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A Combinatorial Model for Lane Merging: Limited Capacity Lanes

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ABSTRACT: A one-lane road briefly expands to two lanes at a stoplight before merging back into one. Some drivers always choose the right lane, while others pick the shorter lane, favoring the right in a tie. In our previous paper, we represented arrival sequences as binary strings that created lattice paths, and analyzed how many vehicles end up in the left lane, drawing heavily on connections to ballot paths. Presently, we consider limited lane capacities: when the right lane fills, a gap can form in the left lane. We calculate the expected size of this gap based on the proportion of each driver type, including new bijections.

Keywords: Ballot numbers; Bijections; Lattice paths; Traffic
2020 Mathematics Subject Classification: 05A15; 05A19

1. Introduction and background

Imagine you are driving on a one-lane road that becomes two lanes, where there is a stoplight, and soon after, the left lane will have to merge into the right. Some drivers will stay in the right lane before the traffic light, regardless of its length. Others will choose the shortest lane, giving preference to the right lane when the lengths are equal.

We model this situation using a binary string called an *arrival sequence*, assuming cars approach the stoplight one at a time with plenty of time to choose their preferred lane. Cars that do not want to merge, and that will always choose the right lane, are denoted with 0 and colored red in diagrams. Cars that prefer the shortest lane (with ties going to the right lane) are denoted by 1 and colored green.

In Figure 1, we illustrate an arrival sequence in the capacity-constrained model: the two-lane segment can hold a fixed number of cars (the capacity m defined below), and once the right lane reaches capacity, subsequent cars cannot enter the left lane. If the arrival sequence is $\mathbf{b} = 0011\underline{1}0011$, then the first car will always choose the right lane, no matter what. In this case, Car 2 is red and will also choose the right lane. The next three cars are green; two will choose the left lane, and the third will choose the right, as the lanes will be equal in length at that point. (Green cars that end up in the right lane will appear as underlined blue digits in arrival sequences throughout the paper, for extra clarity.) Cars 6 and 7 are red and will choose the right lane. Finally, Cars 8 and 9 are green and would like to choose the left lane, but the right lane has reached capacity, so they are stuck behind Car 7. Here we say the lanes have a *capacity* of $m = 5$ and this arrival sequence gives a *gap* of size $g = 3$. We give the following definitions to make the discussion easier.

Definition 1.1. *Fix a lane capacity $m \geq 1$. The **capacity** m is the maximum number of cars that can occupy the* left lane *after the split. An arrival sequence gives a **gap** of size g if the number of cars that reach the left lane is $m - g$.*

We only consider arrival sequences of length $\ell = 2m - 1$ since the maximal number of cars that can end up in the left lane is $m - 1$. Notice this also means that $g \geq 1$.

Definition 1.2. *The car that **fills the right lane to capacity** is called the **block** (or **capacity car**). The cars that arrive after the block and are unable to enter the right lane form the **queue**.*

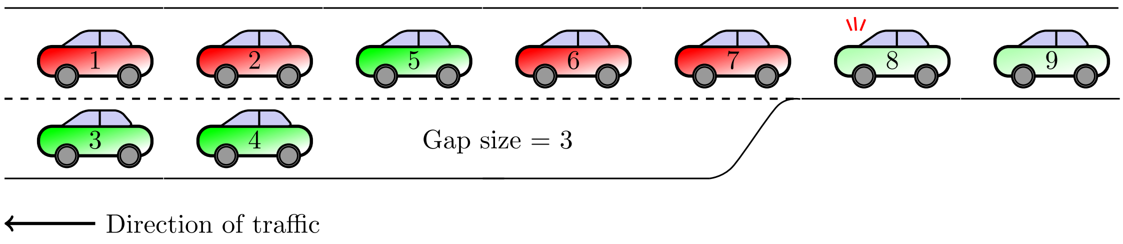

Figure 1: Nine cars waiting to merge for arrival sequence $\mathbf{b} = 001110011$.

In Figure 1, Car 7 is the block, and the queue contains Cars 8 and 9. Notice the queue is always of size $g-1$, since the moment the block occurs, there are m cars in the right lane and $m-g$ cars in the left lane, so exactly $(2m-1)-(m+(m-g)) = g-1$ cars remain, and these are precisely the cars in the queue.

In earlier work [3], we computed the expected number of cars in the left lane given an unrestricted merging model. In this paper, we extend that model by introducing lane capacity constraints – a realistic feature of traffic systems where a fixed number of cars can occupy the two-lane segment near the intersection. This constraint leads to the formation of a gap in the left lane once the right lane fills. In the example above, the gap size is three because the left-lane capacity is $m=5$, but only two green cars made it into the left lane before the red car labeled 7 blocked subsequent green cars from choosing the left lane.

In order to find the expected gap size, we give the following definition.

