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Counting s-Catalan Words According to Total Variation

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ABSTRACT: Given a finite sequence $w = w_1 \cdots w_n$, the (total) variation of w is defined as $\sum_{i=1}^{n-1} |w_{i+1} - w_i|$. In this paper, we study the joint distribution of variation with three other parameters on the set of s-Catalan words of a given length and deduce generating function formulas. Special attention is paid to the cases when $s = p = 1$ and $s = 1$, $p = 0$, where p marks the number of levels in the joint distribution. These cases correspond respectively to the set $\mathcal{C}_n$ consisting of the classical Catalan words of length n and to the subset $\mathcal{E}_n$ of $\mathcal{C}_n$ whose members contain no levels, which are referred to as Motzkin polyominoes. We also provide combinatorial proofs, which make use of statistics on corresponding sets of lattice paths, of the formulas for the sum of the variation values over all members of $\mathcal{C}_n$ or $\mathcal{E}_n$ (and for the cyclic variation as well wherein there is an additional summand $|w_n - w_1|$). As a consequence of our arguments, we are also able to supply a bijective proof of the formula for the total semi-perimeter over $\mathcal{E}_n$, which answers a previously raised question by Baril et al. wherein such a proof was sought.

Keywords: Catalan word; Kernel method; Motzkin polyomino; Total variation
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1. Introduction

Given a word $w = w_1 \cdots w_n$, let $\mathrm{var}(w) = \sum_{i=1}^{n-1} |w_{i+1} - w_i|$ denote the (total) *variation* of w. Let $\mathrm{var}^*(w) = \sum_{i=1}^{n} |w_i - w_{i+1}|$ denote what we will refer to here as the *cyclic variation* of w, where $w_{n+1} = w_1$. For example, if $w = 415772$, then $\mathrm{var}(w) = 3 + 4 + 2 + 5 = 14$ and $\mathrm{var}^*(w) = 14 + 2 = 16$. Mansour [11] studied the variation parameter on k-ary words for a fixed k and derived a formula for the generating function of the distribution. Critical to his proof was the introduction of an additional parameter that tracks $\mathrm{var}(w) + w_1$ for a word w, which he terms the *complete* variation. In [1], Archibald et al. studied the distribution of the variation and descent variation statistics on the set of compositions of a positive integer, where the latter records the value of $\sum_i (w_i - w_{i+1})$ in which the sum ranges only over i such that $w_i > w_{i+1}$. Finally, the cyclic variation statistic has been considered already by several authors (see, e.g., [5–7]) in the context of attempting to find sequences $\sigma = \sigma_1 \cdots \sigma_m$ on a finite alphabet satisfying various restrictions such that what is termed the risk function $\sum_{j=1}^{m} |\sigma_j - \sigma_{j-1}|$ is maximized, where $\sigma_0 = \sigma_m$.

Let s be a fixed positive integer. By an *s-Catalan* word $w = w_1 \cdots w_n$, we mean one over the alphabet of positive integers such that $1 \le w_{i+1} \le w_i + s$ for $1 \le i \le n-1$, with $w_1 = 1$. Let $\mathcal{C}_n^{(s)}$ denote the set of s-Catalan words of length n. It is well known that the cardinality of $\mathcal{C}_n^{(s)}$ is given by the n-th Fuss–Catalan number $\frac{1}{sn+1}\binom{(s+1)n}{n}$. Mansour and Ramírez [12] studied the distribution of the parameters on $\mathcal{C}_n^{(s)}$ tracking the area and semi-perimeter (of the associated bargraphs) and found formulas for the corresponding generating functions. See [10] for a recent variant of $\mathcal{C}_n^{(s)}$ also with the s-Catalan descriptor referring to its connection to the s-binomial coefficients.

Note that $\mathcal{C}_n^{(s)}$ when $s = 1$ corresponds to the set of ordinary Catalan words of length n, which will be denoted by $\mathcal{C}_n$. That $|\mathcal{C}_n| = C_n$ for all $n \ge 1$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th classical Catalan number, occurs for example in [18, Exercise 80]. Catalan words have arisen in the context of the exhaustive generation of Gray codes for sequences satisfying certain growth restrictions [15]. Recently, several combinatorial parameters, among them area, semi-perimeter, number of interior lattice points and capacity, have been studied on $\mathcal{C}_n$;

see, e.g., [4, 13, 14]. Let $\mathcal{E}_n$ denote the subset of $\mathcal{C}_n$ whose members contain no equal adjacent letters. The corresponding bargraphs for the members of $\mathcal{E}_n$ are referred to as *Motzkin polyominoes* in [2], where several combinatorial parameters were studied. By a slight abuse of terminology, we will also apply this same moniker to the words themselves in $\mathcal{E}_n$. It is well known that $|\mathcal{E}_n| = M_{n-1}$ for all $n \geq 1$, where M_n denotes the n-th Motzkin number.

In this paper, we consider the joint distribution of the variation statistic on $\mathcal{C}_n^{(s)}$ for a fixed s with some other combinatorial parameters. By a *level*, *descent* or *ascent* within a word $w = w_1 \cdots w_n$, we mean an index $i \in [n-1]$ such that $w_{i+1} = w_i$, $w_{i+1} < w_i$ or $w_{i+1} > w_i$, respectively. Let $\text{lev}(w)$, $\text{des}(w)$ and $\text{asc}(w)$ denote the number of levels, descents and ascents of w, respectively, and let $\mu(w) = w_n - 1$. Here, we consider the joint distribution $a_n = a_n(p,q,r,t)$ defined by

$$a_n = \sum_{w \in \mathcal{C}_n^{(s)}} p^{\text{lev}(w)} q^{\text{var}(w)} r^{\text{asc}(w)} t^{\mu(w)}, \qquad n \geq 1,$$

and study several special cases of a_n. Note that by taking all of the variables in a_n to be unity except for q, one obtains the var distribution on $\mathcal{C}_n^{(s)}$. Taking $t = q$ in a_n, and observing the fact $\text{var}^*(w) = \text{var}(w) + w_n - 1$ for all $w \in \mathcal{C}_n^{(s)}$, one obtains the joint distribution on $\mathcal{C}_n^{(s)}$ of var^* with levels and ascents. Letting $p = 0$ yields the joint distribution of three parameters on members of $\mathcal{C}_n^{(s)}$ that contain no levels. In particular, when $p = 0$ and $s = 1$, we obtain a distribution of var with asc and μ on the set $\mathcal{E}_n$.

The organization of this paper is as follows. In the next section, we determine the generating function $f(x) = \sum_{n \geq 1} a_n x^n$, where a_n is as defined above. For general s, the generating function f may be expressed as a solution to a certain functional equation whose kernel equation is of degree $s+1$. Hence, in the cases $s = 1$ or 2, one may determine f explicitly in its full generality using the *kernel method* (e.g., [9]). We treat in detail the $s = p = 1$ case, which yields results concerning var and var^* on $\mathcal{C}_n$ such as the corresponding univariate generating functions and formulas for the totals of these parameters on $\mathcal{C}_n$. Letting $p = 0$ instead of $p = 1$ in the preceding gives analogous results on the set $\mathcal{E}_n$.

In the third section, we find combinatorial proofs of formulas obtained in the second for the sum of the var or var^* values taken over all the members of $\mathcal{C}_n$ or $\mathcal{E}_n$ as well as for the sign-balance of var on $\mathcal{C}_n$. To do so, we express the statistics var or var^* over $\mathcal{C}_n$ or $\mathcal{E}_n$ in terms of equivalent statistics on Dyck or Motzkin paths, respectively. We then explain directly the formula found for the total var or var^* using its interpretation as the total of some lattice path statistic on Dyck or Motzkin paths. To do so, we define various bijections on certain sets of marked lattice paths wherein a step of a particular type is distinguished from all others. Finally, upon modifying the ideas used in these bijective arguments, we are able to provide a combinatorial proof of the formula for the total semi-perimeter on $\mathcal{E}_n$, answering a question raised in [2] concerning finding such a proof.

2. Variation parameter on s-Catalan words

Given $1 \leq i \leq n$, let $\mathcal{C}_{n,i}^{(s)}$ denote the subset of $\mathcal{C}_n^{(s)}$ whose members end in i. Define the multivariate generating function for a fixed $s \geq 1$ by

$$F^{(s)}(x;p,q,r,t) = \sum_{n\geq 1} \sum_{i=1}^{ns-s+1} \Bigg(\sum_{\pi \in \mathcal{C}_{n,i}^{(s)}} p^{\text{lev}(\pi)} q^{\text{var}(\pi)} r^{\text{asc}(\pi)} \Bigg) x^n t^{i-1}.$$

Theorem 2.1. *We have*

$$F^{(s)}(x;p,q,r,t) = \frac{x(t_0 - t)}{q - t + qx - (q-t)x\left(p + r\sum_{i=1}^{s}(qt)^i\right)}, \tag{1}$$

where $t_0 = t_0^{(s)}(x,p,r,q)$ is the unique formal power series in x with $\lim_{x\to 0} t_0 = q$ satisfying

$$(qt_0 - 1)(q - q(p-1)x + (px-1)t_0) = qrxt_0(q - t_0)((qt_0)^s - 1). \tag{2}$$

Proof. Given $n \geq 1$ and $1 \leq i \leq (n-1)s+1$, let $f_n^{(i)} = f_n^{(i)}(p,q,r)$ denote the joint distribution polynomial on $\mathcal{C}_{n,i}^{(s)}$ for the statistics tracking levels, total variation and ascents (marked by p, q and r, respectively). If $n \geq 2$, put $f_n^{(i)} = 0$ if $i \leq 0$ or $i \geq (n-1)s$, and note $f_1^{(i)} = \delta_{i,1}$. Upon considering the penultimate letter within a member of $\mathcal{C}_{n+1,i}^{(s)}$, we have

$$f_{n+1}^{(i)} = pf_n^{(i)} + r\sum_{j=1}^{s} q^j f_n^{(i-j)} + \sum_{j=i+1}^{ns-s+1} q^{j-i} f_n^{(j)}, \quad 1 \leq i \leq ns+1. \tag{3}$$

