ecajournal.haifa.ac.il
Enumerative Combinatorics and Applications
ECA **6:1** (2026) Article #S2R2
https://doi.org/10.54550/ECA2026V6S1R2

Proofs of Three Geode Conjectures

Tewodros Amdeberhan† and Doron Zeilberger‡

†*Department of Mathematics, Tulane University, New Orleans, LA 70118, USA*
Email: tamdeber@tulane.edu

‡*Department of Mathematics, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854, USA*
Email: DoronZeil@gmail.com

Received: June 27, 2025, **Accepted**: September 17, 2025, **Published**: September 26, 2025

Abstract: In the May 2025 issue of the Amer. Math. Monthly, Norman J. Wildberger and Dean Rubine introduced a new kind of multi-indexed numbers, which they call 'Geode numbers', obtained from the Hyper-Catalan numbers. They posed three intriguing conjectures about them, which are proved in this note.

Keywords: Constant-terms; Geode array; Hyper-Catalan numbers; Lagrange Inversion; W-Z theory
2020 Mathematics Subject Classification: 05A15; 12D10

1. Introduction

In a recent captivating Monthly article [7], by Norman J. Wildberger and Dean Rubine, the authors utilize a generating series to solve the general univariate polynomial equation. They also explored a "curious factorization" of this hyper-Catalan generating series, and in the penultimate section, they made three conjectures about this algebraic object that they termed the *Geode array.*

In this note, we prove these three conjectures. At least as interesting as the actual statements of the conjectures (now theorems) is *how we proved them*, using several important *tools of the trade.*

The first tool is the *multinomial theorem*

$$(x_1+\cdots+x_r)^n = \sum_{\substack{m_1,\dots,m_r\geq 0\\ m_1+\cdots+m_r=n}} \binom{n}{m_1,\dots,m_r} x_1^{m_1}\cdots x_r^{m_r}. \tag{1}$$

The second tool is *constant-term extraction*, the third is *Wilf-Zeilberger (WZ) algorithmic proof theory* [8] and the last-but-not-least tool is *Lagrange Inversion* [9] that states that: *if $u(t)$ and $\Phi(t)$ are formal power series starting at t^1 and t^0, respectively, then $u(t)=t\Phi(u(t))$ implies*

$$[t^n]u(t) = \frac{1}{n}[z^{n-1}]\Phi(z)^n. \tag{2}$$

Here $[z^n]F(z)$ means the coefficient of z^n in the Laurent expansion of $F(z)$. We shall use the notation $\boldsymbol{CT}_z F(z)$ for the constant-term of $F(z)$.

Example 1.1. *To make the WZ method readily accessible to the unfamiliar reader, let's illustrate how the technique works in proving the known identity $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$. As a first step, we divide both sides to rewrite $\sum_{k=0}^n \binom{n}{k}^2\binom{2n}{n}^{-1} = 1$, identically a constant. Next, define $F(n,k) := \binom{n}{k}^2\binom{2n}{n}^{-1}$. The key here is that the WZ algorithm generates automatically (implemented in the symbolic softwares* Maple *and* Mathematica*) a companion function $G(n,k) := -\binom{n}{k-1}^2\binom{2n+2}{n+1}^{-1}\frac{3n+3-2k}{n+1}$. The theory anticipates that*

$$F(n+1,k)-F(n,k) = G(n,k+1)-G(n,k), \tag{3}$$

which can be checked directly (for instance, divide both sides by $F(n,k)$ and simplify the factorials). Now, sum both sides of (3) over all integers k and note that both $F(n,k)$ and $G(n,k)$ have compact support (they lead *to finite sums only). In addition, the right-hand side vanishes upon summation leaving behind the equation* $\sum_{k=0}^{n+1} F(n+1,k) - \sum_{k=0}^{n} F(n,k) = 0$*; that is, the quantity* $\sum_{k=0}^{n} F(n,k)$ *is independent of* n*. Testing at, say* $n=0$*, shows that this constant value is indeed* 1*. That completes the proof of the desired identity via WZ.*

