ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **6:1** (2026) Article #S2R5
https://doi.org/10.54550/ECA2026V6S1R5

List Coloring the Cartesian Product of a Complete Graph and Complete Bipartite Graph

Hemanshu Kaul[†], Leonardo Marciaga[‡], and Jeffrey A. Mudrock[*]

[†] *Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616.*
Email: kaul@illinoistech.edu

[‡] *Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616.*
Email: lmarciaga@hawk.illinoistech.edu

[*] *Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688.*
Email: mudrock@southalabama.edu

Received: September 20, 2025, **Accepted**: November 2, 2025, **Published**: November 14, 2025

Abstract: We study the list chromatic number of the Cartesian product of a complete graph of order n and a complete bipartite graph with partite sets of size a and b where $a \leq b$, denoted $\chi_\ell(K_n \square K_{a,b})$. At the 2024 Sparse Graphs Coalition's Workshop on algebraic, extremal, and structural methods and problems in graph colouring, Mudrock presented the following question: For each positive integer a, does $\chi_\ell(K_n \square K_{a,b}) = n + a$ if and only if $b \geq (n + a - 1)!^a/(a - 1)!^a$? In this paper, we show the answer to this question is yes by studying $\chi_\ell(H \square K_{a,b})$ when H is strongly chromatic-choosable (a special form of vertex criticality related to chromatic choosability) with the help of the list color function and analytic inequalities such as that of Karamata. Our result can be viewed as a generalization of the well-known result that $\chi_\ell(K_{a,b}) = 1 + a$ if and only if $b \geq a^a$.

Keywords: Cartesian product; Chromatic choosablity; List coloring; List color function
2020 Mathematics Subject Classification: 05C15; 05C30; 05C76

1. Introduction

In this paper, all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking, we follow West [19] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \dots\}$. For $k \in \mathbb{N}$, we write $[k]$ for the set $\{1, \dots, k\}$ and $[0] = \emptyset$. We adopt the convention that $\prod_{i=a}^{b} x_i = 1$ whenever $a > b$. We use AM-GM inequality to mean the Inequality of Arithmetic and Geometric Means. If G is a graph and $S \subseteq V(G)$, we write $G[S]$ for the subgraph of G induced by S. For $v \in V(G)$, we write $d_G(v)$ for the degree of vertex v in the graph G, and we write $N_G(v)$ for the neighborhood of v in G. If G and H are vertex-disjoint graphs, the *join* of G and H, denoted $G \vee H$, is the graph consisting of G, H, and additional edges added so that each vertex in G is adjacent to each vertex in H.

1.1 List Coloring Cartesian Products

List coloring is a variation on the classical vertex coloring problem that was introduced in the 1970s independently by Vizing [17] and Erdős, Rubin, and Taylor [6]. In the classical vertex coloring problem, we seek a *proper k-coloring* of a graph G which is a coloring of the vertices of G with colors from $[k]$ so that adjacent vertices receive different colors. The *chromatic number* of a graph, denoted $\chi(G)$, is the smallest k such that G has a proper k-coloring. For list coloring, we associate a *list assignment* L with a graph G which assigns to each vertex $v \in V(G)$ a list of colors $L(v)$ (we say L is a list assignment for G). The graph G is *L-colorable* if there exists a proper coloring f of G such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to f as a *proper L-coloring* of G). A list assignment L is called a *k-assignment* for G if $|L(v)| = k$ for each $v \in V(G)$. The *list chromatic number* of a graph G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever L is a k-assignment for G. Note that $\chi(G) \leq \chi_\ell(G)$ for all graphs G. We say G is *k-choosable* if $k \geq \chi_\ell(G)$. Additionally, G is said to be *chromatic-choosable* when its list chromatic number equals its chromatic number.

The *Cartesian product* of graphs M and H, denoted $M\Box H$, is the graph with vertex set $V(M)\times V(H)$ and edges created so that (u,v) is adjacent to (u',v') if and only if either $u=u'$ and $vv'\in E(H)$ or $v=v'$ and $uu'\in E(M)$. Throughout this paper, if $G=M\Box H$ and $u\in V(M)$ (resp. $u\in V(H)$), we let V_u be the subset of $V(G)$ consisting of the vertices with first (resp. second) coordinate u. We also let $G_u=G[V_u]$. Similarly, if $S\subseteq V(M)\cup V(H)$, we let $V_S=\bigcup_{s\in S}V_s$ and $G_S=G[V_S]$. By the definition of the Cartesian product of graphs, it is easy to see that G_u is a copy of H (resp. M) when $u\in V(M)$ (resp. $u\in V(H)$). When L is a list assignment for G and $u\in V(G)$, we write L_u for the list assignment for G_u obtained by restricting the domain of L to V_u. Similarly, when $S\subseteq V(M)\cup V(H)$, we write L_S for the list assignment for G_S obtained by restricting the domain of L to V_S.

It is well-known that $\chi(G\Box H)=\max\{\chi(G),\chi(H)\}$. On the other hand, the list chromatic number of the Cartesian product of graphs is not nearly as well understood. In 2006, Borowiecki, Jendrol, Král, and Miškuf [3] showed the following.

Theorem 1.1 ([3]). *For any graphs G and H, $\chi_\ell(G\Box H)\le\min\{\chi_\ell(G)+\mathrm{col}(H),\mathrm{col}(G)+\chi_\ell(H)\}-1$.*

Here $\mathrm{col}(G)$ denotes the *coloring number* of a graph G which is the smallest integer d for which there exists an ordering, $v_1,\ldots,v_n$, of the elements in $V(G)$ such that each vertex v_i has at most $d-1$ neighbors among $v_1,\ldots,v_{i-1}$. It is well known that greedy coloring gives $\chi_\ell(G)\le\mathrm{col}(G)$ for all graphs G. For this paper, it is important to note that Theorem 1.1 implies $\chi_\ell(G\Box K_{a,b})\le\chi_\ell(G)+a$.

It is also proven in [3] that the bound in Theorem 1.1 is tight.

Theorem 1.2 ([3]). *Suppose G is a graph with n vertices. Then, $\chi_\ell(G\Box K_{a,b})=\chi_\ell(G)+a$ whenever $b\ge(\chi_\ell(G)+a-1)^{an}$.*

It is natural to wonder when the bound on b in Theorem 1.2 is best possible. With this in mind, for each $a\in\mathbb{N}$, we let $f_a(G)$ be the smallest b such that $\chi_\ell(G\Box K_{a,b})=\chi_\ell(G)+a$. Note $\chi_\ell(G\Box K_{a,0})=\chi_\ell(G)<\chi_\ell(G)+a$ which implies that $f_a(G)\ge 1$. Second, Theorem 1.2 implies that $f_a(G)\le(\chi_\ell(G)+a-1)^{a|V(G)|}$. This means $f_a(G)$ exists and is a natural number. Also, if G is a disconnected graph with components: $H_1,H_2,\ldots,H_r$, we have $f_a(G)=\max\{f_a(H_i)\colon i\in[r],\chi_\ell(H_i)=\chi_\ell(G)\}$. So, we will restrict our attention to connected graphs from this point forward.