Let $F_n(p,q,r,t)=\sum_{i=1}^{ns-s+1} f_n^{(i)}(p,q,r)t^{i-1}$ for $n\geq 1$. Multiplying both sides of (3) by t^{i-1}, and summing over all $1\leq i\leq ns+1$, we obtain

$$\begin{aligned}F_{n+1}(p,q,r,t)&=p\sum_{i=1}^{ns-s+1} f_n^{(i)}t^{i-1}+r\sum_{i=2}^{ns+1}t^{i-1}\sum_{j=1}^{s}q^jf_n^{(i-j)}+\sum_{i=1}^{ns-s}t^{i-1}\sum_{j=i+1}^{ns-s+1}q^{j-i}f_n^{(j)}\\&=pF_n(p,q,r,t)+r\sum_{j=1}^{s}q^j\sum_{i=j+1}^{ns+1}f_n^{(i-j)}t^{i-1}+\sum_{j=2}^{ns-s+1}f_n^{(j)}\sum_{i=1}^{j-1}q^{j-i}t^{i-1}\\&=pF_n(p,q,r,t)+r\sum_{j=1}^{s}(qt)^j\sum_{i=1}^{ns-s+1}f_n^{(i)}t^{i-1}+\sum_{j=1}^{ns-s+1}f_n^{(j)}\left(\frac{q^j-qt^{j-1}}{q-t}\right)\\&=\frac{q}{q-t}F_n(p,q,r,q)+\left(p-\frac{q}{q-t}+\frac{qrt(1-(qt)^s)}{1-qt}\right)F_n(p,q,r,t),\quad n\geq 1,\end{aligned}\tag{4}$$

with $F_1(p,q,r,t)=1$.

By the definitions, we have $F^{(s)}(x;p,q,r,t)=\sum_{n\geq1}F_n(p,q,r,t)x^n$. Multiplying both sides of (4) by x^{n+1}, and summing over all $n\geq 1$, then gives

$$\left(q-t-(pq-pt-q)x-qrtx(q-t)\frac{1-(qt)^s}{1-qt}\right)F^{(s)}(x;p,q,r,t)=(q-t)x+qxF^{(s)}(x;p,q,r,q).\tag{5}$$

This type of functional equation may be solved using the kernel method. Let

$$K(t;x,p,q,r)=q-t-(pq-pt-q)x-qrtx(q-t)\frac{1-(qt)^s}{1-qt}.$$

Let $t_0=t_0^{(s)}(x,p,q,r)$ be such that $K(t_0;x,p,q,r)=0$. Upon multiplying by $1-qt$, we have that $t=t_0$ is a solution to

$$(qt-1)(q-(p-1)qx+(px-1)t)=qrxt(q-t)((qt)^s-1),\tag{6}$$

and we require that t_0 be the unique root of (6) that is a formal power series in x such that $\lim_{x\to0}t_0=q$. Substituting t_0 for t in (5) annihilates the left-hand side and yields

$$F^{(s)}(x;p,q,r,q)=\frac{t_0-q}{q}.$$

Substituting this back into (5), and solving for $F^{(s)}(x;p,q,r,t)$, yields (1). □

By the *bargraph* of $\pi=\pi_1\cdots\pi_n\in\mathcal{C}_n^{(s)}$, denoted by $b(\pi)$, we mean the polyomino in the (x,y)-plane containing n adjacent vertical columns of unit width flush with the x-axis wherein the i-th column for $1\leq i\leq n$ is of height π_i. Let semi(π) denote the semi-perimeter of $b(\pi)$, which is defined as half the perimeter of the bargraph (including the bottom edge along the x-axis). Comparing the definitions, one has for all $\pi\in\mathcal{C}_n^{(s)}$,

$$\mathrm{var}(\pi)=2(\mathrm{semi}(\pi)-n)-\mu(\pi)-1,\tag{7}$$

where $\mu(\pi)$ denotes the last letter of π (i.e., the height of the final column of $b(\pi)$). In [12], the authors defined the generating function $A^{(s)}(x;p,q;v)$ enumerating nonempty s-Catalan words π jointly according to the parameters tracking the area of $b(\pi)$, semi(π) and $\mu(\pi)-1$ (marked by p, q and v, respectively) and studied the distributions of the area and semi-perimeter statistics on $\mathcal{C}_n^{(s)}$.

By (7), the $p=r=1$ case of our $F^{(s)}(x;p,q,r,t)$ is related to the $p=1$ case of $A^{(s)}(x;p,q;v)$ via the formula

$$F^{(s)}(x;1,q,1,t)=\frac{1}{q^2}A^{(s)}\left(\frac{x}{q^2};1,q^2;\frac{t}{q}\right),$$

with this particular case of $A^{(s)}(x;p,q;v)$ having not been studied in [12].

2.1 The case $s=p=1$

Setting $s=1$ in (6) implies $t_0=t_0^{(1)}(x,p,q,r)$ satisfies

$$qrxt_0^2-(1-(p-q^2r)x)t_0+q-(p-1)qx=0,$$

and hence

$$t_0=\frac{1-(p-q^2r)x-\sqrt{(1-(p-q^2r)x)^2-4q^2rx(1-(p-1)x)}}{2qrx},\tag{8}$$

where we have chosen the negative root for t_0 which is seen to give a power series solution. Using (8), one can show that t_0 may be expressed in terms of the Catalan number generating function $C(x)=\sum_{n\geq 0} C_n x^n$ as

$$t_0=\frac{q^2r-p}{2qr}+\frac{2(p+q^2r)+(4q^2r-(p+q^2r)^2)x}{4qr}C\left(\frac{2(p+q^2r)x+(4q^2r-(p+q^2r)^2)x^2}{4}\right).$$

Letting $t=t_0$ in (5) when $s=1$ gives $F^{(1)}(x;p,q,r,q)=\frac{t_0-q}{q}$.

Substituting this expression for $F^{(1)}(x;p,q,r,q)$ back into (5) then yields the following result.

Theorem 2.2. *The generating function for the joint distribution of the parameters on $\mathcal{C}_n$ for $n\geq 1$ tracking levels, total variation, ascents and the last letter minus one (marked by p, q, r and t, respectively) is given by*

$$F^{(1)}(x;p,q,r,t)=\frac{1-(p+2qrt-q^2r)x-\sqrt{(1-(p-q^2r)x)^2-4q^2rx(1-(p-1)x)}}{2qr(q-t+(qrt^2+(p-q^2r)t-(p-1)q)x)}.$$

Let $A(x;q)$ and $A^*(x;q)$ denote respectively the generating functions for the distributions of the var and var* parameters on $\mathcal{C}_n$ for $n\geq 1$. By the definitions, we have

$$A(x;q)=F^{(1)}(x;1,q,1,1) \text{ and } A^*(x;q)=F^{(1)}(x;1,q,1,q),$$

which yields the following formulas.

Corollary 2.1. *We have*

$$A(x;q)=\frac{1-(1+2q-q^2)x-\sqrt{(1-(1-q^2)x)^2-4q^2x}}{2q(q-1+(1+q-q^2)x)} \tag{9}$$

and

$$A^*(x;q)=\frac{1-(1+q^2)x-\sqrt{(1-(1-q^2)x)^2-4q^2x}}{2q^2x}. \tag{10}$$

Note that $A(x;1)=A^*(x;1)=C(x)-1$, as expected. Using (9) and (10), one can obtain expressions for the totals of var and var* on $\mathcal{C}_n$ as follows. First note

$$\frac{\partial}{\partial q}A(x;q)\Big|_{q=1}=\frac{2x-1}{2x^2}+\frac{2x^2-4x+1}{2x^2\sqrt{1-4x}},$$
$$\frac{\partial}{\partial q}A^*(x;q)\Big|_{q=1}=\frac{x-1}{x}-\frac{3x-1}{x\sqrt{1-4x}}.$$

Recall now $\sum_{n\geq 0}\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}}$; see, e.g., [19, Eqn. (2.5.11)]. Extracting the coefficient of x^n in the last two formulas, and simplifying the resulting binomial expressions, then yields the following result.

Corollary 2.2. *The sums of the var and var* values of all the members of $\mathcal{C}_n$ for $n\geq 2$ are given respectively by $\binom{2n}{n-2}$ and $(n-1)C_n$.*

Note that the sequences $\binom{2n}{n-2}$ and $(n-1)C_n$ coincide with entries A002694 and A276666 in [17]. In the third section, we provide combinatorial proofs of these formulas.

Letting $q=-1$ in (9), and then finding

$$A(x)=\frac{1}{2}(A(x;1)+A(x;-1)) \text{ and } B(x)=\frac{1}{2}(A(x;1)-A(x;-1)),$$

yields the following result.

Corollary 2.3. *The generating function for the number of members of $\mathcal{C}_n$ for $n\geq 1$ having an even or an odd variation value are given respectively by*

$$A(x)=\frac{1-x-(1+x)\sqrt{1-4x}}{2x(2+x)}$$

and

$$B(x)=\frac{1-2x-2x^2-\sqrt{1-4x}}{2x(2+x)}.$$

Let $A(x)=\sum_{n\geq 1}a_nx^n$ and $B(x)=\sum_{n\geq 1}b_nx^n$. Then we have $a_n=$ A000958(n) and $b_n=f_{n+1}$, where f_j denotes the j-th Fine number (e.g., [8] or A000957).