We now bring in the relevant notation adopted in [7] with a caveat that indices are shifted slightly. Consider the equation $0 = 1-\alpha+\sum_{k\geq 1} t_k \alpha^{k+1}$ and denote its series solution by $\alpha = \boldsymbol{S}[t_1, t_2, \dots]$. Letting $\boldsymbol{S}_1 = t_1 + t_2 + \cdots$, Wildberger-Rubine proved [7, Theorem 12] the existence of a (remarkable!) factorization $\boldsymbol{S} - 1 = \boldsymbol{S}_1 \boldsymbol{G}$ and the factor $\boldsymbol{G}[t_1, t_2, \dots]$ (that they dubbed the *Geode series*). Furthermore, we opt to use $G[m_1, m_2, \dots]$ for the coefficient of $t_1^{m_1} t_2^{m_2} \cdots$ in the polyseries $\boldsymbol{G}[t_1, t_2, \dots]$. We are now ready to state the three conjectures (now labeled as theorems) from [7, Page 399] whose proof will be furnished in the next sections.

Theorem 1.1. *For non-negative integers* m_1 *and* m_2*, we have*

$$G[m_1, m_2] = \frac{1}{(2m_1+2m_2+3)(m_1+m_2+1)} \frac{(2m_1+3m_2+3)!}{(m_1+2m_2+2)!m_1!m_2!}.$$

Theorem 1.2. *Denote* $m = m_a + m_{a+1}$*. For integers* $m_a, m_{a+1} \geq 0$ *there holds*

$$\widetilde{G}[m_a, m_{a+1}] = \frac{(am_a + (a+1)(m_{a+1}+1))!}{(a(m+1)+1)(m+1)((a-1)m_a + a(m_{a+1}+1))!m_a!m_{a+1}!}.$$

Theorem 1.3. *For the* $2a$*-variate case, we have*

$$\boldsymbol{G}[-f, f, \dots, -f, f] = \sum_n a^n f^n.$$

2. Proof of Theorem 1.1

For the sake of clarity, let's describe this proof in some detail.

Suppose we are solving the polynomial equation $0 = 1-\alpha+t_1\alpha^2+t_2\alpha^3$ through the formal power series

$$\alpha = \boldsymbol{S}[t_1, t_2] = \sum_{m_1, m_2 \geq 0} C[m_1, m_2]\, t_1^{m_1} t_2^{m_2}.$$

Consequently, the corresponding Geode series becomes $\boldsymbol{G}[t_1, t_2] = \frac{\boldsymbol{S}[t_1,t_2]-1}{t_1+t_2}$. We follow closely [9] to engage the Lagrange Inversion in the extraction of the coefficients $C[m_1, m_2]$ satisfying $n = m_1 + m_2$. Then, the amalgamation of such monomials is given by (2) in the form of

$$\begin{aligned}
\sum_{m_1+m_2=n} C[m_1, m_2]\, t_1^{m_1} t_2^{m_2} &= [Y^n] \left(\sum_{k=1}^{3n+1} \frac{1}{k} [z^{k-1}] \left(1 + Yt_1z^2 + Yt_2z^3\right)^k \right) \\
&= [Y^n] \sum_{m_1, m_2 \geq 0} \frac{\binom{1+2m_1+3m_2}{m_1, m_2, 1+m_1+2m_2}}{1+2m_1+3m_2} Y^{m_1+m_2} t_1^{m_1} t_2^{m_2} \\
&= \sum_{\substack{m_1, m_2 \geq 0 \\ m_1+m_2=n}} \frac{\binom{1+2m_1+3m_2}{m_1, m_2, 1+m_1+2m_2}}{1+2m_1+3m_2} t_1^{m_1} t_2^{m_2} \\
&= \sum_{m_2=0}^{n} \frac{\binom{1+2n+m_2}{n-m_2, m_2, 1+n+m_2}}{1+2n+m_2} t_1^{n-m_2} t_2^{m_2} \\
&= \sum_{k=0}^{n} \frac{\binom{n}{k}\binom{2n+1+k}{n+1+k}}{2n+1+k} t_1^{n-k} t_2^{k}.
\end{aligned}$$

For example, the following reveal both coefficients $C[m_1, m_2]$ and $G[m_1, m_2]$:

$$\begin{aligned}
\sum_{m_1+m_2=3} C[m_1, m_2]\, t_1^{m_1} t_2^{m_2} &= (t_1+t_2)(5t_1^2 + 16t_1t_2 + 12t_2^2), \\
\sum_{m_1+m_2=4} C[m_1, m_2]\, t_1^{m_1} t_2^{m_2} &= (t_1+t_2)(14t_1^3 + 70t_1^2t_2 + 110t_1t_2^2 + 55t_2^3).
\end{aligned}$$