Definition 1.3. *If $\boldsymbol{G}_{m,g}$ is the set of merging paths of length $2m-1$ with capacity m, and gap length g, then let $G(m,g) = |\boldsymbol{G}_{m,g}|$ be the number of such paths and*

$$G(m) = \sum_{g=1}^{m} g \cdot G(m,g).$$

So the expected gap size is

$$\mathbb{E}_m(g) = \frac{G(m)}{2^{2m-1}}.$$

Example 1.1. *For $m = 3, 4$, the histogram in Figure 2 displays how their gap sizes are distributed.*

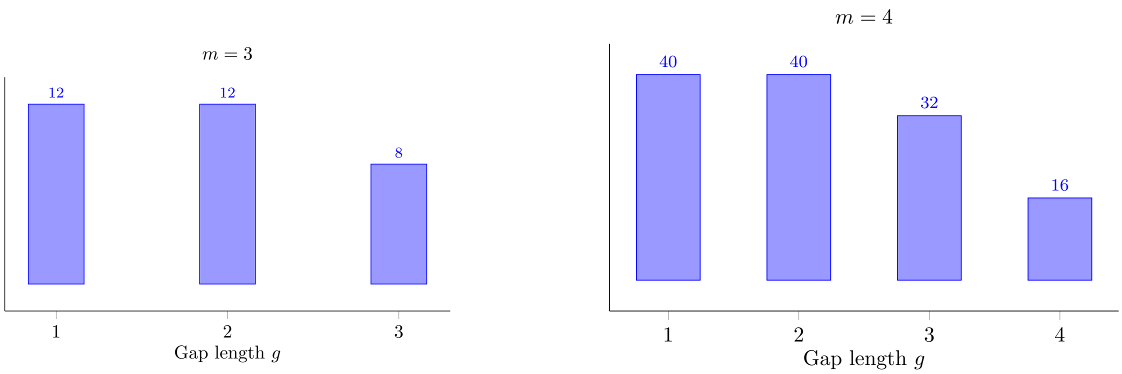

Figure 2: Histogram for $G(m,g)$ where $m=3$ and $m=4$.

So, the expected gap size for $m=3$ is

$$\mathbb{E}_3(g) = \frac{1\cdot 12 + 2\cdot 12 + 3\cdot 8}{12+12+8} = \frac{12+24+24}{32} = \frac{60}{32} = 1.875.$$

and when $m=4$

$$\mathbb{E}_4(g) = \frac{1\cdot 40 + 2\cdot 40 + 3\cdot 32 + 4\cdot 16}{40+40+32+16} = \frac{40+80+96+64}{128} = \frac{280}{128} = 2.1875.$$

Our first main result is **Corollary 3.1**, which answers **Open Question 39** from our previous paper [3]. There, we asked for the expected size of the gap that forms when lane capacity is imposed on the merging model. This corollary provides an exact formula for the expected gap size and shows that it grows asymptotically like

$$\mathbb{E}_m(g) \sim 2\sqrt{m/\pi}.$$

The result highlights a surprising regularity: despite the stochastic arrival process and dynamic lane choices, the average gap exhibits a clean square-root behavior in terms of lane capacity. Our other main results include (1) several closed formulas and recurrences for the number of gap-inducing sequences with fixed red car count, (2) a surprising correspondence between merging paths and classical ballot paths, and (3) a refined enumeration of sequences by gap size and bounce structure.

The remainder of the paper is organized as follows. In Section 2, we formalize the connection between arrival sequences and merging paths, a family of lattice paths which provide a geometric visualization of how gaps form. Section 3 focuses on enumerative results for fixed lane capacities, culminating in an exact and asymptotic formula for the expected gap size. In Section 4, we refine our analysis by fixing the number of red cars in the arrival sequence and explore the behavior of the function $G(m, k, g)$. Section 5 develops connections between merging paths and classical ballot paths, leading to several closed formulas and bijective proofs. In Section 6, we consider refined counts based on the number of bounces in a merging path and derive further structural results. Finally, Section 7 investigates asymptotic behavior and gives exact expressions for the expected gap size when the number of red cars is fixed. We conclude with open problems and conjectures in Section 8.

Building on the rich history of lattice path enumeration, Sections 2 and 5 introduce two important families: ballot paths and merging paths. Ballot paths are lattice paths that never dip below the diagonal and count the number of ways to tally a two-candidate election so that the leading candidate stays ahead throughout the count [1, 2, 12, 13]. Merging paths, as defined in our previous paper [3], generalize ballot paths. For a comprehensive history of lattice path enumeration, see Humphreys' survey [5], which covers applications ranging from games [16] and electrostatics [9] to number theory [8,14] and statistics [6,11]. There is also a well-developed body of literature on bijections between lattice paths and other combinatorial structures [7, 10, 15].