Given $m\geq 0$, let

$$A_m(x)=\sum_{n\geq 1}\#\{\pi\in\mathcal{C}_n:\mathrm{var}(\pi)=m\}x^n,$$
$$A_m^*(x)=\sum_{n\geq 1}\#\{\pi\in\mathcal{C}_n:\mathrm{var}^*(\pi)=m\}x^n.$$

That is, $A_m(x)=[q^m]A(x;q)$ and $A_m^*(x)=[q^m]A^*(x;q)$. Note $A_m^*(x)=0$ if m is odd. We have that $A_m(x)$ and $A_m^*(x)$ may be expressed in terms of the Narayana polynomials as follows.

Proposition 2.1. *Let $N_n(x)$ denote the n-th Narayana polynomial and set*

$$u_n(x)=\sum_{j=1}^{\lfloor\frac{n+2}{2}\rfloor}(-1)^j\binom{n-j+1}{j-1}\left(\frac{x(1-x)}{(1+x)^2}\right)^j.$$

If $m\geq 1$, then

$$A_m(x)=-\left(\frac{1+x}{1-x}\right)^{m+2}u_m(x)+\sum_{k=0}^{\lfloor(m-1)/2\rfloor}\frac{x^k(1+x)^{m-2k+1}}{(1-x)^{m+2}}N_k(x)u_{m-2k-1}(x),$$
$$A_{2m}^*(x)=\frac{x^mN_m(x)}{(1-x)^{2m+1}},$$

with $A_0(x)=A_0^*(x)=\frac{x}{1-x}$.

Proof. Recall (e.g., [3, Eqn. (7)]) that the generating function for the Narayana polynomials is given by

$$\sum_{n\geq 0}N_n(x)q^n=\frac{1+q-qx-\sqrt{1-2(1+x)q+(1-x)^2q^2}}{2q}.$$

Upon substituting $q\mapsto\frac{q^2x}{(1-x)^2}$, and then multiplying by $\frac{q}{1-x}$, we have

$$t_0^{(1)}(x,1,q,1)=\sum_{n\geq 0}\frac{x^nN_n(x)}{(1-x)^{2n+1}}q^{2n+1}.$$

Further, it is well known (e.g., A208342) that

$$\frac{x}{q-1+x(1+q-q^2)}=\sum_{n\geq 0}\left(\frac{1+x}{1-x}\right)^{n+2}u_n(x)q^n.$$

By the fact $A(x;q)=\frac{x(t_0^{(1)}(x,1,q,1)-1)}{q-1+x(1+q-q^2)}$, we have

$$A(x;q)=\left(\sum_{n\geq 0}\left(\frac{1+x}{1-x}\right)^{n+2}u_n(x)q^n\right)\left(-1+\sum_{n\geq 0}\frac{x^nN_n(x)}{(1-x)^{2n+1}}q^{2n+1}\right).$$

Computing the coefficient of q^m in this convolution yields the desired formula for $A_m(x)$. The second formula follows from the fact $A^*(x;q)=\frac{t_0^{(1)}(x,1,q,1)-q}{q}$. $\square$

Note, for example, from the preceding formula for $A_m(x)$, we have

$$A(x;q)=\frac{x}{1-x}+\frac{x^2}{(1-x)^2}q+\frac{2x^3}{(1-x)^3}q^2+\frac{3x^4}{(1-x)^4}q^3+\frac{(5x+1)x^4}{(1-x)^5}q^4+\frac{(8x+2)x^5}{(1-x)^6}q^5+\cdots.$$

For the last result of this subsection, we describe when the maximum variation is achieved on $\mathcal{C}_n$.

Proposition 2.2. *If $n\geq 3$, then the maximum variation on $\mathcal{C}_n$ of $2n-4$ is attained by the word $12\cdots n1$, with this maximal word being unique for $n\geq 4$.*

Proof. Clearly, $\mathrm{var}(12\cdots(n-1)1)=2n-4$. Now let $x=x_1\cdots x_n\in\mathcal{C}_n$ and suppose there exists $i_0\in[n-2]$ such that $x_{i_0}\geq x_{i_0+1}$. Define a Catalan word $y=y_1\cdots y_n$ by

$$y_i=\begin{cases}x_i, & \text{if } 1\leq i\leq i_0;\\ x_i+x_{i_0}-x_{i_0+1}+1, & \text{if } i_0+1\leq i\leq n-1;\\ 1, & \text{if } i=n.\end{cases}$$

Then

$$\begin{aligned}\mathrm{var}(y)&=\sum_{i=1}^{i_0-1}|y_{i+1}-y_i|+|y_{i_0+1}-y_{i_0}|+\sum_{i=i_0+1}^{n-1}|y_{i+1}-y_i|\\ &=\sum_{i=1}^{i_0-1}|x_{i+1}-x_i|+|x_{i_0+1}+x_{i_0}-x_{i_0+1}+1-x_{i_0}|\\ &\quad+\sum_{i=i_0+1}^{n-2}|x_{i+1}-x_i|+|1-(x_{n-1}+x_{i_0}-x_{i_0+1}+1)|\\ &=\mathrm{var}(x)+1+|x_{n-1}+x_{i_0}-x_{i_0+1}|-|x_{i_0+1}-x_{i_0}|-|x_n-x_{n-1}|\\ &=\mathrm{var}(x)+1+x_{n-1}+x_{i_0}-x_{i_0+1}-(x_{i_0}-x_{i_0+1})-|x_n-x_{n-1}|\\ &=\mathrm{var}(x)+1+\begin{cases}x_{n-1}-1, & \text{if } x_n=x_{n-1}+1;\\ x_n, & \text{if } x_n\leq x_{n-1},\end{cases}\\ &\geq\mathrm{var}(x)+1.\end{aligned}$$

This shows that in order for a Catalan word $x_1\cdots x_n$ to have maximum variation, it is necessary that $x_1\cdots x_{n-1}=12\cdots(n-1)$. In this case, the variation is maximized if x_n is as far as possible from $n-1$. If $n=3$, then we may take $x_3=1$ or $x_3=3$, whereas if $n\geq4$, the maximum is attained only by $x_n=1$. □

Thus, the degree of the variation distribution polynomial on $\mathcal{C}_n$ is $2n-4$ for $n\geq3$, with leading coefficient equal to one if $n\geq4$. Similar reasoning shows that the maximum cyclic variation on $\mathcal{C}_n$ of $2n-2$ is achieved uniquely by $12\cdots n$ for all $n\geq1$.

2.2 The case $s=1$ and $p=0$

Let $\mathcal{E}_n$ denote the subset of $\mathcal{C}_n$ whose members contain no levels. Recall $|\mathcal{E}_n|=M_{n-1}$, where M_n denotes the n-th Motzkin number. Various parameters, such as area, semi-perimeter and number of interior points, were considered on the set $\mathcal{E}_n$ by Baril et al. in [2].

As particular cases of the results above, one obtains formulas for the distribution of var and var* on $\mathcal{E}_n$. Setting $p=0$ in Theorem 2.2 gives

$$F^{(1)}(x;0,q,r,t)=\frac{1-(2qt-q^2)rx-\sqrt{1-2q^2rx+q^2r(q^2r-4)x^2}}{2qr(q-t+(rt^2-qrt+1)qx)}.$$

Note that

$$F^{(1)}(x;0,1,1,1)=\frac{1-x-\sqrt{1-2x-3x^2}}{2x}=\sum_{n\geq1}M_{n-1}x^n,$$

as expected.

Let $D(x;q)=F^{(1)}(x;0,q,1,1)$ and $D^*(x;q)=F^{(1)}(x;0,q,1,q)$, which are seen to enumerate the nonempty members of $\mathcal{E}_n$ according to var and var*, respectively.

Corollary 2.4. *We have*

$$D(x;q)=\frac{1-(2q-q^2)x-\sqrt{1-2q^2x+q^2(q^2-4)x^2}}{2q(q-1-(q-2)qx)} \tag{11}$$

and

$$D^*(x;q)=\frac{1-q^2x-\sqrt{1-2q^2x+q^2(q^2-4)x^2}}{2q^2x}. \tag{12}$$

Taking the q-partial derivatives in (11) and (12), we have

$$\frac{\partial}{\partial q}D(x;q)\Big|_{q=1} = (1+x)\left(\frac{1}{\sqrt{1-2x-3x^2}} - \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}\right),$$
$$\frac{\partial}{\partial q}D^*(x;q)\Big|_{q=1} = \frac{1-x-2x^2}{x\sqrt{1-2x-3x^2}} - \frac{1}{x}.$$

Let G_n denote the n-th grand Motzkin number, see A002426, and recall $\sum_{n\geq 0} G_n x^n = \frac{1}{\sqrt{1-2x-3x^2}}$. Extracting the coefficient of x^n in the last two formulas, we obtain the following result.

Corollary 2.5. *The sums of the var and var* values of all the members of $\mathcal{E}_n$ for $n \geq 1$ are given respectively by $G_n + G_{n-1} - M_n - M_{n-1}$ and $G_{n+1} - G_n - 2G_{n-1}$.*

In the third section, we provide a combinatorial proof of Corollary 2.5 making use of the interpretation of M_n and G_n in terms of lattice paths.

2.3 The case $s = 2$

In this subsection, we solve explicitly the $s = 2$ case of Theorem 2.1.