As a first step, we reprove that the linear term $t_2 + t_3$ divides the polynomial

$$P_n(t_1, t_2) := \sum_{k=0}^{n} \frac{\binom{n}{k}\binom{2n+1+k}{n+1+k}}{2n+1+k} t_1^{n-k} t_2^k.$$

This is equivalent to proving that $P_n(-t_2, t_2) = 0$, which, in turn, is equivalent to the following identity:

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}\binom{2n+1+k}{n+1+k}}{2n+1+k} = 0.$$

To continue, we invoke the role of the WZ method. Define the functions $F(n,k) := (-1)^k \frac{\binom{n}{k}\binom{2n+1+k}{n+1+k}}{2n+1+k}$ and also $H(n,k) := -F(n,k) \cdot \frac{k(n+1+k)}{n(2n+1)}$ to verify $F(n,k) = H(n,k+1) - H(n,k)$. The rest is routine [8].

Our next step will actually find $G[m_1, m_2]$. For that we perform the division $\frac{P_n(t_1,t_2)}{t_1+t_2}$ to obtain (algebraically) that

$$\begin{aligned}
[t_1^{n-1-i} t_2^i] \left(\frac{P_n(t_1, t_2)}{t_1 + t_2} \right) &= \sum_{j=0}^{i} (-1)^{i-j} \frac{\binom{n}{j}\binom{2n+1+j}{n+1+j}}{2n+1+j} \\
&= (-1)^i [H(n, i+1) - H(n, 0)] \\
&= (-1)^i H(n, i+1) \\
&= \frac{1}{2n+1} \binom{n-1}{i} \binom{2n+1+i}{n+1+i}
\end{aligned}$$

which leads to (an equivalent form of) the first conjecture [7] on $G[m_1, m_2]$, here stated as Theorem 1.1.

3. Proof of Theorem 1.2

Now that the reader, hopefully, is getting accustomed to our proof-procedure as depicted in Section 2, let's move on to the next conjecture [7, Page 399] which does generalize the one we just finished proving. For brevity, denote $\widetilde{G} = \widetilde{G}[m_a, m_{a+1}] = G[0, 0, \ldots, m_a, m_{a+1}]$. Again, we revive the Lagrange Inversion (2). Suppose $n = m_a + m_{a+1}$. Then the total content of such monomials is encapsulated by

$$\begin{aligned}
\sum_{m_a + m_{a+1} = n} \widetilde{G}\, t_a^{m_2} t_{a+1}^{m_3} &= \frac{[Y^n]}{t_a + t_{a+1}} \sum_{k=1}^{(a+1)n+1} \frac{1}{k} [z^{k-1}] \left(1 + Y t_a z^a + Y t_{a+1} z^{a+1}\right)^k \\
&= \frac{[Y^n]}{t_a + t_{a+1}} \sum_{m_a, m_{a+1} \geq 0} \frac{\binom{1+am_a+(a+1)m_{a+1}}{m_a, m_{a+1}, 1+(a-1)m_a+am_{a+1}} Y^{m_a+m_{a+1}} t_a^{m_a} t_{a+1}^{m_{a+1}}}{1 + am_a + (a+1)m_{a+1}} \\
&= \sum_{\substack{m_a, m_{a+1} \geq 0 \\ m_a + m_{a+1} = n}} \frac{\binom{1+am_a+(a+1)m_{a+1}}{m_a, m_{a+1}, 1+(a-1)m_a+am_{a+1}}}{1 + am_a + (a+1)m_{a+1}} \frac{t_a^{m_a} t_{a+1}^{m_{a+1}}}{t_a + t_{a+1}} \\
&= \sum_{m_{a+1}=0}^{n} \frac{\binom{1+an+m_{a+1}}{n-m_{a+1}, m_{a+1}, 1+(a-1)n+m_3}}{1 + an + m_{a+1}} \frac{t_a^{n-m_{a+1}} t_{a+1}^{m_{a+1}}}{t_a + t_{a+1}} \\
&= \sum_{k=0}^{n} \frac{\binom{n}{k}\binom{an+1+k}{(a-1)n+1+k}}{an+1+k} \frac{t_a^{n-k} t_{a+1}^k}{t_a + t_{a+1}}.
\end{aligned}$$