1.2 The List Color Function and Strong Chromatic-Choosability

Let $P(G,k)$ be the *chromatic polynomial* of the graph G; that is, $P(G,k)$ is equal to the number of proper k-colorings of G. It is known that $P(G,k)$ is a polynomial in k (see [2]). In 1990 [12], this notion was extended to list coloring as follows. If L is a list assignment for G, we use $P(G,L)$ to denote the number of proper L-colorings of G. The *list color function* $P_\ell(G,k)$ is the minimum value of $P(G,L)$ where the minimum is taken over all possible k-assignments L for G. Since a k-assignment could assign the same k colors to every vertex in a graph, it is clear that $P_\ell(G,k)\le P(G,k)$ for each $k\in\mathbb{N}$. In general, the list color function of a graph can differ significantly from its chromatic polynomial for small values of k. However, for large values of k, Dong and Zhang [4] (improving upon results in [5], [16], and [18]) showed that for any graph G with at least 2 edges, $P_\ell(G,k)=P(G,k)$ whenever $k\ge|E(G)|-1$.

In the case G is a complete graph or a cycle, it is well known (see [15]) that $P(C_n,k)=(k-1)^n+(-1)^n(k-1)$ and $P(K_n,k)=\prod_{i=0}^{n-1}(k-i)$. It is easy to see that for each $n,k\in\mathbb{N}$, $P(K_n,k)=P_\ell(K_n,k)$, and it was shown in [11] that for each $n,k\in\mathbb{N}$, $P(C_n,k)=P_\ell(C_n,k)$. Also, the list color function of certain Cartesian products of graphs was recently utilized in [10].

A graph is *k-vertex critical* if its chromatic number is k and the removal of any vertex in the graph decreases the chromatic number of the graph. In [8], the first and third named authors introduced the related notion of strong chromatic-choosability and used the list color function to compute f_a with $a=1$ for graphs that are strongly chromatic-choosable (Theorem 1.3 below). A graph G is *strongly k-chromatic-choosable* if it is k-vertex critical and every $(k-1)$-assignment L for which G is not L-colorable has the property that the lists are the same on all vertices. List assignments that assign the same list of colors to every vertex of a graph are called *constant*. We say G is *strongly chromatic-choosable* if it is strongly $\chi(G)$-chromatic-choosable. Note that if G is strongly k-chromatic-choosable, then the only reason G is not $(k-1)$-choosable is that a proper $(k-1)$-coloring of G does not exist. Simple examples of strongly chromatic-choosable graphs include complete graphs, odd cycles, and the join of a complete graph and odd cycle (see [1] and [8] for many other examples).

Theorem 1.3 ([8]). *Let M be a strongly k-chromatic-choosable graph. Then, $f_1(M)=P_\ell(M,k)$.*

1.3 Motivating Question

The following general upper bound on $f_a(G)$ was proven in [9].

Theorem 1.4 ([9]). *For any graph G and $a\in\mathbb{N}$, $f_a(G)\le(P_\ell(G,\chi_\ell(G)+a-1))^a$.*

Notice Theorem 1.3 shows the bound in Theorem 1.4 is tight when $a = 1$ and G is strongly chromatic-choosable. However, it is not the case that $f_a(G) = (P_\ell(G, \chi_\ell(G) + a - 1))^a$ for all graphs G and $a \in \mathbb{N}$ since it is easy to see that $f_1(C_{2n+2}) = 1$, yet $P_\ell(C_{2n+2}, 2) = 2$. This observation leads to the following open question.

Question 1.1 ([9]). *For what graphs does $f_a(G) = (P_\ell(G, \chi_\ell(G) + a - 1))^a$ for each $a \in \mathbb{N}$?*

In [9], some partial progress was made on Question 1.1, specifically in the case where our attention is restricted to strongly chromatic-choosable graphs.

Theorem 1.5 ([9]). *If M is strongly chromatic-choosable and $\chi(M) \geq a + 1$, then $f_a(M) = (P_\ell(M, \chi_\ell(M) + a - 1))^a$.*

It is unknown whether there are any strongly chromatic-choosable graphs M for which $f_a(M) < (P_\ell(M, \chi_\ell(M) + a - 1))^a$. Consequently, the following question, which was presented by the third named author at the 2024 Sparse Graphs Coalition's Workshop on algebraic, extremal, and structural methods and problems in graph colouring [14], is open and served as the main motivation for this paper.

Question 1.2 ([9, 14]). *Is it the case that $f_a(K_n) = (P_\ell(K_n, n + a - 1))^a = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$ for each $n, a \in \mathbb{N}$?*

In what follows, we show that the answer to this question is yes. In Subsection 2.1 we present several important lemmas and observations that are used in the proof of our main result. Importantly, the results in Subsection 2.1 apply to all strongly chromatic-choosable graphs; so, they may be of independent interest since they could be used to explore whether all strongly chromatic-choosable graphs satisfy the condition in Question 1.1. After proving several technical inequalities, which include the use of the AM-GM inequality and Karamata's Inequality, in Subsection 2.2, we complete the proof of our main result in Subsection 2.3, which we now state.

Theorem 1.6. *For each $n, a \in \mathbb{N}$, $\chi_\ell(K_n \square K_{a,b}) = n + a$ if and only if $b \geq (n + a - 1)!^a/(a - 1)!^a$. That is, $f_a(K_n) = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$ for each $n, a \in \mathbb{N}$.*

It is worth mentioning that when $n = 1$, Theorem 1.6 says $\chi_\ell(K_{a,b}) = \chi_\ell(K_1 \square K_{a,b}) = 1 + a$ if and only if $b \geq a^a$ which is a well-known list coloring result.

2. Proof of Theorem 1.6

We now introduce some notation and terminology that will be used for the remainder of this paper. Suppose a, b, and $k \geq 2$ are positive integers and M is a strongly k-chromatic-choosable graph. Suppose $H = M \square K_{a,b}$, $V(M) = \{v_1, \ldots, v_n\}$, and the partite sets of the copy of $K_{a,b}$ used to form H are $X = \{x_1, \ldots, x_a\}$ and $Y = \{y_1, \ldots, y_b\}$. Suppose L is an arbitrary $(k + a - 1)$-assignment for H.

Notice that for Theorem 1.6 we are specifically interested in the case in which $M = K_n$ for some $n \geq 2$. Note that K_n is a strongly n-chromatic-choosable graph. If $n \geq a + 1$, $f_a(K_n) = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$ by Theorem 1.5. Therefore, to prove Theorem 1.6, we may suppose from this point forward that $n \leq a$ when $M = K_n$. Also, notice that by Theorem 1.4, $f_a(K_n) \leq \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$. So, proving Theorem 1.6 amounts to showing that if $b < \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$, then H is $(n + a - 1)$-choosable.

2.1 Important Tools

We begin by pointing out certain conditions on L for which it is easy to construct a proper L-coloring of H.

Lemma 2.1 ([9]). *Suppose M is a strongly k-chromatic-choosable graph and $H = M \square K_{a,b}$. Suppose that L is a $(k + a - 1)$-assignment for H such that there exist l, i, and j with $i \neq j$ and $L(v_l, x_i) \cap L(v_l, x_j) \neq \emptyset$. Then, there is a proper L-coloring of H.*

From Lemma 2.1 we immediately get the following observation.

Observation 2.1. *Suppose M is a strongly k-chromatic-choosable graph and $H = M \square K_{a,b}$. Suppose that L is a $(k + a - 1)$-assignment for H such that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for each $i \in [n]$. If H is L-colorable for any such L, then H is $(k + a - 1)$-choosable.*

So, Theorem 1.6 reduces to proving that there is a proper L-coloring of H when $M = K_n$, $b < \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$, and the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for all $i \in [n]$. From now on, we will assume that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for all $i \in [n]$.

Recall that for any $u \in V(M)$ (resp. $u \in X \cup Y$), we let V_u be the subset of $V(H)$ consisting of the vertices with first (resp. second) coordinate u. We also let $H_u = H[V_u]$. Similarly, if $S \subseteq V(M) \cup (X \cup Y)$, we let $V_S = \bigcup_{s \in S} V_s$, $H_S = H[V_S]$, and L_S be the list assignment for H_S obtained by restricting the domain of L to V_S.