Theorem 2.3. *Let*

$$\alpha = \frac{1-q^2}{q}, \qquad \beta = \frac{px-1}{q^2rx} - 1, \qquad \gamma = \frac{1-(p-1)x}{qrx},$$

and

$$\mathfrak{p} = \beta - \frac{\alpha^2}{3}, \qquad \mathfrak{q} = \frac{2\alpha^3}{27} - \frac{\alpha\beta}{3} + \gamma.$$

Then

$$F^{(2)}(x;p,q,r,t) = \frac{x(t_0-t)}{q-t+qx-(q-t)x(p+r(qt+q^2t^2))}, \tag{13}$$

where

$$t_0 = t_0^{(2)}(x,p,q,r) = -\frac{\alpha}{3} + 2\sqrt{-\frac{\mathfrak{p}}{3}}\cos\left(\frac{1}{3}\arccos\left(\frac{3\mathfrak{q}}{2\mathfrak{p}}\sqrt{-\frac{3}{\mathfrak{p}}}\right) + \frac{4\pi}{3}\right). \tag{14}$$

Proof. Formula (13) follows from (1) once t_0 is known, so we need only show (14). Setting $s = 2$ in (2), and canceling the common factor $qt_0 - 1$, we obtain the cubic equation

$$q - q(p-1)x + (px-1)t_0 = qrxt_0(q-t_0)(qt_0+1),$$

which may be rewritten as

$$t_0^3 + \alpha t_0^2 + \beta t_0 + \gamma = 0,$$

where α, β and γ are as stated. By the standard substitution $t_0 = z - \alpha/3$, one obtains the depressed equation $z^3 + \mathfrak{p}z + \mathfrak{q} = 0$, where $\mathfrak{p}$ and $\mathfrak{q}$ are as stated, which may be solved using, e.g., [16, p. 104]. We select the unique solution for t_0 such that $t_0\mid_{x=0} = \lim_{x\to 0^+} t_0^{(2)}(x,p,q,r) = q$, which is required in order for the $q = t$ case of $F^{(2)}(x;p,q,r,t)$ to be zero when $x = 0$. This yields the form of t_0 stated in (14) and completes the proof. ☐

Let $K(x;q) = F^{(2)}(x;1,q,1,1)$ and $K^*(x;q) = F^{(2)}(x;1,q,1,q)$.

Corollary 2.6. *We have*

$$K(x;q) = \frac{x(t_0-1)}{q-1+qx-(q-1)x(1+q+q^2)} \tag{15}$$

and

$$K^*(x;q) = \frac{t_0-q}{q}, \tag{16}$$

where in both cases $t_0 = t_0(x,q)$ is as in (14) with $\mathfrak{p}$ and $\mathfrak{q}$ given by

$$\mathfrak{p} = \frac{1}{q^2}\left(\frac{x(1-q^2)-1}{x} - \frac{(1-q^2)^2}{3}\right),$$
$$\mathfrak{q} = \frac{2(1-q^2)^3}{27q^3} - \frac{(1-q^2)\big(x(1-q^2)-1\big)}{3q^3x} + \frac{1}{qx}.$$

Note that when $q=1$, the generating functions $K(x;q)$ and $K^*(x;q)$ both reduce to $-\frac{2}{\sqrt{3x}}\sin(\theta)-1$, where

$$\theta=\frac{1}{3}\arcsin\left(-\frac{3\sqrt{3x}}{2}\right),$$

which is in accord with the fact $-\frac{2}{\sqrt{3x}}\sin(\theta)=\sum_{n\geq 0}\frac{1}{2n+1}\binom{3n}{n}x^n$. Differentiation of (15) and (16) with respect to q, and setting $q=1$, yields after some algebra the formulas

$$\frac{\partial}{\partial q}K(x;q)\Big|_{q=1}=\frac{2(4-9x)}{3\sqrt{4-27x}}\cos\theta-\frac{2(x^2+x-1)}{x\sqrt{3x}}\sin\theta+\frac{3-4x}{3x},$$
$$\frac{\partial}{\partial q}K^*(x;q)\Big|_{q=1}=\frac{2}{3}\left(\frac{4-9x}{\sqrt{4-27x}}\cos\theta+\frac{\sqrt{3}(2-x)}{\sqrt{x}}\sin\theta+1\right).$$

Extracting the coefficient of x^n for $n\geq 1$ then implies the following result.

Corollary 2.7. *The sums of the var and var* values of all the members of* $\mathcal{C}_n^{(2)}$ *for* $n\geq 1$ *are given respectively by*

$$\frac{8n+7}{3(2n+1)}\binom{3n}{n}-\frac{1}{2n+3}\binom{3n+3}{n+1}-\frac{6n-4}{2n-1}\binom{3n-3}{n-1}$$

and

$$2\binom{3n-2}{n}-\frac{2}{2n+1}\binom{3n}{n}.$$

We note that the total var* on $\mathcal{C}_n^{(2)}$ coincides with twice the sequence $A308677(n-1)$ for $n\geq 2$, with A308677 having apparently arisen before in connection with the pattern avoidance problem on Stirling permutations.

3. Combinatorial proofs

In this section, we provide combinatorial proofs of some of the results from the prior and, to this end, make use of the lattice path representation of a member of $\mathcal{C}_n$. Let $\mathcal{D}_n$ denote the set of lattice paths from $(0,0)$ to $(2n,0)$ using $u=(1,1)$ and $d=(1,-1)$ steps that never go below the x-axis. Members of $\mathcal{D}_n$ are referred to as *Dyck paths* (of semi-length n). We now recall a simple bijection, e.g., [4], between $\mathcal{C}_n$ and $\mathcal{D}_n$. Given $\pi=\pi_1\cdots\pi_n\in\mathcal{C}_n$, let $\imath(\pi)$ denote the member of $\mathcal{D}_n$ whose i-th up step from the left ends at height π_i for $1\leq i\leq n$. This uniquely determines the positions of the down steps and hence gives rise to a member of $\mathcal{D}_n$. For example, if $n=6$ and $\pi=123231\in\mathcal{C}_6$, then $\imath(\pi)=u^3d^2u^2d^3ud\in\mathcal{D}_6$.

By identifying members of $\mathcal{C}_n$ (and later members of $\mathcal{E}_n$) in terms of lattice paths, it is possible to explain the formulas for the totals of var and var* on $\mathcal{C}_n$ and $\mathcal{E}_n$, the sign balance of var on $\mathcal{C}_n$ and the maximum achievable var value on $\mathcal{C}_n$. To do so, it is also convenient to find a statistic on $\mathcal{D}_n$ that is equivalent to the var parameter on $\mathcal{C}_n$. By a *peak* within a member of $\mathcal{D}_n$, we mean an occurrence of a u being directly followed by d. One can show, see [4], that $\text{semi}(\pi)+\text{peak}(\imath(\pi))=2n+1$ for all $\pi\in\mathcal{C}_n$, where $\text{semi}(\pi)$ and $\text{peak}(\imath(\pi))$ denote the semi-perimeter of π and the number of peaks in $\imath(\pi)$. Thus, we have

$$\begin{aligned}\text{var}(\pi)&=2(\text{semi}(\pi)-n)-\mu(\pi)-1=2(n+1-\text{peak}(\imath(\pi)))-\mu(\pi)-1\\&=2n-2\text{peak}(\imath(\pi))-\mu(\pi)+1,\end{aligned}$$

where $\mu(\pi)$ denotes the final letter of π. Since $\text{peak}(\lambda)=n-UU(\lambda)$ for $\lambda\in\mathcal{D}_n$, where $UU(\lambda)$ is the number of occurrences of two consecutive u steps in λ, we get

$$\text{var}(\pi)=2UU(\imath(\pi))-\mu(\pi)+1,\qquad \pi\in\mathcal{C}_n. \tag{17}$$

Formula (17) may also be argued directly as follows. Consider pairing each u step within $\imath(\pi)$ that is not part of a peak with the nearest d to its right of the same height. One can show (we leave as an exercise) that each paired d contributes 1 towards exactly one of the differences $|\pi_i-\pi_{i+1}|$ in var$^*(\pi)$ for which $\pi_i>\pi_{i+1}$, where $\pi=\pi_1\cdots\pi_n$. Further, each uu in $\imath(\pi)$ corresponds to an ascent in π, and hence to one of the differences $|\pi_i-\pi_{i+1}|$ in var$^*(\pi)$ for which $\pi_{i+1}=\pi_i+1$. Since $UU(\imath(\pi))$ equals the number of paired u's as described above, it follows that $2UU(\imath(\pi))$ gives var$^*(\pi)$, from which we subtract $|\pi_n-\pi_1|=\mu(\pi)-1$ to obtain var(π).

Combining the arguments for (7) and (17), we have established here

$$2(\text{semi}(\pi)-n)-\mu(\pi)-1=2n-2\text{peak}(\imath(\pi))-\mu(\pi)+1,$$

as both sides have been shown independently to equal $\mathrm{var}(\pi)$ for all π. Thus, we have obtained a new proof of the aforementioned fact $\mathrm{semi}(\pi)+\mathrm{peak}(\imath(\pi))=2n+1$, which was demonstrated in [4, Theorem 3.2] by a recursive bijective argument.