As a first step, we justify that the linear term $t_a + t_{a+1}$ divides the polynomial

$$P_n(t_a, t_{a+1}) := \sum_{k=0}^{n} \frac{\binom{n}{k}\binom{an+1+k}{(a-1)n+1+k}}{an+1+k} t_a^{n-k} t_{a+1}^k.$$

This is tantamount to $P_n(-t_{a+1}, t_{a+1}) = 0$ which is equivalent to the identity that

$$\sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k}\binom{an+1+k}{(a-1)n+1+k}}{an+1+k} = 0.$$

Again, apply the Wilf-Zeilberger approach with $F(n,k) := \frac{(-1)^k\binom{n}{k}\binom{an+1+k}{(a-1)n+1+k}}{an+1+k}$ and $H(n,k) := -F(n,k)\cdot\frac{k((a-1)n+1+k)}{n(an+1)}$ to verify $F(n,k) = H(n,k+1) - H(n,k)$. The rest is trivial.

Our next step will actually determine $\widetilde{G}[m_a, m_{a+1}]$. To this effect, let's divide $\frac{P_n(t_a,t_{a+1})}{t_a+t_{a+1}}$ to obtain (routinely) that

$$\begin{aligned}
[t_a^{n-1-i}t_{a+1}^i]\left(\frac{P_n(t_a,t_{a+1})}{t_a+t_{a+1}}\right) &= \sum_{j=0}^{i}(-1)^{i-j}\frac{\binom{n}{j}\binom{an+1+j}{(a-1)n+1+j}}{an+1+j}\\
&= (-1)^i[H(n,i+1)-H(n,0)] = (-1)^i H(n,i+1)\\
&= \frac{1}{an+1}\binom{n-1}{i}\binom{an+1+i}{(a-1)n+1+i}
\end{aligned}$$

which proves the desired conjecture on $\widetilde{G}[m_a, m_{a+1}]$.

4. Proof of Theorem 1.3

The proof of this last conjecture [7, Page 399] is a bit more complicated.

To begin, we make a slight alteration by writing $(-1)^i t_i$ instead of the customary plain t_i [7]. Thanks to the Lagrange Inversion (2), we have

$$\begin{aligned}
&[Y^n]\left(\sum_{k=1}^{\infty}\frac{1}{k}\,[z^{k-1}]\left(1 - Yt_1z^2 + Yt_2z^3 - \cdots - Yt_{2a-1}z^{2a} + Yt_{2a}z^{2a+1}\right)^k\right)\\
=&[Y^n]\sum_{m_1,\ldots,m_{2a}\geq 0}\frac{(-1)^{m_1+\cdots+m_{2a-1}}\binom{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}{m_1,m_2,\ldots,m_{2a},1+m_1+2m_2+\cdots+(2a)m_{2a}}(Yt_1)^{m_1}\cdots(Yt_{2a})^{m_{2a}}}{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}\\
=&\sum_{\substack{m_1,\ldots,m_{2a}\geq 0\\ m_1+\cdots+m_{2a}=n}}\frac{(-1)^{m_1+\cdots+m_{2a-1}}\binom{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}{m_1,m_2,\ldots,m_{2a},1+m_1+2m_2+\cdots+(2a)m_{2a}}t_1^{m_1}\cdots t_{2a}^{m_{2a}}}{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}.
\end{aligned}$$

First, consider the case $a = 1$ and refer back to Theorem 1.1 (and its proof), to gather that if $t_1 = -f$ and $t_2 = f$ then, as expected, we arrive at

$$f^{n-1}\sum_{m=0}^{n-1}\frac{(-1)^{n-1-m}}{2n+1}\binom{n-1}{m}\binom{2n+1+m}{n+1+m} = f^{n-1}$$

as justified by the *WZ-certificate* [8] given by

$$R(n,m) := \frac{m(8mn+10n^2+6m+15n+6)}{2(2n+3)(n+1)(n-m)}.$$

Second, we go back to study the above-posed calculations when $a > 1$. To set the stage, substitute

$$t_1 = t_2 = \cdots = t_{2a-1} = f$$

while leaving out t_{2a} as an indeterminate. The outcome takes the form

$$\sum_{\substack{m_1,\ldots,m_{2a}\geq 0\\ m_1+\cdots+m_{2a}=n}}\frac{(-1)^{m_1+m_3+\cdots+m_{2a-1}}\binom{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}{m_1,m_2,\ldots,m_{2a},1+m_1+2m_2+\cdots+(2a)m_{2a}}f^{n-m_{2a}}t_{2a}^{m_{2a}}}{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}.$$