Let f be a proper L_X-coloring of H_X. We say f is a *bad coloring for H_{y_i}* if there is no proper L'-coloring for H_{y_i} where L' is the list assignment for H_{y_i} given by $L'(v_j, y_i) = L_{y_i}(v_j, y_i) - \{f(v_j, x_l) : l \in [a]\}$ for each $j \in [n]$. Additionally, we say that f is an *$(n-1)$-to-1 coloring* if for each color q in the range of f, $\left|f^{-1}(q)\right| \leq n-1$.

Our next lemma relates the notion of bad coloring to the existence of a proper L-coloring of H.

Lemma 2.2 ([9]). *Suppose $H = M \square K_{a,b}$ with $a, b \in \mathbb{N}$ and L is a list assignment for H. Suppose $\mathcal{C}_X$ is the set of all proper L_X-colorings of H_X. For each $f \in \mathcal{C}_X$ there exists an $l \in [b]$ such that f is a bad coloring for H_{y_l} if and only if there is no proper L-coloring of H.*

Let $\mathcal{H} = \{H_{y_1}, \ldots, H_{y_b}\}$. Suppose $\mathcal{C}_X$ is the set of all proper L_X-colorings of H_X. The above lemma implies that when H has no proper L-coloring, we can define a function $\mathfrak{F}\colon \mathcal{C}_X \to \mathcal{H}$ such that $\mathfrak{F}(c) = H_{y_l}$, where H_{y_l} is the element of $\mathcal{H}$ with lowest index $l \in [b]$ such that c is a bad coloring for H_{y_l}. Note that if we can show $\left|\mathfrak{F}^{-1}(H_{y_l})\right| \leq q$ for each $l \in [b]$, $\left|\mathcal{C}_X\right|/q \leq b$.

Lemma 2.3 ([9]). *Suppose M is strongly k-chromatic-choosable and $H = M \square K_{a,1}$. Let L be a $(k+a-1)$-assignment for H such that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for each $i \in [n]$. Let $\mathcal{B}$ be the set of proper L_X-colorings of H_X that are bad for H_{y_1}. Then, $|\mathcal{B}| \leq 2^{k-1}$.*

Assuming the same setup as Lemma 2.3, we now prove that if there is an $(n-1)$-to-1 proper L_X-coloring of H_X that is bad for H_{y_1}, then it is in fact the only bad coloring for H_{y_1}.

Lemma 2.4. *Suppose M is strongly k-chromatic-choosable, and $H = M \square K_{a,1}$. Let L be a $(k+a-1)$-assignment for H such that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for each $i \in [n]$. Let $\mathcal{B}$ be the set of proper L_X-colorings of H_X that are bad for H_{y_1}, and let $\mathcal{B}_I$ consist of all the elements of $\mathcal{B}$ that are $(n-1)$-to-1. If $\mathcal{B}_I$ is non-empty, then $|\mathcal{B}_I| = 1$ and $\mathcal{B} = \mathcal{B}_I$.*

Proof. Suppose for the sake of contradiction that there exist two different colorings $c, c' \in \mathcal{B}$ such that $c \in \mathcal{B}_I$. Since M is strongly k-chromatic-choosable and both c and c' are bad for H_{y_1}, there exist two sets of colors K, K', both of size $k-1$, such that for all $i \in [n]$,

$$L(v_i, y_1) - \{c(v_i, x_1), \ldots, c(v_i, x_a)\} = K, \tag{1}$$

and

$$L(v_i, y_1) - \{c'(v_i, x_1), \ldots, c'(v_i, x_a)\} = K'.$$

Note that $K, K' \subseteq L(v_i, y_1)$ for each $i \in [n]$. We will obtain a contradiction when $K = K'$ by showing that $c = c'$. We will obtain a contradiction when $K \neq K'$ by showing that $c \notin \mathcal{B}_I$.

If $K = K'$, then for all $i \in [n]$,

$$\{c(v_i, x_1), \ldots, c(v_i, x_a)\} = \{c'(v_i, x_1), \ldots, c'(v_i, x_a)\}.$$

Fix an arbitrary $j \in [a]$ and $s \in [n]$. Since the lists $L(v_s, x_1), \ldots, L(v_s, x_a)$ are pairwise disjoint, for any $l \in [a]$ such that $l \neq j$, we have $c(v_s, x_j) \neq c'(v_s, x_l)$. Therefore $c(v_s, x_j) = c'(v_s, x_j)$. As j and s were arbitrary, we conclude that $c = c'$.

Now assume $K \neq K'$. Since $|K| = |K'|$, there exists a color $q \in K' - K$. Note that for all $i \in [n]$, $q \in L(v_i, y_1)$ since $K' \subseteq L(v_i, y_1)$. Then, for all $i \in [n]$, $q \in \{c(v_i, x_1), \ldots, c(v_i, x_a)\}$ by (1) since $q \notin K$. This means for each $i \in [n]$, there exists a $j \in [a]$ that satisfies $c(v_i, x_j) = q$. Therefore, $|c^{-1}(q)| \geq n$ contradicting $c \in \mathcal{B}_I$.

Thus, it is impossible to have two different colorings in $\mathcal{B}$ when one of them is in $\mathcal{B}_I$. ☐

Now suppose $H = M \square K_{a,b}$ and L is a $(k+a-1)$-assignment for H that satisfies the conditions we have established. In the next lemma, we give a sufficient condition in terms of a bound on b for there to be a proper L-coloring of H. The bound on b is in terms of the number of proper L_X-colorings of H_X and the number of $(n-1)$-to-1 proper L_X-colorings of H_X.

Lemma 2.5. *Suppose M is strongly k-chromatic-choosable, and $H = M \square K_{a,b}$. Let L be a $(k+a-1)$-assignment for H such that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for each $i \in [n]$. Let $\mathcal{C}_X$ be the set of all proper L_X-colorings of H_X, and let $\mathcal{I}_X \subseteq \mathcal{C}_X$ be the set of $(n-1)$-to-1 colorings in $\mathcal{C}_X$. If*

$$b < |\mathcal{I}_X| + \frac{|\mathcal{C}_X| - |\mathcal{I}_X|}{2^{k-1}}$$

then H has a proper L-coloring.

Proof. We prove the contrapositive. Let $\mathcal{H} = \{H_{y_1}, \ldots, H_{y_b}\}$. Let $\mathfrak{F}\colon \mathcal{C}_X \to \mathcal{H}$ be the function given by $\mathfrak{F}(c) = H_{y_l}$ where H_{y_l} is the element of $\mathcal{H}$ with lowest index $l \in [b]$ such that c is a bad coloring for H_{y_l}. Such a y_l always exists due to Lemma 2.2. We also let $\overline{\mathcal{I}_X} = \mathcal{C}_X - \mathcal{I}_X$.