Using (17), one can also obtain an alternative explanation of Proposition 2.2 as follows. Since $UU(\imath(\pi))$ equals n minus the number of runs of u in $\imath(\pi)$, the $2UU(\imath(\pi))$ term in (17) is clearly at most $2n-2$, with equality if and only if $\imath(\pi)=u^nd^n$, i.e., $\pi=12\cdots n$, in which case we get $\mathrm{var}(\pi)=n-1$. Thus, among all other $\pi\in\mathcal{C}_n$, we have that $UU(\imath(\pi))$ is at most $n-2$, and hence the difference of $2UU(\imath(\pi))$ with $\mu(\pi)-1$ is at most $2n-4$, with equality possible only if π ends in 1. Since π ending in 1 accounts for one of the runs of d in $\imath(\pi)$, in order for $\imath(\pi)$ to contain only two runs altogether, we must have $\imath(\pi)=u^{n-1}d^{n-1}ud$, i.e., $\pi=12\cdots(n-1)1$. Thus, the maximum variation on $\mathcal{C}_n$ of $2n-4$ when $n\geq 4$ is achieved only by $\pi=12\cdots(n-1)1$, as desired.

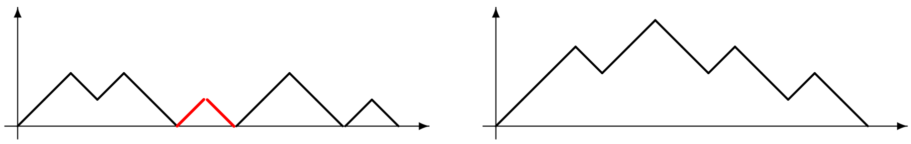
We now provide a bijective explanation of Corollary 2.3 concerning the sign-balance of the var statistic on $\mathcal{C}_n$.

Combinatorial proof of Corollary 2.3:

By (17), the parity of $\mathrm{var}(\pi)$ for $\pi\in\mathcal{C}_n$ is opposite that of the parity of the last letter of π. Let $\mathcal{C}_n^{(e)}$ and $\mathcal{C}_n^{(o)}$ denote the subsets of $\mathcal{C}_n$ containing those members where the last letter is even or odd, respectively. We have via $\imath$ that $\mathcal{C}_n^{(e)}$ and $\mathcal{C}_n^{(o)}$ correspond to the subsets of $\mathcal{D}_n$ ending in an even or odd number of down steps, respectively. By symmetry, this is equivalent to beginning with an even or odd number of up steps, the subsets of $\mathcal{D}_n$ of which we represent by $\mathcal{D}_n^{(e)}$ and $\mathcal{D}_n^{(o)}$, respectively. To establish Corollary 2.3, it then suffices to show $|\mathcal{D}_n^{(e)}|=f_{n+1}$ and $|\mathcal{D}_n^{(o)}|=\ell_n$ for $n\geq 1$, where f_n denotes the n-th Fine number and $\ell_n=A000958(n)$.

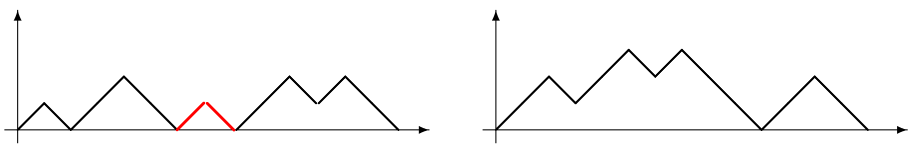
We now recall combinatorial interpretations for the sequences f_{n+1} and ℓ_n. By a *low ud* or *udu*, we mean an occurrence of the respective string of steps in which the first step begins on the x-axis. Then it is well known that f_{n+1} and ℓ_n enumerate the subsets of $\mathcal{D}_n$ in which no low ud or udu occurs, respectively. Let $\mathcal{F}_n$ denote the subset of $\mathcal{D}_n$ in which no low ud occurs. To establish the second statement in Corollary 2.3, it is enough to define a bijection j between $\mathcal{D}_n^{(e)}$ and $\mathcal{F}_n$.

To do so, suppose $\lambda\in\mathcal{D}_n^{(e)}$ and let $j(\lambda)=\lambda$ if λ does not contain a low ud. Otherwise, we may write $\lambda=\alpha\mathbf{ud}\beta$, where the ud in bold is the leftmost low ud of λ. We decompose the section α further as $\alpha=u^md\alpha'$ for some even $m\geq 2$, where α' does not contain a low ud. In this case, we let $j(\lambda)=u^{m+1}d\beta d\alpha'$, which is seen to belong to $\mathcal{F}_n\cap\mathcal{D}_n^{(o)}$. Note that a member of $\mathcal{F}_n$ must start with at least two up steps in order to avoid a low ud, which implies the mapping j is onto $\mathcal{F}_n$ and hence a bijection, as desired. See Figure 1 below for an example of the mapping j where $n=7$ and $m=2$.



$\lambda=u^2dud^2\mathbf{ud}u^2d^2ud\in\mathcal{D}_7^{(e)}$ $\qquad$ $j(\lambda)=u^3du^2d^2ud^2ud^2\in\mathcal{F}_7$

Figure 1: Lattice paths λ and $j(\lambda)$, where the leftmost low ud of λ is in red.



$\pi=udu^2d^2\mathbf{ud}u^2dud^2\in\mathcal{D}_7^{(o)}$ $\qquad$ $h(\pi)=u^2du^2dud^3u^2d^2\in\mathcal{D}_7^{(e)}$

Figure 2: Lattice paths π and $h(\pi)$, where π contains a non-initial low ud.

Let $\mathcal{L}_n$ denote the subset of $\mathcal{D}_n$ whose members contain no low udu. To complete the proof of Corollary 2.3, we define a bijection between $\mathcal{D}_n^{(o)}$ and $\mathcal{L}_n$, where we may assume $n\geq 2$. Note that a member of $\mathcal{L}_n$ may end in a ud, with this being the only possible position for a low ud, due to avoidance of low udu. Upon removing a possible terminal low ud from a member of $\mathcal{L}_n$, we have $\mathcal{L}_n=\mathcal{F}_n\cup\mathcal{F}_{n-1}$. By the prior bijection j, we have that the sets $\mathcal{F}_n\cup\mathcal{F}_{n-1}$ and $\mathcal{D}_n^{(e)}\cup\mathcal{D}_{n-1}^{(e)}$ are equinumerous. Thus, to complete the proof, it suffices to identify a bijection h between $\mathcal{D}_n^{(o)}$ and $\mathcal{D}_n^{(e)}\cup\mathcal{D}_{n-1}^{(e)}$.

Let $\pi\in\mathcal{D}_n^{(o)}$ and first suppose π contains at least one low ud outside of a possible initial ud. Then we may write $\pi=u^{2k-1}d\alpha\mathbf{ud}\beta$, where $k\geq 1$, the ud in bold is low, the section α does not contain a low ud and β is possibly empty. In this case, we let $h(\pi)=u^{2k}d\beta d\alpha$, which belongs to $\mathcal{D}_n^{(e)}$ and does not contain a low ud; see Figure 2 for an example. Now assume π starts with ud and contains no other low ud. Then we have $\pi=ud\alpha$, where $\alpha\in\mathcal{F}_{n-1}$, in which case, we delete the initial ud and apply the inverse of the previous mapping j (in the case $n-1$) to the section α to obtain $h(\pi)=j^{-1}(\alpha)\in\mathcal{D}_{n-1}^{(e)}$. Finally, suppose π does not contain a low ud. In this case, we let $h(\pi)=j^{-1}(\pi)$, which is seen to belong to $\mathcal{D}_n^{(e)}$ and contain a low ud. Combining the three preceding cases, each of which is seen to be reversible, one has that h provides the desired bijection between $\mathcal{D}_n^{(o)}$ and $\mathcal{D}_n^{(e)}\cup\mathcal{D}_{n-1}^{(e)}$. □

We now explain bijectively the formulas found above for the total var and var* on $\mathcal{C}_n$.

Combinatorial proof of Corollary 2.2:

For a statistic s defined on either $\mathcal{C}_n$ or $\mathcal{D}_n$, we will denote the sum of the values of s over all the members of either set by $\text{tot}_n(s)$. We first show $\text{tot}_n(\text{var})=\binom{2n}{n-2}$ for $n\geq 2$. Let DD stand for the statistic on $\mathcal{C}_n$ tracking the number of occurrences of two consecutive downsteps and note UU and DD are identically distributed on $\mathcal{D}_n$. By (17), we have

$$\text{tot}_n(\text{var})=\text{tot}_n(DD)+\text{tot}_n^*(DD), \tag{18}$$

where $\text{tot}_n^*(DD)$ counts all dd's in $\mathcal{D}_n$ excluding those occurring within terminal runs of d. Let $\mathcal{D}_n'$ denote the set of marked members of $\mathcal{D}_n$ wherein some d is marked, not the first within a run of d, and let $\mathcal{D}_n^*\subseteq\mathcal{D}_n'$ consist of those paths wherein the marked d does not lie within the terminal run of d. Then we have $|\mathcal{D}_n'|=\text{tot}_n(DD)$ and $|\mathcal{D}_n^*|=\text{tot}_n^*(DD)$, and, by (18), it suffices to show

$$|\mathcal{D}_n'|+|\mathcal{D}_n^*|=\binom{2n}{n-2}, \qquad n\geq 2. \tag{19}$$

To show (19), first note that $\pi\in\mathcal{D}_n'$ is expressible as $\pi=\alpha d\mathbf{d}\beta$, where the marked d is in bold and β is possibly empty. Let $\kappa(\pi)=u\widehat{\alpha}d\widehat{\beta}$, where $\widehat{\sigma}$ is obtained from a sequence σ of steps within a lattice path by reversing the order of the steps and then replacing each u with d and d with u. Let $\mathcal{P}_n$ denote the set of lattice paths from $(0,0)$ to $(2n,-2)$ using u and d and starting with u. Note that $\kappa(\pi)\in\mathcal{P}_n$ for all $\pi\in\mathcal{D}_n'$, with κ being reversed by considering the leftmost occurrence of the minimum height $m\leq -2$ achieved by a path in $\mathcal{P}_n$. Note that m within $\kappa(\pi)$ is first achieved just after the d step separating the sections $\widehat{\alpha}$ and $\widehat{\beta}$. Thus, we have $|\mathcal{D}_n'|=|\mathcal{P}_n|=\binom{2n-1}{n-2}$, so to complete the proof of (19), we must show $|\mathcal{D}_n^*|=\binom{2n-1}{n-3}$, where we may assume $n\geq 3$.