At this point, divide out the current polynomial (in t_{2a}) by the linear factor

$$-t_1 + t_2 - \cdots - t_{2a-3} + t_{2a-1} - t_{2a-1} + t_{2a} = t_{2a} - f$$

and then replace t_{2a} by f. That leads to the sum

$$f^{n-1}\sum_{i=0}^{n-1}\sum_{m_{2a}=0}^{i}\sum_{\substack{m_1,\ldots,m_{2a}\geq 0\\ m_1+\cdots+m_{2a}=n}}\frac{(-1)^{1+m_1+m_3+\cdots+m_{2a-1}}\binom{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}{m_1,m_2,\ldots,m_{2a},1+m_1+2m_2+\cdots+(2a)m_{2a}}}{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}.$$

Therefore, our main task that remains is to prove the identity declared by

$$\sum_{i=0}^{n-1} \sum_{\substack{m_1,\dots,m_{2a-1}\geq 0 \\ m_1+\cdots+m_{2a}=n \\ 0\leq m_{2a}\leq i}} \frac{(-1)^{1+m_1+m_3+\cdots+m_{2a-1}}\binom{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}}{m_1,m_2,\dots,m_{2a},1+m_1+2m_2+\cdots+(2a)m_{2a}}}{1+2m_1+3m_2+\cdots+(2a+1)m_{2a}} = a^{n-1}.$$

To put this more succinctly, introduce some notation. Let $\mathcal{P}$ denote the set of all integer partitions λ, written as $\lambda = (\lambda_1, \lambda_2, \dots)$ or $\lambda = 1^{m_1}2^{m_2}\dots(2a)^{m_{2a}}$. The size of λ is denoted by $|\lambda| = \lambda_1 + \lambda_2 + \cdots = m_1 + 2m_2 + \cdots + (2a)m_{2a}$ while we use $\ell(\lambda) = m_1 + m_2 + \cdots + m_{2a}$ for the length of the partition. So, the claim stands at

$$\sum_{\substack{\lambda\in\mathcal{P} \\ \ell(\lambda)=n \\ \lambda_1\leq 2a}} (-1)^{1+|\lambda|} \cdot \frac{(n-m_{2a})\binom{n}{m_1,\dots,m_{2a}}\binom{|\lambda|+n+1}{|\lambda|+1}}{|\lambda|+n+1} = a^{n-1}. \tag{4}$$

We find it more convenient to split up this assertion into two separate claims

$$(-1)^1 \sum_{\substack{\lambda\in\mathcal{P} \\ \ell(\lambda)=n \\ \lambda_1\leq 2a}} (-1)^{|\lambda|}\binom{n}{m_1,\dots,m_{2a}}\binom{|\lambda|+n}{|\lambda|+1} = 0, \tag{5}$$

$$\sum_{\substack{\mu\in\mathcal{P} \\ \ell(\mu)=n-1 \\ \mu_1\leq 2a}} (-1)^{|\mu|}\binom{n-1}{m_1,\dots,m_{2a}}\binom{|\mu|+2a+n}{|\mu|+2a+1} = a^{n-1}. \tag{6}$$

One arrives at (5) due to

$$\frac{n\binom{|\lambda+n+1}{|\lambda|+1}}{|\lambda+n+1} = \binom{|\lambda+n}{|\lambda|+1}$$

and (6) arises because of $m_{2a}\binom{n}{m_1,\dots,m_{2a}}\frac{(|\lambda+n)!}{(|\lambda+1)!n!} = \binom{n-1}{m_1,\dots,m_{2a}-1}\binom{|\lambda|+n}{|\lambda|+1}$ and then we reindex $m'_{2a} = m_{2a}-1$ to convert $|\lambda| = |\mu|+2a$ where $\ell(\mu) = n-1$.

In fact, let's generalize (5) and (6) by introducing an extra parameter x.