By Lemma 2.4, when the domain of $\mathfrak{F}$ is restricted to $\mathcal{I}_X$, the resulting function is injective. Thus, $|\mathfrak{F}(\mathcal{I}_X)| = |\mathcal{I}_X|$. On the other hand, Lemma 2.3 implies that for each element in $\mathfrak{F}(\overline{\mathcal{I}_X})$, there are at most 2^{k-1} distinct elements from $\overline{\mathcal{I}_X}$ that are mapped to it. Hence $2^{k-1}\left|\mathfrak{F}(\overline{\mathcal{I}_X})\right| \geq \left|\overline{\mathcal{I}_X}\right|$ implying $\left|\mathfrak{F}(\overline{\mathcal{I}_X})\right| \geq \left|\overline{\mathcal{I}_X}\right|/2^{k-1}$. By Lemma 2.4, the images $\mathfrak{F}(\mathcal{I}_X)$ and $\mathfrak{F}(\overline{\mathcal{I}_X})$ are disjoint. Therefore, $|\mathfrak{F}(\mathcal{C}_X)| = |\mathfrak{F}(\mathcal{I}_X)| + \left|\mathfrak{F}(\overline{\mathcal{I}_X})\right|$ which implies

$$|\mathfrak{F}(\mathcal{C}_X)| \geq |\mathcal{I}_X| + \frac{\left|\overline{\mathcal{I}_X}\right|}{2^{k-1}} = |\mathcal{I}_X| + \frac{|\mathcal{C}_X| - |\mathcal{I}_X|}{2^{k-1}}.$$

On the other hand, since $\mathfrak{F}(\mathcal{C}_X) \subseteq \mathcal{H}$, $|\mathfrak{F}(\mathcal{C}_X)| \leq |\mathcal{H}| = b$; hence, the result follows. □

We can now describe our strategy for proving Theorem 1.6. We suppose $M = K_n$, $n \leq a$, and $H = M \square K_{a,b}$ where $b = P_\ell(M, n+a-1)^a - 1$. Then, we suppose that there is an $(n+a-1)$-assignment L for H for which there is no proper L-coloring of H. Observation 2.1 allows us to assume that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for each $i \in [n]$. We show that under these conditions $P_\ell(M, n+a-1)^a \leq |\mathcal{I}_X| + (|\mathcal{C}_X| - |\mathcal{I}_X|)/2^{n-1}$. Then, Lemma 2.5 implies $f_a(M) \geq P_\ell(M, n+a-1)^a$. This along with Theorem 1.4 implies Theorem 1.6.

2.2 Technical Lemmas

For the remainder of the paper, assume $M = K_n$ and $2 \leq n \leq a$. Note that M is strongly n-chromatic-choosable. Also, assume $H = M \square K_{a,b}$, where $b = P_\ell(K_n, n+a-1)^a - 1 = (\prod_{i=0}^{n-1}(n+a-1-i))^a - 1$. Assume L is an $(n+a-1)$-assignment for H such that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint for each $i \in [n]$. Finally, assume $\mathcal{C}_X$ is the set of all proper L_X-colorings of H_X, and assume $\mathcal{I}_X$ is the set of $(n-1)$-to-1 colorings in $\mathcal{C}_X$.

Lemma 2.6. *Suppose $c \in \mathcal{C}_X$ and s is any color in the range of c. If there are distinct vertices $(v_i, x_j), (v_{i'}, x_{j'}) \in c^{-1}(s)$, then $i \neq i'$ and $j \neq j'$. Consequently, $\left|c^{-1}(s)\right| \leq n$.*

Proof. Suppose there are distinct vertices $(v_i, x_j), (v_{i'}, x_{j'}) \in c^{-1}(s)$. Note that if $j = j'$, then $i \neq i'$ and $c(v_i, x_j) = c(v_{i'}, x_j)$. This contradicts the fact that c is proper. Thus, $j \neq j'$. Finally, if $i = i'$, then $s \in L(v_i, x_j) \cap L(v_i, x_{j'})$ which contradicts the fact that $L(v_i, x_j)$ and $L(v_i, x_{j'})$ are disjoint. □

We now establish some notation. For each $q \in \bigcup_{j=1}^n L(v_1, x_j)$, we let

$$s_q = \begin{cases} 1, & \text{if there exists } c \in \mathcal{C}_X \text{ such that } \left|c^{-1}(q)\right| = n; \\ 0, & \text{otherwise.} \end{cases}$$

Additionally, for each $\mathbf{q} = (q_1, \ldots, q_a) \in \prod_{j=1}^a L(v_1, x_j)$, we let $s(\mathbf{q}) = \sum_{j=1}^a s_{q_i}$. We also define $\mathcal{C}_{X,\mathbf{q}}$ as the set of all proper colorings c of H_X such that for all $j \in [a]$, $c(v_1, x_j) = q_j$, and $\mathcal{I}_{X,\mathbf{q}}$ as the set of $(n-1)$-to-1 proper colorings c of H_X such that for all $j \in [a]$, $c(v_1, x_j) = q_j$.

We can readily observe that

$$\sum_{\mathbf{q} \in \prod_{j=1}^a L(v_1, x_j)} |\mathcal{I}_{X,\mathbf{q}}| = |\mathcal{I}_X| \qquad \text{and} \qquad \sum_{\mathbf{q} \in \prod_{j=1}^a L(v_1, x_j)} |\mathcal{C}_{X,\mathbf{q}}| = |\mathcal{C}_X|.$$

We will use these identities to bound $|\mathcal{C}_X|$ and $|\mathcal{I}_X|$. First, we need a technical lemma.

Lemma 2.7. *Let $\mathbf{q} = (q_1, \ldots, q_a)$ be a fixed element of $\prod_{j=1}^a L(v_1, x_j)$. If $s_{q_t} = 1$ for some $t \in [a]$, then $q_t \notin L(v_i, x_t)$ for each $i \in \{2, \ldots, n\}$.*

Proof. For the sake of contradiction, suppose that $q_t \in L(v_r, x_t)$ for some $r \in \{2, \ldots, n\}$. Since $s_{q_t} = 1$, there exists a coloring $c \in \mathcal{C}_X$ such that $\left|c^{-1}(q_t)\right| = n$ and $c(v_1, x_t) = q_t$. Lemma 2.6 implies $\{i \in [n] :$ there is a $j \in [a]$ such that $c(v_i, x_j) = q_t\} = [n]$. So, there exists a $t' \in [a]$ such that $c(v_r, x_{t'}) = q_t$, and the properness of c implies that $t \neq t'$. Therefore, $L(v_r, x_{t'}) \cap L(v_r, x_t) \neq \emptyset$, a contradiction. □

Lemma 2.8. *Let $\mathbf{q} = (q_1, \ldots, q_a)$ be a fixed element of $\prod_{j=1}^a L(v_1, x_j)$, and let $s = s(\mathbf{q})$. Then*

$$|\mathcal{C}_{X,\mathbf{q}}| \geq a^{a-s}(n+a-1)^s \prod_{i=2}^{n-1}(n+a-i)^a.$$

Proof. We establish the desired bound by giving a lower bound on the number of ways to greedily complete a proper L_X-coloring of H_X, c, where for each $j \in [a]$, $c(v_1, x_j) = q_j$ (i.e., the vertices $(v_1, x_1), \ldots, (v_1, x_a)$ are precolored according to $\mathbf{q}$).

Suppose $j \in [a]$. Now, consider the number of ways we can greedily construct a proper L_{x_j}-coloring of H_{x_j} that colors (v_1, x_j) with q_j. If $s_{q_j} = 1$, then $q_j \notin L(v_i, x_j)$ for each $i \in \{2, \ldots, n\}$ by Lemma 2.7. Consequently, there are at least $\prod_{i=2}^{n}(n + a - i + 1)$ ways to greedily complete a proper L_{x_j}-coloring of H_{x_j}. On the other hand, if $s_{q_j} = 0$, we can only guarantee that there are at least $\prod_{i=2}^{n}(n + a - i)$ ways to greedily complete a proper L_{x_j}-coloring of H_{x_j}. It follows that

$$|\mathcal{C}_{X,\mathbf{q}}| \geq \left(\prod_{i=2}^{n}(n + a - i + 1)\right)^{s} \left(\prod_{i=2}^{n}(n + a - i)\right)^{a-s} = a^{a-s}(n + a - 1)^{s} \prod_{i=2}^{n-1}(n + a - i)^{a}.$$

□

Next, we define an auxiliary graph that we will use in the next two lemmas to get to a lower bound on $|\mathcal{I}_{X,\mathbf{q}}|$.