To accomplish this, it is easier to enumerate the members of the complementary set $S=\mathcal{D}_n'\backslash\mathcal{D}_n^*$. Note that the image of S under κ consists of those paths in $\mathcal{P}_n$ for which there are no d steps after the first occurrence of the minimum height m. We then seek to show $|\kappa(S)|=\binom{2n-1}{n-2}-\binom{2n-1}{n-3}$. Let V denote the subset of $\mathcal{P}_n$ consisting of those paths that never dip below the line $y=-2$. If $\rho\in\mathcal{P}_n$ does indeed go below the line $y=-2$ at some point, then reflecting in the line $y=-3$ the subpath of ρ consisting of all steps to the right of the leftmost step of ρ achieving a height of -3 yields an arbitrary lattice path from $(0,0)$ to $(2n,-4)$ starting with u, of which there are $\binom{2n-1}{n-3}$. Thus, by subtraction, we have $|V|=\binom{2n-1}{n-2}-\binom{2n-1}{n-3}$, and so we seek to define a bijection between $\kappa(S)$ and V.

To do so, let $\lambda\in\kappa(S)$, which we decompose as $\lambda=u^rd\gamma\mathbf{d}u^s$, where $r\geq 1$, $s\geq 0$ and the minimum height of λ is achieved (only) after the d in bold. We then transform λ to $\widetilde{\lambda}=u^{r+s+1}d\gamma$ wherein the $(r+1)$-st u within the initial run of u is marked. Note $\widetilde{\lambda}$ is seen to be a (marked) member of $\mathcal{D}_n$ starting with at least two u steps wherein some u, not the first, within the initial run of u is marked. We decompose the section $d\gamma$ of $\widetilde{\lambda}$ as $d\gamma^{(1)}d\gamma^{(2)}\cdots d\gamma^{(r+s+1)}$, where $\gamma^{(i)}$ for each $i\in[r+s+1]$ is a possibly empty Dyck path (when viewed as starting from the origin). We then let

$$\psi(\lambda)=(u\gamma^{(1)}d)\cdots(u\gamma^{(r)}d)(d\gamma^{(r+1)}d)(u\gamma^{(r+2)}d)\cdots(u\gamma^{(r+s+1)}d).$$

Note that $\psi(\lambda)$ is seen to belong to $\mathcal{P}_n$, and indeed V, for all $\lambda\in\kappa(S)$. Moreover, the mapping ψ may be reversed, as $\widetilde{\lambda}$, and hence λ, may be recovered by considering the subpath within a member of V lying between the leftmost d steps ending at heights -1 and -2. Thus, ψ provides the desired bijection between $\kappa(S)$ and V, which completes the proof of the formula for $\text{tot}_n(\text{var})$.

We now prove the second assertion in Corollary 2.2. By (17) and the fact $\text{var}^*=\text{var}+\mu-1$, we have $\text{tot}_n(\text{var}^*)=2\text{tot}_n(UU)=2|\mathcal{P}_n|$. Let $\mathcal{R}_n$ denote the set of lattice paths from $(0,0)$ to $(2n,-2)$ using u and d. Consider the subset W of $\mathcal{R}_n$ consisting of those paths π such that either (i) $\pi\in\mathcal{P}_n$ or (ii) π starts with d and returns at least once to the x-axis. Upon reflecting in the x-axis the subpath that consists of all steps to the left of and including the first return to the x-axis within π satisfying (ii), one has that members of W for which (i) holds are equinumerous with those satisfying (ii), and hence $|W|=2|\mathcal{P}_n|$.

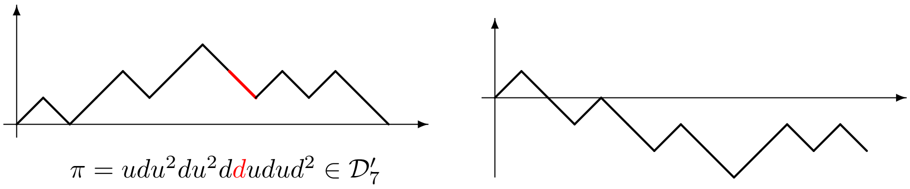
We find a second expression for $|W|$ by considering its complement within R_n as follows. Let $\pi\in\mathcal{D}_n$, with one of its down steps marked, and suppose $\pi=\alpha\mathbf{d}\beta$, where the marked d is in bold. Then the mapping $\pi\mapsto\widehat{\alpha}d\widehat{\beta}$

provides a bijection between members of $\mathcal{D}_n$ so marked and $\mathcal{R}_n$, whence $|\mathcal{R}_n| = nC_n$. Note that $R_n \backslash W$ consists of paths from $(0,0)$ to $(2n,-2)$ starting with d and never returning to the x-axis. Upon changing the initial d to a u within a member of $\mathcal{R}_n\backslash W$, moving this letter to the end and reflecting the resulting path in the line $y=-1/2$, one obtains a member of $\mathcal{D}_n$. This operation is reversible, and hence $|\mathcal{R}_n\backslash W| = C_n$. Thus, we have

$$\mathrm{tot}_n(\mathrm{var}^*) = |W| = |\mathcal{R}_n| - |\mathcal{R}_n\backslash W| = nC_n - C_n = (n-1)C_n,$$

which completes the proof of Corollary 2.2. □

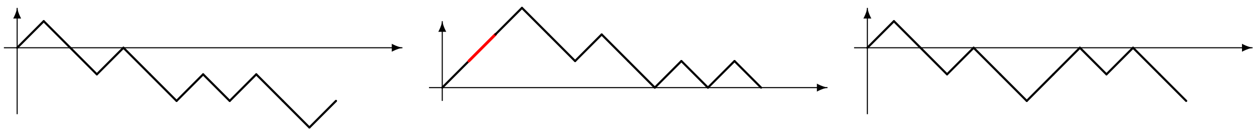
We illustrate in Figures 3 and 4 below the bijections κ and ψ used above in the proof of the first assertion of Corollary 2.2.



$\pi = udu^2du^2ddudud^2 \in \mathcal{D}'_7$

$\kappa(\pi) = ud^2ud^2ud^2u^2dud \in \mathcal{P}_7$

Figure 3: Lattice paths π and $\kappa(\pi)$, where the marked step of π is in red.



$\lambda = ud^2ud^2udud^2u \in \kappa(S)$ $\quad$ $\widetilde{\lambda} = uuud^2ud^2udud \in \mathcal{D}_6$ $\quad$ $\psi(\lambda) = ud^2ud^2u^2dud^2 \in V$

Figure 4: Lattice paths λ, $\widetilde{\lambda}$ and $\psi(\lambda)$, where the marked step of $\widetilde{\lambda}$ is in red.

We now turn to finding a lattice path statistic that is equivalent to the variation parameter on $\mathcal{E}_n$. Let $\mathcal{M}_n$ denote the set of lattice paths from $(0,0)$ to $(n,0)$ using $u=(1,1)$, $d=(1,-1)$ and $h=(1,0)$ steps that never dip below the x-axis. Recall $|\mathcal{M}_n| = M_n$, where M_n is the n-th Motzkin number. A bijection between $\mathcal{E}_n$ and $\mathcal{M}_{n-1}$, which we will denote here by σ, was found in [2] and is defined recursively by

$$\sigma(\pi) = \begin{cases} \varepsilon, & \text{if } \pi = 1; \\ h\sigma(\pi'), & \text{if } \pi = 1(\pi'+1); \\ u\sigma(\pi')d\sigma(\pi''), & \text{if } \pi = 1(\pi'+1)\pi'', \end{cases}$$

where ε is the empty lattice path, π' and π'' are nonempty Motzkin polyominoes and $\rho+1$ for a sequence ρ is obtained by adding 1 to each entry of ρ. For example, if $\pi = 12342312 \in \mathcal{E}_8$, then we have

$$\sigma(\pi) = u\sigma(12312)d\sigma(12) = uu\sigma(12)d\sigma(12)dh\sigma(1) = uuhdhdh \in \mathcal{M}_7.$$

One can show, see [2, Theorem 4.2], the relation

$$\mathrm{semi}(\pi) + U(\sigma(\pi)) = 2n, \qquad \pi \in \mathcal{E}_n, \tag{20}$$

where $U(\lambda)$ denotes the number of up steps in the lattice path λ. In our example above, where $n=8$, we have $\mathrm{semi}(\pi) + U(\sigma(\pi)) = 14 + 2 = 16$.

An h step within a member of $\mathcal{M}_n$ will be described as *low* if it has height zero (i.e., lies along the x-axis), and as *high*, otherwise. Let $H(\lambda)$, $H'(\lambda)$ and $H^*(\lambda)$ denote the number of h, low h and high h steps, respectively, within a lattice path λ. One may verify $\mu(\pi) = H'(\sigma(\pi)) + 1$ for all $\pi \in \mathcal{E}_n$.

