show that the operator $D_{z_1,\dots,z_m}$ preserves the rationality of a formal power series.

Proposition 5.3. *Suppose that ℓ and m are integers with $2 \le m \le \ell$ and let $z_1, \dots, z_\ell$ be indeterminates. If $f(z_1, \dots, z_\ell)$ is a formal power series in $z_1, \dots, z_\ell$ that is a rational function in $z_1, \dots, z_\ell$, then $D_{z_1,\dots,z_m} f(z_1, \dots, z_\ell)$ is a rational function in $z_1, \dots, z_\ell$.*

Proof. If $m = 2$, the statement follows from (14). Suppose that the statement holds for $m \ge 2$. Then by Lemma 5.2 the case of $m + 1$ also holds. The proof then follows by induction. ☐

Finally, we can prove the rationality of the generating function studied in the previous section.

Corollary 5.4. *For fixed positive integers k and ℓ, the generating function*

$$\sum_{b \ge 0} J(b, k, \ell) x^b$$

is a rational function in the variable x.

Proof. By Theorem 4.5 and Proposition 5.3, we have

$$\sum_{b \ge 0} J(b, k, \ell) x^b = [z_1^k \cdots z_\ell^k] f(z_1, \dots, z_\ell, x) \tag{15}$$

for a formal power series $f(z_1, \dots, z_\ell, x)$ in $z_1, \dots, z_\ell, x$ that is a rational function in these indeterminates. If $g(z)$ is a formal power series in z that is a rational function in z and some other indeterminates, say $u_1, \dots, u_r$, then by the quotient rule, $[z^k]g(z) = k! g^{(k)}(0)$ is a rational function in $u_1, \dots, u_r$. Therefore the right-hand side of (15) is a rational function in x as desired. ☐

By Corollary 5.4, for fixed positive integers k and ℓ, the sequence $\{J(b, k, \ell)\}_{b \ge 0}$ satisfies a linear recurrence relation. However, due to the complexity of its generating function formula in Theorem 4.5, finding an explicit recurrence relation appears to be challenging. In [11], the authors used holonomic methods to find recurrence relations for the number of multiset derangements. It would be interesting to see if their method can be applied to the above sequence.

6. Conclusion

In this paper, we found an expression for the generating function

$$\sum_{b\geq 0} J(b,k,\ell)x^b \tag{16}$$

for the number of multiplex juggling card sequences when the capacity k and the length ℓ are fixed. As a consequence, we showed that this generating function is a rational function in x. Equivalently, the sequence $\{J(b,k,\ell)\}_{b\geq 0}$ satisfies a linear recurrence relation.

Note that there are three parameters in the number $J(b,k,\ell)$ and the generating function in (16) keeps track of b. Therefore, it is natural to consider the following two generating functions:

$$\sum_{k\geq 0} J(b,k,\ell)y^k, \tag{17}$$

$$\sum_{\ell\geq 0} J(b,k,\ell)z^\ell. \tag{18}$$

Since $J(b,k,\ell) = J(b,\infty,\ell)$ for $k \geq b$, it is immediate that the generating function (17) is a rational function in y. Using the transfer matrix method [17, Section 4.7], one can show that the generating function in (18) is also a rational function in z. It would be very interesting to see if this rationality continues to hold for the multivariate generating function, keeping track of all three parameters b, k, and ℓ.

Problem 6.1. Determine whether the following is a rational function in the three variables x, y, z:

$$\sum_{b\geq 0}\sum_{k\geq 0}\sum_{\ell\geq 0} J(b,k,\ell)x^b y^k z^\ell.$$

Note that $J(b,k,\ell)$ can be used to compute the number of ways to juggle b balls with capacity k for beats $1,\dots,\ell$ without any restrictions on the initial and final states of the balls. In order to enumerate periodic multiplex juggling patterns, we need to consider the number $J_0(b,k,\ell)$ of ℓ-card sequences $(C_1,\dots,C_\ell) \in \mathcal{J}(b,k,\ell)$ such that the departure composition of C_1 is equal to the arrival composition of C_ℓ. It would be interesting to extend our results to $J_0(b,k,\ell)$. We end this paper with the following problems.

Problem 6.2. For fixed k and ℓ, find a formula for the generating function

$$\sum_{b\geq 0} J_0(b,k,\ell)x^b.$$

Problem 6.3. Determine whether the following is a rational function in the three variables x, y, z:

$$\sum_{b\geq 0}\sum_{k\geq 0}\sum_{\ell\geq 0} J_0(b,k,\ell)x^b y^k z^\ell.$$

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