Definition 2.1. *For each $\mathbf{q} = (q_1, \ldots, q_a)$ in $\prod_{j=1}^{a} L(v_1, x_j)$, we define a graph $M_{\mathbf{q}}$. Let $V(M_{\mathbf{q}}) = V_X$. The edge set of $M_{\mathbf{q}}$ is such that the following conditions hold. For each $j \in [a]$, if $s_{q_j} = 0$, the set $\{(v_i, x_j) \colon i \in [n]\}$ is a clique in $M_{\mathbf{q}}$. Otherwise, (v_1, x_j) is adjacent to each vertex in the set $\{(v_2, x_{j'}) \colon j' \in [a], j' \neq j\}$, and $\{(v_i, x_j) \colon 2 \leq i \leq n\}$ is a clique in $M_{\mathbf{q}}$.*

Lemma 2.9. *Suppose $\mathbf{q} = (q_1, \ldots, q_a) \in \prod_{j=1}^{a} L(v_1, x_j)$. Let $\mathcal{C}'_{\mathbf{q}}$ be the set of proper L_X-colorings c of $M_{\mathbf{q}}$ such that for all $j \in [a]$, $c(v_1, x_j) = q_j$. Then $\mathcal{C}'_{\mathbf{q}} \subseteq \mathcal{I}_{X,\mathbf{q}}$.*

Proof. Suppose f is an arbitrary element of $\mathcal{C}'_{\mathbf{q}}$. We claim that f is an $(n-1)$-to-1 proper L_X-coloring of H_X.

We begin by showing that f is a proper L_X-coloring of H_X. Suppose $i, i' \in [n]$ with $i < i'$ and $j \in [a]$. If $s_{q_j} = 0$, we immediately have that $f(v_i, x_j) \neq f(v_{i'}, x_j)$ since $(v_i, x_j)(v_{i'}, x_j) \in E(M_{\mathbf{q}})$. So, we may suppose that $s_{q_j} = 1$. If $i \geq 2$, we once again have $f(v_i, x_j) \neq f(v_{i'}, x_j)$ since $(v_i, x_j)(v_{i'}, x_j) \in E(M_{\mathbf{q}})$. So, suppose that $i = 1$. Lemma 2.7 tells us that q_j is not in any of the lists: $L(v_2, x_j), \ldots, L(v_n, x_j)$ which implies $f(v_1, x_j) \neq f(v_{i'}, x_j)$. Thus, f is a proper L_X-coloring of H_X.

Finally, we must show that f uses no color more than $(n-1)$ times. For the sake of contradiction, suppose there is a γ such that $|f^{-1}(\gamma)| = n$. By Lemma 2.6, $\{i \in [n] : \text{there is a } j \in [a] \text{ such that } f(v_i, x_j) = \gamma\} = [n]$.

Without loss of generality, suppose $f(v_1, x_1) = \gamma$. This means that $q_1 = \gamma$. Since f is a proper L_X-coloring of H_X, we know $s_\gamma = 1$. We also know there is an $\omega \in [a]$ such that $f(v_2, x_\omega) = \gamma$. By the manner in which the edges of $M_{\mathbf{q}}$ are defined, it must be that $\omega = 1$. However, Lemma 2.7 tells us $\gamma \notin L(v_2, x_1)$ which is a contradiction. □

Lemma 2.10. *Let $\mathbf{q} = (q_1, \ldots, q_a)$ be a fixed element of $\prod_{j=1}^{a} L(v_1, x_j)$, and let $s = s(\mathbf{q})$. Let $\mathcal{C}'_{\mathbf{q}}$ be the set of proper L_X-colorings c of $M_{\mathbf{q}}$ such that for all $j \in [a]$, $c(v_1, x_j) = q_j$. For each $j \in [a]$ let $d_j = |L(v_2, x_j) \cap (\{q_i : s_{q_i} = 1\} \cup \{q_j\})|$. The following statements hold.*

i) We have $d_j \leq s + 1$ for each $j \in [a]$ and $\sum_{j=1}^{a} d_j \leq a$.

ii) It is the case that

$$|\mathcal{C}'_{\mathbf{q}}| \geq \prod_{j=1}^{a}(n + a - 1 - d_j) \left(\prod_{i=3}^{n}(n + a - i + 1)\right)^{s} \left(\prod_{i=3}^{n}(n + a - i)\right)^{a-s}.$$

Proof. The first inequality of Statement (i) follows from the definition of d_j. The second inequality follows from the fact that $L(v_2, x_1), \ldots, L(v_2, x_a)$ are pairwise disjoint and $|\{q_1, \ldots, q_a\}| = a$.

Now, we turn our attention to Statement (ii). We prove our desired bound by describing a procedure for greedily constructing a proper L_X-coloring, c, of $M_{\mathbf{q}}$ and bounding the number of ways each step can be completed.

First, for each $j \in [a]$, let $c(v_1, x_j) = q_j$ (i.e., the vertices $(v_1, x_1), \ldots, (v_1, x_a)$ are colored according to $\mathbf{q}$). This can be done in one way. Then, for each $j \in [a]$ color (v_2, x_j) with some $a_j \in L(v_2, x_j) - (\{q_i : s_{q_i} = 1\} \cup \{q_j\})$. This can be done in at least $\prod_{j=1}^{a}(n + a - 1 - d_j)$ ways. Notice our coloring is now complete if $n = 2$, and the desired bound holds when $n = 2$. So, we may assume $n \geq 3$.

Suppose $j \in [a]$. Now, consider the number of ways we can greedily construct a proper L_{x_j}-coloring of H_{x_j} that colors (v_1, x_j) with q_j and (v_2, x_j) with a_j. If $s_{q_j} = 1$, then $q_j \notin L(v_i, x_j)$ for each $i \in \{3, \ldots, n\}$ by

Lemma 2.7. Consequently, there are at least $\prod_{i=3}^{n}(n+a-1-(i-2))$ ways to greedily complete a proper L_{x_j}-coloring of H_{x_j}. On the other hand, if $s_{q_j}=0$, we can only guarantee that there are at least $\prod_{i=3}^{n}(n+a-1-(i-1))$ ways to greedily complete a proper L_{x_j}-coloring of H_{x_j}. It follows that

$$|\mathcal{C}'_{\mathbf{q}}| \geq \prod_{j=1}^{a}(n+a-1-d_j)\left(\prod_{i=3}^{n}(n+a-i+1)\right)^{s}\left(\prod_{i=3}^{n}(n+a-i)\right)^{a-s}.$$

□

2.2.1 Applying Karamata's Inequality

Lemmas 2.9 and 2.10 yield a lower bound for $|\mathcal{I}_{X,\mathbf{q}}|$ that depends on $d_1,\dots,d_a$. We will now use the celebrated Karamata's Inequality [7] to obtain a lower bound that depends only on $n,a,$ and s. The statement of this inequality first requires a definition. Let $\mathbf{a}=(a_i)_{i=1}^{n}$ and $\mathbf{b}=(b_i)_{i=1}^{n}$ be two finite sequences of real numbers with $n\geq 2$. We say that $\mathbf{a}$ *majorizes* $\mathbf{b}$, written $\mathbf{a}\succ\mathbf{b}$, if the following three conditions hold:

i) $a_1\geq\cdots\geq a_n$ and $b_1\geq\cdots\geq b_n$;

ii) $a_1+\cdots+a_k\geq b_1+\cdots+b_k$ for each $k\in[n-1]$;

iii) $a_1+\cdots+a_n=b_1+\cdots+b_n$.