2. Merging paths and gap formation

Definition 2.1. *To each arrival sequence, we can assign a (decorated) lattice path, which we call a* ***merging path****. This path is created by assigning an up-step to each red car (0), a right-step to each green car (1) when the lanes are uneven, and a decorated up-step (**bounce**) for a green car when the lanes are even.*

For instance, the merging path for the arrival sequence $\mathbf{b} = 00111001$ is also shown in Figure 3. When a green car ends up in the right lane, we say that the merging path *bounces* off the diagonal, and we decorate the corresponding up-step by highlighting it with a bold blue arrow. Merging paths with no bounces are the famous *ballot paths* (i.e., lattice paths that do not cross below the diagonal). Hence, merging paths generalize the ballot paths. Ballot paths are used to enumerate the number of ways the ballots in a two-candidate election can be counted so that the winning candidate is never trailing.

If an arrival sequences gives a gap of size g, then when the block occurs, there are m cars in the right lane (each contributing an up-step to the merging path) and $m - g$ cars in the left lane (each contributing a right-step to the merging path). Moreover, the block must be either a red car or a bouncing green car, and so we have proven the following.

Proposition 2.1. *If* $\mathbf{b}$ *is an arrival sequence that gives a gap of* g*, then the corresponding merging path reaches the point* $(m - g, m)$ *with an up-step, where* m *is the lane capacity. In other words, it passes through the point* $(m - g, m - 1)$.

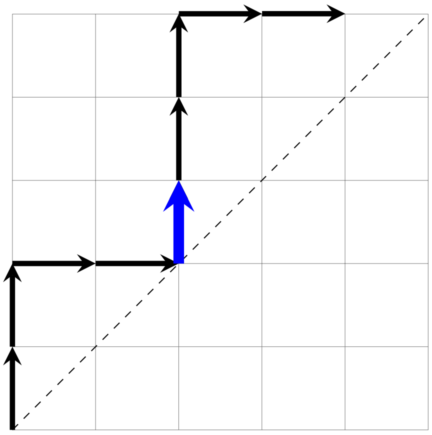

Figure 3: Merging path for the arrival sequence $\mathbf{b} = 001110011$.

We will calculate the expected gap size over all arrival sequences of length $2m - 1$ in Corollary 3.1. The following theorem calculates the numerator of that expected value calculation.

3. Expected gap size

Theorem 3.1. *Let $G(m)$ be the sum of the gaps for the merging sequences with capacity m and length $2m-1$. Then*

$$G(m) = \sum_{g=1}^{m} g2^g \binom{2m-g-1}{m-g} = m\binom{2m}{m}.$$

Proof. The merging path must reach $(m-g, m-1)$ and the $g-1$ cars in the queue can be any set of green and red cars, so the number of arrival sequences in $\mathbf{G}_{m,g}$ is $M_{m-g}(m-1)2^{g-1}$ where $M_x(y)$ is the number of merging paths reaching (x, y). From Theorem 6 of [3] we know the numbers $M_x(y)$ have the following closed formulas:

$$M_x(y) = 2\binom{x+y}{x} \text{ for } x < y, \text{ and } M_y(y) = \binom{2y}{y}.$$

Substituting, multiplying by g, and summing over $g \geq 1$, we have

$$G(m) = \binom{2(m-1)}{m-1} + \sum_{g=2}^{m} g2^g \binom{2m-g-1}{m-g} = \sum_{g=1}^{m} g2^g \binom{2m-g-1}{m-g}.$$

We just proved $G(m) = \sum\limits_{g=1}^{m} g2^g \binom{2m-g-1}{m-g}$. To prove that $G(m) = m\binom{2m}{m}$ we will now consider the generating function

$$F(x, y) = \sum_{n,m\geq 0} \left[\sum_{g=1}^{m} g2^g \binom{n+m-g-1}{m-g}\right] x^m y^n = \frac{2x(1-x)}{(1-2x)^2(1-x-y)}.$$

We are interested in the diagonal of this generating function, that is, the coefficient of $x^m y^m$. A method for extracting the diagonal of a generating function can be found in Section 6.3 of Stanley's text [15][p. 179]. Briefly, create the Laurent series $G(t, s) = F(s, t/s)$ so diag $F = [s^0]G$. This often requires a partial fraction decomposition of G, regarded as a function of s, and using Cauchy's Integration Theorem. Thus,

$$\text{diag } F = [s^0]F(s, t/s) = \frac{1}{2\pi i}\int_{|s|=\rho} F(s, t/s)\frac{ds}{s},$$

where G converges on some circle $|s| = \rho > 0$. Applying this method to F, we obtain

$$\text{diag } F = \frac{2t}{(1-4t)^{3/2}}.$$

Finally, we extract the coefficient of t^m to conclude that

$$G(m) = \sum_{g=1}^{m} g2^g \binom{2m-g-1}{m-g} = m\binom{2m}{m}. \qquad \square$$

The following corollary is immediate after dividing the result in Theorem 3.1 by 2^{2m-1} and using Stirling's Approximation.