For $\pi \in \mathcal{E}_n$, we thus have by (7) and (20),

$$\begin{aligned} \mathrm{var}(\pi) &= 2(\mathrm{semi}(\pi) - n) - \mu(\pi) - 1 \\ &= 2n - 2U(\sigma(\pi)) - \mu(\pi) - 1 \\ &= 2n - (n-1-H(\sigma(\pi))) - (H'(\sigma(\pi)) + 1) - 1 \\ &= n - 1 + H(\sigma(\pi)) - H'(\sigma(\pi)) \\ &= n - 1 + H^*(\sigma(\pi)) \qquad (21) \end{aligned}$$

and

$$\begin{aligned}\mathrm{var}^*(\pi) &= \mathrm{var}(\pi) + \mu(\pi) - 1 = \mathrm{var}(\pi) + H'(\sigma(\pi)) \\ &= n - 1 + H(\sigma(\pi)).\end{aligned} \tag{22}$$

For the remainder of this section, let $\mathrm{tot}_n(s)$ denote the sum of the values of s taken over all the members of $\mathcal{E}_n$ or over the corresponding lattice paths comprising $\mathcal{M}_{n-1}$, where s denotes a statistic defined on either one of these sets. Using the expressions for var and var* in terms of statistics on $\mathcal{M}_{n-1}$, one can explain bijectively the formulas found in Corollary 2.5 for $\mathrm{tot}_n(\mathrm{var})$ and $\mathrm{tot}_n(\mathrm{var}^*)$.

Combinatorial proof of Corollary 2.5:

Let $\mathcal{J}_n$ denote the set of marked members of $\mathcal{M}_{n-1}$, wherein a selected step is marked, and if the selected step is a high h, then it may be marked in one of two ways. By (21), we have $\mathrm{tot}_n(\mathrm{var}) = |\mathcal{J}_n|$ for all $n \geq 1$. Let $\mathcal{G}_n$ denote the set of (unrestricted) lattice paths from $(0,0)$ to $(n,0)$ using u, d and h steps. Members of $\mathcal{G}_n$ are referred to as *grand* Motzkin paths and it is well known that $|\mathcal{G}_n| = G_n$ for all $n \geq 0$. Let

$$\mathcal{Q}_n = (\mathcal{G}_n \backslash \mathcal{M}_n) \cup (\mathcal{G}_{n-1} \backslash \mathcal{M}_{n-1}).$$

To establish the first assertion in Corollary 2.5, it suffices to define a bijection between $\mathcal{J}_n$ and $\mathcal{Q}_n$, where we may assume $n \geq 2$.

To do so, suppose $\pi \in \mathcal{J}_n$, and we write $\pi = \alpha\mathbf{x}\beta$, where $\mathbf{x} \in \{u, d, h\}$ denotes the marked step of π. Given a sequence ρ of steps within a lattice path, let $\widehat{\rho}$ be obtained from ρ by reversing the order of the steps and replacing each u with d and d with u, leaving any h steps unchanged. We define f on $\mathcal{J}_n$ by

$$f(\pi) = \begin{cases} u\widehat{\alpha}d\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{h}\beta, \text{ with } \mathbf{h} \text{ high and marked in the second way;} \\ d\widehat{\alpha}u\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{h}\beta, \text{ with } \mathbf{h} \text{ marked in the first way if high;} \\ h\widehat{\alpha}d\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{u}\beta; \\ \widehat{\alpha}u\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{d}\beta. \end{cases}$$

The lattice path $f(\pi)$ has endpoint $(n,0)$ in the first three cases and endpoint $(n-1,0)$ in the last and is seen to go below the x-axis at some point in each case, whence $f(\pi) \in \mathcal{Q}_n$ for all π. To reverse f in the first and third cases, consider the leftmost step achieving the minimum height within $\lambda \in \mathcal{G}_n \backslash \mathcal{M}_n$ starting with u or h and act accordingly performing the inverse procedure in each case. On the other hand, if $\lambda \in \mathcal{G}_n \backslash \mathcal{M}_n$ starting with d or $\lambda \in \mathcal{G}_{n-1} \backslash \mathcal{M}_{n-1}$, then consider the rightmost step achieving the minimum height to reverse f. Thus, the mapping f is seen to provide the desired bijection between $\mathcal{J}_n$ and $\mathcal{Q}_n$, which establishes the first assertion in Corollary 2.5.

To prove the formula for $\mathrm{tot}_n(\mathrm{var}^*)$, we first introduce some further terminology as follows. By a *unit* within a member $\mathcal{G}_n$, we mean a sequence of steps starting and ending on the x-axis whose first step is u or d. A *unit* will be described as *positive* or *negative* depending on whether the first step is u or d. Hence, a positive unit is of the form $u\rho d$, where ρ is a possibly empty Motzkin path, with the initial u starting from the x-axis. Note that reflection of a positive unit in the x-axis gives a negative unit.

We first show that there are $G_{n-1} - M_{n-1}$ members of $\mathcal{G}_{n+1}$ starting with at least two units such that the first unit is positive and the second is negative. To do so, we note that $\lambda \in \mathcal{G}_{n-1} \backslash \mathcal{M}_{n-1}$ implies $\lambda = \lambda' d\lambda''$, where λ' is a possibly empty Motzkin path. Then it is seen that the mapping $\lambda \mapsto u\lambda' d^2\lambda''$ is a bijection between $\mathcal{G}_{n-1} \backslash \mathcal{M}_{n-1}$ and members of $\mathcal{G}_{n+1}$ starting with a positive, followed by a negative, unit, which establishes the assertion. By reflection of the appropriate units in the x-axis, there are also $G_{n-1} - M_{n-1}$ members of $\mathcal{G}_{n+1}$ starting with at least two units wherein the signs of the first two units are specified. Thus, there are $4(G_{n-1} - M_{n-1})$ members of $\mathcal{G}_{n+1}$ altogether that start with at least two units.

Let $\mathcal{K}_n$ denote the set of marked members of $\mathcal{M}_{n-1}$ wherein a selected step is marked, and if the selected step is an h, then it may be marked in one of two ways. By (22), we have $\mathrm{tot}_n(\mathrm{var}^*) = |\mathcal{K}_n|$. Let $\mathcal{L}_n$ denote the subset of $\mathcal{G}_{n+1}$ whose members start with a unit followed by a low h or start with two units of the same sign. Note that there are G_n members of $\mathcal{G}_{n+1}$ starting with h, $2M_{n-1}$ members that consist of a single unit (positive or negative) and $2(G_{n-1} - M_{n-1})$ that start with at least two units, the first two of which are of opposite sign. By subtraction, we have

$$|\mathcal{L}_n| = G_{n+1} - G_n - 2M_{n-1} - 2(G_{n-1} - M_{n-1}) = G_{n+1} - G_n - 2G_{n-1}.$$

Thus, to complete the proof, it suffices to define a bijection between $\mathcal{K}_n$ and $\mathcal{L}_n$. Before doing so, we need to define some preliminary operations as follows. Given $\rho \in \mathcal{G}_n$ starting with u or d, let $g_1(\rho) \in \mathcal{G}_{n+1}$ be obtained from ρ by inserting a low h directly following the first unit of ρ. Now suppose $\rho \in \mathcal{G}_{n-1} \backslash \mathcal{M}_{n-1}$ is given by $\rho = \rho_1 d\rho_2 u\rho_3$, where ρ_1 and $-\rho_2$ are Motzkin paths and ρ_3 is a grand Motzkin path. Then let

$$g_2(\rho) = (u\rho_1 d)(u(-\rho_2)d)\rho_3 \text{ and } g_3(\rho) = (d(-\rho_1)u)(d\rho_2 u)\rho_3,$$

both of which are seen to belong to $\mathcal{G}_{n+1}$ and start with two units of the same sign.

Let $\pi \in \alpha\mathbf{x}\beta \in \mathcal{K}_n$, where $\mathbf{x} \in \{u,d,h\}$ denotes the marked step. Define the mapping g on $\mathcal{K}_n$ by

$$g(\pi) = \begin{cases} g_1(u\widehat{\alpha}d\widehat{\beta}), & \text{if } \pi = \alpha\mathbf{h}\beta, \text{ where } \mathbf{h} \text{ is marked in the first way;} \\ g_1(d\widehat{\alpha}u\widehat{\beta}), & \text{if } \pi = \alpha\mathbf{h}\beta, \text{ where } \mathbf{h} \text{ is marked in the second way;} \\ g_2(\widehat{\alpha}d\widehat{\beta}), & \text{if } \pi = \alpha\mathbf{u}\beta; \\ g_3(\widehat{\alpha}u\widehat{\beta}), & \text{if } \pi = \alpha\mathbf{d}\beta. \end{cases}$$

Note that the first two cases of g cover all members of $\mathcal{L}_n$ starting with a unit followed by a low h, whereas the last two cases cover those members starting with at least two units, the first two of which must be of the same sign. One may verify that g is reversible in each case and thus provides the desired bijection between $\mathcal{K}_n$ and $\mathcal{L}_n$. □

We give a couple of examples of the mapping g from the last proof in Figures 5 and 6 below.

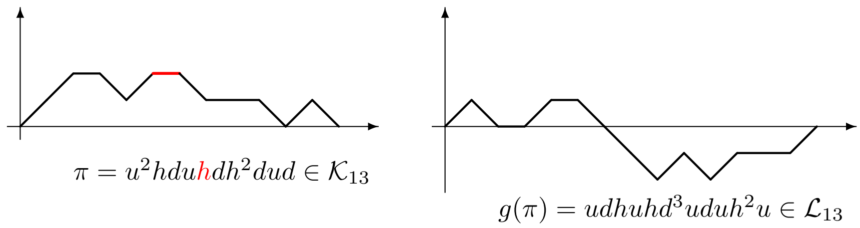


Figure 5: Example of the mapping g when $n = 13$, where the h in red is marked the first way.