Lemma 2.11 ([7]). *Suppose $n\geq 2$. Let $\mathbf{a}=(a_i)_{i=1}^{n}$ and $\mathbf{b}=(b_i)_{i=1}^{n}$ be finite sequences of real numbers from an interval $(\alpha,\beta)\subseteq\mathbb{R}$. If $\mathbf{a}\succ\mathbf{b}$ and $f\colon(\alpha,\beta)\to\mathbb{R}$ is a concave function, then*

$$\sum_{i=1}^{n}f(a_i)\leq\sum_{i=1}^{n}f(b_i).$$

Lemma 2.12. *Let n,m,k,C be positive integers such that $n\geq 2$, $m>k$ and $C>k$. Let $\mathcal{X}$ be the set of n-tuples $(x_1,\dots,x_n)\in\mathbb{Z}^n$ such that $0\leq x_i\leq k$ for all $i\in[n]$ and $x_1+\cdots+x_n\leq m$. Suppose $m=kq+r$ where q and r are nonnegative integers such that $0\leq r<k$. Then the following statements hold.*

i) If $n\leq q$, then

$$\min_{(x_1,\dots,x_n)\in\mathcal{X}}\prod_{i=1}^{n}(C-x_i)=(C-k)^n.$$

ii) If $n\geq q+1$, then

$$\min_{(x_1,\dots,x_n)\in\mathcal{X}}\prod_{i=1}^{n}(C-x_i)=(C-k)^q(C-r)C^{n-(q+1)}.$$

Proof. For all $(x_1,\dots,x_n)\in\mathcal{X}$, let

$$P(x_1,\dots,x_n)=\prod_{i=1}^{n}(C-x_i).$$

First, suppose $n\leq q$. Let $\mathbf{y}=(y_1,\dots,y_n)$ be the n-tuple given by $y_i=k$ for each $i\in[n]$. Since $kn\leq kq\leq m$, $\mathbf{y}\in\mathcal{X}$. For any $(x_1,\dots,x_n)\in\mathcal{X}$, $C-x_i\geq C-k$ for each $i\in[n]$. So,

$$P(x_1,\dots,x_n)\geq P(\mathbf{y})=(C-k)^n$$

which completes the proof of Statement (i).

From now on, suppose $n\geq q+1$. We claim that the minimum value of $P(x_1,\dots,x_n)$ over $\mathcal{X}$ is attained at a point $(z_1,\dots,z_n)$ in which $z_1+\cdots+z_n=m$. Note that if $z_1+\cdots+z_n<m$, there is a $t\in[n]$ such that $z_t<k$; otherwise,

$$m>z_1+\cdots+z_n=kn\geq kq+k>kq+r=m$$

which is a contradiction. Let $(w_1,\dots,w_n)$ be the n-tuple given by $w_i=z_i$ when $i\neq t$ and $w_t=z_t+1$. Then $(w_1,\dots,w_n)\in\mathcal{X}$, since $w_t=z_t+1\leq k$ and $w_1+\cdots+w_n\leq m$. Furthermore,

$$P(w_1,\dots,w_n)<P(z_1,\dots,z_n).$$

Therefore, if $z_1+\cdots+z_n<m$, the minimum value of P is not attained at $(z_1,\dots,z_n)$. Hence, the minimum occurs at an element in the set $\mathcal{X}'$, where $\mathcal{X}'$ is the set of all elements of $\mathcal{X}$ whose coordinates sum to m.

Let $f(x) = \log(C - x)$ for all $x \in (-1, C)$. Suppose $(x_1, \ldots, x_n) \in \mathcal{X}$. Since $x_i \le k < C$ for all $i \in [n]$, $f(x_i)$ is real for each $i \in [n]$. Also, for each $(x_1, \ldots, x_n) \in \mathcal{X}$, $\log P(x_1, \ldots, x_n) = \sum_{i=1}^{n} f(x_i)$.

Consider the n-tuple $(x_1^*, \ldots, x_n^*)$ where $x_i^* = k$ for each $i \in [q]$, $x_{q+1}^* = r$, and $x_i^* = 0$ otherwise. Additionally, for a given $(x_1, \ldots, x_n) \in \mathcal{X}'$, let $x_1', \ldots, x_n'$ be an ordering of the numbers $x_1, \ldots, x_n$ such that $x_1' \ge \cdots \ge x_n'$. We claim $(x_1^*, \ldots, x_n^*) \succ (x_1', \ldots, x_n')$. To see why, note: for all $l \in [q]$, $\sum_{i=1}^{l} x_i^* = kl \ge \sum_{i=1}^{l} x_i'$, for all $l \in \{q+1, \ldots, n-1\}$ $\sum_{i=1}^{l} x_i^* = m \ge \sum_{i=1}^{l} x_i'$, and $\sum_{i=1}^{n} x_i^* = m = \sum_{i=1}^{n} x_i'$.

Since $f(x)$ is concave on $(-1, C)$, Karamata's inequality yields

$$\sum_{i=1}^{n} f(x_i) = \sum_{i=1}^{n} f(x_i') \ge \sum_{i=1}^{n} f(x_i^*) = q \log(C - k) + \log(C - r) + (n - q - 1) \log C$$

which implies

$$P(x_1, \ldots, x_n) \ge P(x_1^*, \ldots, x_n^*) = (C - k)^q (C - r) C^{n-q-1}$$

as desired. □

Lemma 2.13. *Let $\mathbf{q} = (q_1, \ldots, q_a)$ be a fixed element of $\prod_{j=1}^{a} L(v_1, x_j)$, and let $s = s(\mathbf{q})$. Then*

$$|\mathcal{I}_{X,\mathbf{q}}| \ge (n + a - s - 2)^{\frac{a}{s+1}} (n + a - 1)^{\frac{sa}{s+1}} \left(\prod_{i=3}^{n} (n + a - i + 1) \right)^{s} \left(\prod_{i=3}^{n} (n + a - i) \right)^{a-s}.$$

Proof. Using the notation of Lemma 2.10, by Statement (ii) of Lemma 2.10 it suffices to show that

$$\prod_{j=1}^{a} (n + a - 1 - d_j) \ge (n + a - s - 2)^{\frac{a}{s+1}} (n + a - 1)^{\frac{sa}{s+1}}.$$

Suppose $a = (s + 1)q + r$, where q and r are nonnegative integers such that $0 \le r < s + 1$. Suppose $s = 0$. Then, Statement (i) of Lemma 2.10 and Lemma 2.12 imply

$$\prod_{j=1}^{a} (n + a - 1 - d_j) \ge (n + a - 2)^a$$

as desired. Now suppose $0 < s \le a$. Statement (i) of Lemma 2.10 and Lemma 2.12 imply

$$\prod_{j=1}^{a} (n + a - 1 - d_j) \ge (n + a - 2 - s)^q (n + a - 1 - r)(n + a - 1)^{a-(q+1)}.$$

By the AM-GM inequality,

$$n + a - 1 - r = \frac{r(n + a - 2 - s) + (s + 1 - r)(n + a - 1)}{s + 1} \ge (n + a - 2 - s)^{\frac{r}{s+1}} (n + a - 1)^{\frac{s+1-r}{s+1}}.$$