Claim 1: For positive integers n, a and an indeterminate x, we have

$$\sum_{\substack{\lambda\in\mathcal{P} \\ \ell(\lambda)=n \\ \lambda_1\leq 2a}} (-1)^{|\lambda|}\binom{n}{m_1,\dots,m_{2a}}\binom{|\lambda|+n+x}{n-1} = 0.$$

Claim 2: For positive integers n, a and an indeterminate x, we have

$$\sum_{\substack{\lambda\in\mathcal{P} \\ \ell(\lambda)=n-1 \\ \lambda_1\leq 2a}} (-1)^{|\lambda|}\binom{n-1}{m_1,\dots,m_{2a}}\binom{|\lambda|+n+x}{n-1} = a^{n-1}.$$

Claim 2 implies Claim 1: Assuming $n = k_1 + \cdots + k_r$, we apply the multinomial recurrence

$$\binom{n}{k_1,\dots,k_r} = \binom{n-1}{k_1-1,\dots,k_r} + \cdots + \binom{n-1}{k_1,\dots,k_r-1} \tag{7}$$

followed by appropriate reindexing (observe: if m_i in λ drops to $m_i - 1$ in μ then $|\lambda| = |\mu| + i$) so that

$$\begin{aligned}&\sum_{\substack{\lambda\in\mathcal{P} \\ \ell(\lambda)=n \\ \lambda_1\leq 2a}} (-1)^{|\lambda|}\binom{n}{m_1,\dots,m_{2a}}\binom{|\lambda|+n+x}{n-1} \\ =&\sum_{i=1}^{2a}\sum_{\substack{\lambda\in\mathcal{P} \\ \ell(\lambda)=n \\ \lambda_1\leq 2a}} (-1)^{|\lambda|}\binom{n-1}{m_1,\dots,m_i-1,\dots m_{2a}}\binom{|\lambda|+n+x}{n-1}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2a} \sum_{\substack{\mu\in\mathcal{P}\\ \ell(\mu)=n-1\\ \mu_1\le 2a}} (-1)^{|\mu|+i} \binom{n-1}{m_1,\dots,m_i',\dots m_{2a}} \binom{|\mu|+n+(x+i)}{n-1}\\
&= \sum_{i=1}^{2a} (-1)^i \sum_{\substack{\mu\in\mathcal{P}\\ \ell(\mu)=n-1\\ \mu_1\le 2a}} (-1)^{|\mu|} \binom{n-1}{m_1,\dots,m_i',\dots m_{2a}} \binom{|\mu|+n+(x+i)}{n-1}\\
&= a^{n-1}\sum_{i=1}^{2a}(-1)^i = 0.
\end{aligned}
$$

Proof of Claim 2: Let's now utilize the multinomial theorem (1) and constant-term extraction. Start by noting the constant-term extraction

$$
\binom{|\lambda|+n+x}{n-1} = \binom{m_1+2m_2+\cdots+(2a)m_{2a}+n+x}{n-1} = \boldsymbol{CT}_z\left[\frac{(1+z)^{m_1+2m_2+\cdots+(2a)m_{2a}+n+x}}{z^{n-1}}\right].
$$

Insert this into the left-hand side of Claim 2, take $\boldsymbol{CT}_z$ outside the sum, factor out the inside and reapply the multinomial theorem in reverse (1) to get

$$
\begin{aligned}
&\sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)=n-1\\ \lambda_1\le 2a}} (-1)^{|\lambda|} \binom{n-1}{m_1,\dots,m_{2a}} \binom{|\lambda|+n+x}{n-1}\\
=&\boldsymbol{CT}_z\left[\frac{(1+z)^{n+x}}{z^{n-1}} \sum \binom{n-1}{m_1,\dots,m_{2a}} (-1-z)^{m_1}(-1-z)^{2m_2}\cdots(-1-z)^{(2a)m_{2a}}\right]\\
=&\boldsymbol{CT}_z\left[\frac{(1+z)^{n+x}}{z^{n-1}} \left\{-(1+z)^1+(1+z)^2-(1+z)^3+\cdots+(1+z)^{2a}\right\}^{n-1}\right].
\end{aligned}
$$

Next, follow through with the geometric series expansion to obtain

$$
\begin{aligned}
\sum_{\substack{\lambda\in\mathcal{P}\\ \ell(\lambda)=n-1\\ \lambda_1\le 2a}} (-1)^{|\lambda|} \binom{n-1}{m_1,\dots,m_{2a}} \binom{|\lambda|+n+x}{n-1} =&\boldsymbol{CT}_z\left[(-1)^{n-1}\frac{(1+z)^{2n+x-1}}{z^{n-1}} \left\{\frac{1-(1+z)^{2a}}{2+z}\right\}^{n-1}\right]\\
=&\boldsymbol{CT}_z\left[\frac{(1+z)^{2n+x-1}}{(2z)^{n-1}} \left\{\frac{z\sum_{k=1}^{2a}\binom{2a}{k}z^{k-1}}{1+\frac{z}{2}}\right\}^{n-1}\right] = a^{n-1}.
\end{aligned}
$$

The proof is indeed complete.