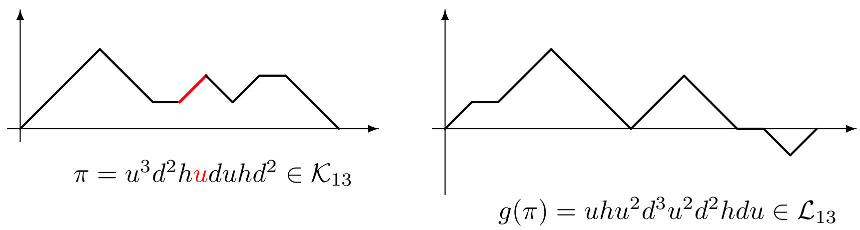


Figure 6: Example of the mapping g when $n = 13$, where the marked step of π is in red.

The preceding arguments can be modified to obtain combinatorially an expression for the total semi-perimeter over all the members of $\mathcal{E}_n$. In our proof, we will make use of (20). For completeness, we provide here an alternative proof of (20) to the inductive argument given in [2, Theorem 4.2], which perhaps demonstrates its truth more heuristically. To do so, we consider the effect on the semi-perimeter when a column is appended to a nonempty Motzkin polyomino. Suppose $\pi \in \mathcal{E}_n$ for some $n > 1$ is obtained by appending a column of height x to $\pi' \in \mathcal{E}_{n-1}$ whose final column has height y. One may verify

$$\text{semi}(\pi) = \begin{cases} \text{semi}(\pi') + 1, & \text{if } x < y; \\ \text{semi}(\pi') + 2, & \text{if } x = y + 1. \end{cases} \tag{23}$$

We now track the corresponding change in the number of up steps in going from $\sigma(\pi')$ to $\sigma(\pi)$. If $x < y$, then $\sigma(\pi')$ must contain at least one h step, as π' has last letter $y > 1$, and $\sigma(\pi)$ is obtained from $\sigma(\pi')$ in this case by changing one of the h steps of $\sigma(\pi')$ to a u and appending a terminal d step. To see this, note that within $\sigma(\pi')$, the second case of σ, which includes addition of an h step, is applied in conjunction with the rightmost occurrence of x. Since $x < y$, this added h is seen to be low. However, this h becomes a u, d pair in $\sigma(\pi)$, where the d is the final step, as the third case of the mapping σ now applies to the section of π occurring to the right of and including the second-to-last occurrence of x.

On the other hand, if $x = y + 1$, then $\sigma(\pi)$ is obtained from $\sigma(\pi')$ by appending a low h. This is clear if $\pi' = 12\cdots(n-1)$, so assume π' contains a descent. Let $\pi' = \pi''z(z+1)\cdots y$, where $z \leq y$ is the rightmost descent bottom of π'. If $z = y$, then $\sigma(\pi')$ ends in d, by the preceding, with the first $n-2$ letters of $\sigma(\pi)$ coinciding with $\sigma(\pi')$. But $\sigma(12) = h$ must now be appended to this, due to π ending in $y(y+1)$ instead of just y. If $z < y$, then similar reasoning shows that $\sigma(\pi')$ and $\sigma(\pi)$ are obtained from $\sigma(\pi''z)$ by appending $y - z$ or $y - z + 1$ low h steps, respectively, and the same conclusion concerning $\sigma(\pi)$ and $\sigma(\pi')$ is reached.

Thus, we have that each descent of π gives rise to a distinct u, d pair in $\sigma(\pi)$, with all u, d pairs in $\sigma(\pi)$ arising in this manner. Further, each time the added column yields a descent (i.e., when $x < y$), only 1 is added to the semi-perimeter instead of 2. Since $\text{semi}(1) = 2$, then (23) implies $\text{semi}(\pi) = 2n - \text{des}(\pi)$ for all $\pi \in \mathcal{E}_n$, where $\mathrm{des}(\pi)$ denotes the number of descents of π. Note that if 2 were added each time a column is appended, this would coincide with the case $\pi = 12\cdots n$, which has semi-perimeter $2n$. Since $\mathrm{des}(\pi) = U(\sigma(\pi))$ for all $\pi \in \mathcal{E}_n$, the proof of (20) is complete.

We now turn to finding an expression for $\mathrm{tot}_n(\mathrm{semi})$. By (7) and the fact $\mathrm{var}^*(\pi) = \mathrm{var}(\pi) + \mu(\pi) - 1$ for all π, we get $\mathrm{var}^*(\pi) = 2(\mathrm{semi}(\pi) - n) - 2$ and hence

$$\mathrm{semi}(\pi) = n + 1 + \frac{1}{2}\mathrm{var}^*(\pi).$$

Thus, by Corollary 2.5, we have

$$\begin{aligned}\mathrm{tot}_n(\mathrm{semi}) &= (n+1)M_{n-1} + \frac{1}{2}(G_{n+1} - G_n - 2G_{n-1}) = \frac{1}{2}(G_{n+1} + G_{n-1}) \\ &= G_n + 2G_{n-1} - M_{n-1}, \end{aligned} \tag{24}$$

upon making use of the facts $(n+1)M_{n-1} = \frac{1}{2}(G_n + 3G_{n-1})$ and $G_{n+1} = 2G_n + 3G_{n-1} - 2M_{n-1}$ for all $n \geq 1$. The expression in (24) may be realized as well by totaling $\mathrm{semi}(\pi) = 2n - U(\sigma(\pi))$ over all the members of $\mathcal{E}_n$ and recalling that there are $G_{n-1} - M_{n-1}$ up steps altogether in $\mathcal{M}_{n-1}$.

Formula (24) was also found in [2, Corollary 4.5] by differentiation of the appropriate generating function, and the question of finding a direct combinatorial proof was raised. We conclude by providing such a proof of this formula.

Combinatorial proof of (24)*:*

Let

$$\mathcal{Y}_n = \mathcal{K}_n \cup \mathcal{Y}'_n \cup \mathcal{M}^{(1)}_{n-1} \cup \mathcal{M}^{(2)}_{n-1},$$

where $\mathcal{K}_n$ is as in the proof of Corollary 2.5, $\mathcal{Y}'_n$ consists of members of $\mathcal{M}_{n-1}$ in which a d step is marked and $\mathcal{M}^{(i)}_{n-1}$ for $i = 1, 2$ denotes a (disjoint) copy of the set $\mathcal{M}_{n-1}$. Observe that, for all $\pi \in \mathcal{E}_n$,

$$\begin{aligned}\mathrm{semi}(\pi) &= 2n - U(\sigma(\pi)) = 2n - (n - 1 - D(\sigma(\pi)) - H(\sigma(\pi))) \\ &= n + 1 + D(\sigma(\pi)) + H(\sigma(\pi)).\end{aligned}$$

Let s and t denote the statistics on $\mathcal{M}_{n-1}$ defined by $n - 1 + H$ and $n + 1 + D + H$, respectively. Recall $\mathrm{tot}_n(s) = |\mathcal{K}_n|$, and hence $\mathrm{tot}_n(\mathrm{semi}) = \mathrm{tot}_n(t) = |\mathcal{Y}_n|$. Let

$$\mathcal{Z}_n = \mathcal{G}_n \cup \mathcal{G}_{n-1} \cup (\mathcal{G}_{n-1} \backslash \mathcal{M}_{n-1}),$$

and hence $|\mathcal{Z}_n| = G_n + 2G_{n-1} - M_{n-1}$. Thus, to establish (24), it is enough to define a bijection between $\mathcal{Y}_n$ and $\mathcal{Z}_n$, where it may be assumed $n > 1$.

Define the mapping ℓ on $\mathcal{Y}_n$ by

$$\ell(\pi) = \begin{cases} u\widehat{\alpha}d\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{h}\beta \in \mathcal{K}_n, \text{ where } \mathbf{h} \text{ is marked in the first way;} \\ d\widehat{\alpha}u\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{h}\beta \in \mathcal{K}_n, \text{ where } \mathbf{h} \text{ is marked in the second way;} \\ h\widehat{\alpha}u\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{d}\beta \in \mathcal{K}_n; \\ h\pi, & \text{if } \pi \in \mathcal{M}^{(1)}_{n-1}; \\ \widehat{\alpha}d\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{u}\beta \in \mathcal{K}_n; \\ \pi, & \text{if } \pi \in \mathcal{M}^{(2)}_{n-1}; \\ \widehat{\alpha}u\widehat{\beta}, & \text{if } \pi = \alpha\mathbf{d}\beta \in \mathcal{Y}'_n, \end{cases}$$

where the marked steps within the various members of $\mathcal{Y}_n$ are indicated in bold.

Note that the first four cases of ℓ, taken together, are onto the subset $\mathcal{G}_n$ of $\mathcal{Z}_n$, the next two cases of ℓ are onto $\mathcal{G}_{n-1}$, with the last case covering $\mathcal{G}_{n-1} \backslash \mathcal{M}_{n-1}$. Further, we have that the third and fourth cases of ℓ combined yield all of the members of $\mathcal{G}_n$ starting with h, upon considering whether or not such a member of $\mathcal{G}_n$ goes below the x-axis. The fifth and sixth cases of ℓ are similarly differentiated with regard to their images in $\mathcal{G}_{n-1}$. Since ℓ is seen to be reversible in all cases, it provides the desired bijection between $\mathcal{Y}_n$ and $\mathcal{Z}_n$, which completes the proof of (24). □

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