Therefore,

$$\begin{aligned}\prod_{j=1}^{a} (n + a - 1 - d_j) &\ge (n + a - 2 - s)^{q + \frac{r}{s+1}} (n + a - 1)^{a - q - \frac{r}{s+1}} \\ &= (n + a - s - 2)^{\frac{a}{s+1}} (n + a - 1)^{\frac{sa}{s+1}}\end{aligned}$$

as desired. □

2.3 Completing the proof of Theorem 1.6

In this section we will prove that $P_\ell(M, n + a - 1)^a \le |\mathcal{I}_X| + (|\mathcal{C}_X| - |\mathcal{I}_X|)/2^{n-1}$ which by Lemma 2.5 will complete our proof of Theorem 1.6. Note that $P_\ell(M, n + a - 1)^a = \prod_{i=1}^{n} (n + a - i)^a$, and

$$\frac{|\mathcal{C}_X| + (2^{n-1} - 1) |\mathcal{I}_X|}{2^{n-1}} = \frac{1}{2^{n-1}} \sum_{\mathbf{q} \in \prod_{j=1}^{a} L(v_1, x_j)} (|\mathcal{C}_{X,\mathbf{q}}| + (2^{n-1} - 1) |\mathcal{I}_{X,\mathbf{q}}|).$$

Since the sum on the right has $(n+a-1)^a$ terms, showing that $|\mathcal{C}_{X,\mathbf{q}}|+(2^{n-1}-1)\,|\mathcal{I}_{X,\mathbf{q}}| \geq 2^{n-1}\prod_{i=2}^{n}(n+a-i)^a$ for each $\mathbf{q} \in \prod_{j=1}^{a} L(v_1, x_j)$ will imply the desired inequality. If $s = s(\mathbf{q})$ for some $\mathbf{q} \in \prod_{j=1}^{a} L(v_1, x_j)$, Lemmas 2.8 and 2.13 along with some simplification tell us that proving

$$\left(1+\frac{n-1}{a}\right)^s + (2^{n-1}-1)\left[\left(1-\frac{s}{n+a-2}\right)\left(1+\frac{1}{n+a-2}\right)^s\right]^{\frac{a}{s+1}}\left(1+\frac{n-2}{a}\right)^s \geq 2^{n-1} \tag{2}$$

will imply $|\mathcal{C}_{X,\mathbf{q}}| + (2^{n-1}-1)\,|\mathcal{I}_{X,\mathbf{q}}| \geq 2^{n-1}\prod_{i=2}^{n}(n+a-i)^a$. So, our goal is to prove (2) whenever $2 \leq n \leq a$ and $0 \leq s \leq a$.

We will first prove (2) when $n \geq 3$. Our first lemma shows that when s is large, the first term on the left side (2) is large enough to justify (2).

Lemma 2.14. *Suppose $a \geq n \geq 3$ and $s > 0.73(n+a-2)$. Then,*

$$\left(1+\frac{n-1}{a}\right)^s > 2^{n-1}.$$

Proof. We will use the following well-known inequality ([13], page 267): for any real numbers $x, n > 0$ it is the case that

$$\left(1+\frac{x}{n}\right)^{n+\frac{x}{2}} \geq e^x.$$

Since $n+a-2 \geq a+(n-1)/2$, the above inequality implies

$$\left(1+\frac{n-1}{a}\right)^{n+a-2} \geq \left(1+\frac{n-1}{a}\right)^{a+(n-1)/2} \geq e^{n-1}.$$

Therefore,

$$\left(1+\frac{n-1}{a}\right)^s > \left(1+\frac{n-1}{a}\right)^{0.73(n+a-2)} \geq e^{0.73(n-1)} > 2^{n-1}.$$

□

Suppose $x \in \mathbb{R}$. One can easily show with basic calculus-based arguments that $\ln(1+x) \geq x - x^2/2$ whenever $x \geq 0$ and $\ln(1+x) \geq x - 1.1x^2$ whenever $x \geq -0.73$. Also, by the AM-GM Inequality,

$$\left(1+\frac{n-1}{a}\right)^s + (2^{n-1}-1)\left[\left(1-\frac{s}{n+a-2}\right)\left(1+\frac{1}{n+a-2}\right)^s\right]^{\frac{a}{s+1}}\left(1+\frac{n-2}{a}\right)^s \geq 2^{n-1}$$

$$\left[\left(1+\frac{n-1}{a}\right)^{s(s+1)}\left(1+\frac{n-2}{a}\right)^{(2^{n-1}-1)s(s+1)}\left[\left(1-\frac{s}{n+a-2}\right)\left(1+\frac{1}{n+a-2}\right)^s\right]^{(2^{n-1}-1)a}\right]^{\frac{1}{(s+1)2^{n-1}}}.$$

We will need these facts to handle small values of s when $n \geq 3$.

Lemma 2.15. *Suppose $n \geq 3$ and $0 \leq s \leq 0.73(n+a-2)$. Then*

$$\left(1+\frac{n-1}{a}\right)^{s(s+1)}\left(1+\frac{n-2}{a}\right)^{(2^{n-1}-1)s(s+1)}\left[\left(1-\frac{s}{n+a-2}\right)\left(1+\frac{1}{n+a-2}\right)^s\right]^{(2^{n-1}-1)a} \geq 1.$$

Notice that Lemma 2.14, along with our application of the AM-GM inequality and Lemma 2.15 will imply (2) when $n \geq 3$.

Proof. Clearly, the desired inequality is equivalent to

$$\begin{aligned} &s(s+1)\ln\left(1+\frac{n-1}{a}\right) + (2^{n-1}-1)s(s+1)\ln\left(1+\frac{n-2}{a}\right) \\ &+ (2^{n-1}-1)a\ln\left(1-\frac{s}{n+a-2}\right) + (2^{n-1}-1)as\ln\left(1+\frac{1}{n+a-2}\right) \geq 0. \end{aligned}$$

We will now prove this inequality for $n \geq 4$, and then we will handle $n = 3$ separately. Let $M = 2^{n-1} - 1$ and $b = s(s+1)$. For $n \geq 4$, we see

$$\begin{aligned}
&s(s+1)\ln\left(1+\frac{n-1}{a}\right) + (2^{n-1}-1)s(s+1)\ln\left(1+\frac{n-2}{a}\right)\\
&+ (2^{n-1}-1)a\ln\left(1-\frac{s}{n+a-2}\right) + (2^{n-1}-1)as\ln\left(1+\frac{1}{n+a-2}\right)\\
&\geq Mb\ln\left(1+\frac{n-2}{a}\right) + Ma\left(\ln\left(1-\frac{s}{n+a-2}\right) + s\ln\left(1+\frac{1}{n+a-2}\right)\right)\\
&\geq Mb\left(\frac{n-2}{a} - \frac{(n-2)^2}{2a^2}\right) + Ma\left(\left(\frac{-s}{n+a-2} - \frac{1.1s^2}{(n+a-2)^2}\right) + \left(\frac{s}{n+a-2} - \frac{s}{2(n+a-2)^2}\right)\right)\\
&\geq Mb\left(\frac{n-2}{a} - \frac{(n-2)^2}{2a^2}\right) + Ma\left(\frac{-1.1b}{(n+a-2)^2}\right)\\
&= Mb\left(\frac{(10n-31)a^3 + 5(n-2)^2(3a^2-(n-2)^2)}{10a^2(a+n-2)^2}\right) \geq 0
\end{aligned}$$

as desired. Now, suppose that $n = 3$ and $b = s(s+1)$. Then,

$$\begin{aligned}
&s(s+1)\ln\left(1+\frac{2}{a}\right) + 3s(s+1)\ln\left(1+\frac{1}{a}\right) + 3a\ln\left(1-\frac{s}{a+1}\right) + 3as\ln\left(1+\frac{1}{a+1}\right)\\
&\geq b\left(\frac{2}{a} - \frac{2}{a^2} + \frac{3}{a} - \frac{3}{2a^2}\right) + 3a\left(\frac{-1.1b}{(a+1)^2}\right)\\
&= b\left(\frac{a(17a^2-20) + 5(13a^2-7)}{10a^2(a+1)^2}\right) \geq 0
\end{aligned}$$

as desired. $\square$

Finally, to complete our proof of Theorem 1.6, we must prove (2) when $n = 2$. In particular, we must prove the following inequality.