5. Conclusion

In this last section, we have elected to leave the reader with some final but motivating pointers.

Remark 5.1. *On [7, Page 399], it is stated that* "With $k-2$ leading zeros, we conjecture that $G[0,\dots,m_k]$ is a two-parameter Fuss-Catalan number." *For Fuss-Catalan numbers, see [2], [5]. In light of the conjectures we already proved, the current claim is rather obvious (for further discussion on the topic the reader is directed to [4]).*

Remark 5.2. *One can prove both Theorem 1.1 and 1.2 with the following observation. It suffices to explain this for Theorem 1.1. Since $C[m_1,m_2]$ are known from the Lagrange Inversion and because we have and explicit conjectured formula $G[m_1,m_2]$ due to [7], all that is required is to verify the relation $G[m_1-1,m_2]+G[m_1,m_2-1]=C[m_1,m_2]$. This, however, is routine. Of course, the proofs in Section s1 and 2 do not assume knowing $C[m_1,m_2]$ and $G[m_1,m_2]$* a priori*: they are pure* derivations *from scratch.*

Remark 5.3. *We offer (the proof is analogous to Theorem 1.2 but omitted) the assertion that*

$$
G[0,\dots,0,m_s,0,\dots,m_t] = \frac{1}{n}\sum_{j=0}^{i}(-1)^{i-j}\binom{n}{j}\binom{(s+1)n+(t-s)j}{n-1},
$$

where we used $m_s=n-1-i, m_t=i$.

Remark 5.4. *We also offer (the proof is analogous to Theorem 1.3 but omitted) the assertion that for a generalized* $2a$*-variate case, we have*

$$\boldsymbol{G}[-c_a f, c_1 f, -c_1 f, c_2 f, -c_2 f, \cdots, c_{a-1} f, -c_{a-1} f, c_a f] = \sum_n (2ac_a - c_1 - c_2 - \cdots - c_a)^n f^n.$$

6. Acknowledgments

After the completion of this work, Dean Rubine informed us that he has independently proved a couple of the conjectures with a different method [6] and one of them with the help of Ira Gessel [3]. The authors here together with Manuel Kauers [1] have explored the structure of the Geode in its generality. The authors are grateful to the referees for their valuable and constructive suggestions that have improved this paper.

References

[1] T. Amdeberhan, M. Kauers, and D. Zeilberger, *The challenge of computing Geode numbers*, to appear in Palestine J. Math.

[2] J-C. Aval, *Multivariate Fuss-Catalan numbers*, Discrete Math. 308:20 (2008), 4660–4669.

[3] I. Gessel, *Lattice paths and the Geode*, `https://arxiv.org/abs/2507.09405`.

[4] S. R. Mane, *Multiparameter Fuss–Catalan numbers with application to algebraic equations*, `https://arxiv.org/pdf/1607.04144`.

[5] W. Młotkowski and K. A. Penson, *A Fuss-type family of positive definite sequences*, Colloq. Math. 151:2 (2018), 289–304.

[6] D. Rubine, *Hyper-Catalan and Geode Recurrences and Three Conjectures of Wildberger*, `https://arxiv.org/abs/2507.04552`.

[7] N. J. Wildberger and D. Rubine, *A Hyper-Catalan Series Solution to Polynomial Equations, and the Geode*, Amer. Math. Monthly 132:5 (2025), 383–402. `https://www.tandfonline.com/doi/full/10.1080/00029890.2025.2460966`.

[8] H. S. Wilf and D. Zeilberger, *Rational functions certify combinatorial identities*, J. Amer. Math. Soc. 3:1 (1990), 147–158. `https://www.ams.org/journals/jams/1990-03-01/S0894-0347-1990-1007910-7/S0894-0347-1990-1007910-7.pdf`.

[9] D. Zeilberger, *Lagrange Inversion Without Tears (Analysis) (based on Henrici)*, `https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/lag.pdf`.