Lemma 2.16. *For $a \geq 2$ and $0 \leq s \leq a$,*

$$\left(1+\frac{1}{a}\right)^s + \left[\left(1-\frac{s}{a}\right)\left(1+\frac{1}{a}\right)^s\right]^{\frac{a}{s+1}} \geq 2.$$

Proof. Throughout the proof, let $r = s/a$, and note that $0 \leq r \leq 1$. Using $\ln(1+x) \geq x - x^2/2$ for positive real values of x, we obtain

$$\left(1+\frac{1}{a}\right)^s \geq e^{r-\frac{r}{2a}} \geq e^{0.75r}.$$

Now, for some real $\tau \in (0.5, \infty)$ notice that if r is such that $\ln(1-r) \geq -r - \tau r^2$, we have

$$\left[\left(1-\frac{s}{a}\right)\left(1+\frac{1}{a}\right)^s\right]^{\frac{a}{s+1}} \geq e^{\frac{a}{s+1}\left(-r-\tau r^2\right)} e^{\frac{a}{s+1}\left(r-\frac{r}{2a}\right)} = e^{\frac{1}{s+1}\left(-\tau rs - \frac{r}{2}\right)} = e^{\frac{-\tau}{s+1}\left(rs + \frac{r}{2\tau}\right)} \geq e^{\frac{-\tau}{s+1}(rs+r)} = e^{-\tau r}.$$

So, whenever r is such that $\ln(1-r) \geq -r - \tau r^2$,

$$\left(1+\frac{1}{a}\right)^s + \left[\left(1-\frac{s}{a}\right)\left(1+\frac{1}{a}\right)^s\right]^{\frac{a}{s+1}} \geq e^{0.75r} + e^{-\tau r}.$$

Finally, one can use basic ideas from calculus to verify the following facts. When $r \geq 0.93$, $e^{0.75r} \geq 2$. When $0.78 \leq r \leq 0.93$, $\ln(1-r) \geq -r - 2r^2$ and $e^{0.75r} + e^{-2r} \geq 2$. When $0.53 \leq r \leq 0.78$, $\ln(1-r) \geq -r - 1.25r^2$ and $e^{0.75r} + e^{-1.25r} \geq 2$. When $0.34 \leq r \leq 0.53$, $\ln(1-r) \geq -r - r^2$ and $e^{0.75r} + e^{-r} \geq 2$. Finally, when $0 \leq r \leq 0.34$, $\ln(1-r) \geq -r - 0.75r^2$ and $e^{0.75r} + e^{-0.75r} \geq 2$. This completes the proof. $\square$

Having proven (2), we are ready to bring all the ingredients together and give a short proof of Theorem 1.6, which we restate.

Theorem 1.6. *For each $n, a \in \mathbb{N}$, $\chi_\ell(K_n \square K_{a,b}) = n + a$ if and only if $b \geq (n+a-1)!^a/(a-1)!^a$. That is, $f_a(K_n) = \left(\frac{(n+a-1)!}{(a-1)!}\right)^a$ for each $n, a \in \mathbb{N}$.*

Proof. By Theorem 1.4, $f_a(K_n) \leq P_\ell(M, n+a-1)^a$. Thus, it suffices to show that $f_a(K_n) \geq P_\ell(M, n+a-1)^a$; that is, we want to show that if $M = K_n$ and $H = M \square K_{a,b}$, where $b = P_\ell(M, n+a-1)^a - 1$, then $\chi_\ell(M \square K_{a,b}) \leq n + a - 1$.

Suppose for the sake of contradiction that there exists an $(n + a - 1)$-assignment L for H for which there is no proper L-coloring of H. By Observation 2.1, we may assume that the lists $L(v_i, x_1), \ldots, L(v_i, x_a)$ are pairwise disjoint, for all $i \in [n]$. By (2),

$$|\mathcal{I}_X| + \frac{|\mathcal{C}_X| - |\mathcal{I}_X|}{2^{n-1}} \geq P_\ell(M, n+a-1)^a > b.$$

By Lemma 2.5, there is a proper L-coloring of H which is a contradiction. □

References

[1] A. Bernshteyn, H. Kaul, J. A. Mudrock, G. Sharma, *On strongly and robustly critical graphs*, preprint, arXiv: 2408.04538.

[2] G. D. Birkhoff, *A determinant formula for the number of ways of coloring a map*, Ann. of Math, 14 (1912), 42–46.

[3] M. Borowiecki, S. Jendrol, D. Král, and J. Miškuf, *List coloring of cartesian products of graphs*, Discrete Math. 306 (2006), 1955–1958.

[4] F. Dong and M. Zhang, *How large can $P(G, L) - P(G, k)$ be for k-assignments L*, J. Comb. Theory Ser. B 161 (2023), 109–119.

[5] Q. Donner, *On the number of list-colorings*, J. Graph Theory 16 (1992), 239–245.

[6] P. Erdős, A. L. Rubin, and H. Taylor, *Choosability in graphs*, Congr. Numer. 26 (1979), 125–127.

[7] Z. Kadelburg, D. Ðukić, M. Lukić, and I. Matić, *Inequalities of Karamata, Schur and Muirhead, and some applications*, Teaching of Mathematics 8(1) (2005), 31–45.

[8] H. Kaul and J. Mudrock, *Criticality, the list color function, and list coloring the Cartesian product of graphs*, J. Combin. 12 (2021) 479–514.

[9] H. Kaul and J. Mudrock, *List coloring a Cartesian product with a complete bipartite factor*, Graphs Combin. 35 (2019), 1571–1583.

[10] H. Kaul and J. Mudrock, em Counting packings of list-colorings of graphs, Enumer. Combin. Appl. 5:3 (2025), Article S2R20.

[11] R. Kirov and R. Naimi, *List coloring and n-monophilic graphs*, Ars Combin. 124 (2016), 329–340.

[12] A. V. Kostochka and A. Sidorenko, *Problem session*, Fourth Czechoslovak Symposium on Combinatorics, Prachatice, June 1990.

[13] D. S. Mitrinovic, *Analytic Inequalities*, Springer-Verlag, New York, 1970.

[14] J. A. Mudrock, *Problem 6*, Sparse Graphs Coalition Workshop Series, 2024: Algebraic, extremal, and structural methods and problems in graph colouring, online, February 2024. `https://sparse-graphs.mimuw.edu.pl/doku.php?id=sessions:2024sessions:2024session1`

[15] R. C. Read, *An introduction to chromatic polynomials*, J. Comb. Theory 4 (1968), 52–71.

[16] C. Thomassen, *The chromatic polynomial and list colorings*, J. Comb. Theory Ser. B 99 (2009), 474–479.

[17] V. G. Vizing, *Coloring the vertices of a graph in prescribed colors*, Diskret. Analiz. no. 29, Metody Diskret. Anal. v Teorii Kodovi Skhem 101 (1976), 3–10.

[18] W. Wang, J. Qian, and Z. Yan, *When does the list-coloring function of a graph equal its chromatic polynomial*, J. Comb. Theory Ser. B 122 (2017), 543–549.

[19] D. B. West, *Introduction to Graph Theory.* Upper Saddle River, NJ: Prentice Hall, 2001.