Corollary 3.1. *The expected gap size $\mathbb{E}_m(g)$ for all arrival sequences of length $2m-1$ and capacity m is*

$$\mathbb{E}_m(g) = \frac{m\binom{2m}{m}}{2^{2m-1}} \sim 2\sqrt{m/\pi}.$$

4. Arrival sequences with exactly k red cars

In this section, we refine our analysis by fixing the number of red cars in the arrival sequence. Specifically, we classify all sequences of length $2m-1$ with exactly k red cars (0s) according to the gap size they produce, and we study the behavior of the function $G(m, k, g)$, which counts how many such sequences yield a gap of size g.

Definition 4.1. *Let $\mathbf{G_{m,k,g}}$ be the set of arrival sequences of length $2m-1$ containing k 0s, that give a gap of g. Let $G(m, k, g) = |\mathbf{G_{m,k,g}}|$, the number of such arrival sequences.*

g	0	1	2	3	4	5	6	7	8	9	10	11	
2		1	2	1									$m = 2$
1	1	2	1										
3			1	3	3	1							$m = 3$
2		1	4	5	2								
1	1	4	5	2									
4				1	4	6	4	1					$m = 4$
3			1	6	12	10	3						
2		1	6	14	14	5							
1	1	6	14	14	5								
5					1	5	10	10	5	1			$m = 5$
4				1	8	22	28	17	4				
3			1	8	27	43	32	9					
2		1	8	27	48	42	14						
1	1	8	27	48	42	14							
6						1	6	15	20	15	6	1	$m = 6$
5					1	10	35	60	55	26	5		
4				1	10	44	96	109	62	14			
3			1	10	44	110	151	104	28				
2		1	10	44	110	165	132	42					
1	1	10	44	110	165	132	42						
	0	1	2	3	4	5	6	7	8	9	10	11	k

Table 1: The values of $G(m, k, g)$ from $m = 2$ to $m = 6$.

The following table gives the nonzero values of $G(m, k, g)$ up to $m = 6$.

Theorem 4.1. *For any $m, k > 0$,*

$$G(m, k, m-1) = \binom{m}{k-m+1}.$$

Proof. A gap of size $m-1$ can only occur if the arrival sequence begins with m red cars or a green car followed by $m-1$ red cars. In the first case, there are $k-m$ red cars among the $m-1$ cars in the queue that can appear in any order. In the second case, there are $k-m+1$ red cars among the $m-1$ cars in the queue that can appear in any order. This gives a total of

$$\binom{m-1}{k-m} + \binom{m-1}{k-m+1} = \binom{m}{k-m+1}$$

arrival sequences. □

The bottom two rows of each parallelogram of numbers in Table 1 are identical, giving us the following result.

Theorem 4.2. *For any $m, k > 0$,*

$$G(m, k, 1) = G(m, k+1, 2).$$

Proof. We start by noting that in order to get a gap of size 1, the penultimate car in the arrival sequence must be green. Changing that car to red gives one more red car and increases the gap to 2. Conversely, if the gap is 2, then the block must be a red car. Switching it to a green car allows it to go into the left lane, decreasing the gap by 1 and decreasing the number of red cars by 1. □

5. Ballot path connections and bijections

Next, we notice that the numbers appearing on the right side of each parallelogram of numbers in Table 1 are the ballot numbers. To prove this, we need the following lemmas giving important relationships between the number of green and red cars in the lanes and in the queue, and the number of bounces in the merging path.

Lemma 5.1. *Let $p \in \mathbf{G_{m,k,g}}$. Let b be the number of bounces in the merging path for p before the block, x be the number of green cars before the block, y be the number of green cars in the queue, and z be the number of red cars in the queue. We obtain the system of equations*

$$x + y = 2m - k - 1 \quad (1)$$
$$y + z = g - 1 \quad (2)$$
$$x - b = m - g. \quad (3)$$

For reference, see Figure 1, where $b = 1$, $x = 3$, $y = 2$, and $z = 0$. As $m = 5$, $k = 4$, and $g = 3$, we can see all three equalities hold true.

Lemma 5.2. *Let $p \in \mathbf{G_{m,k,g}}$. If b is the number of bounces in the merging path for p before the block, then*

$$b \geq m - k.$$

Proof. Subtracting (3) from (1) in Lemma 5.1 gives $y + b = m - k + g - 1$. The result follows since (2) implies $y \leq g - 1$. □

Theorem 5.1. *If $k \geq m + g$, then $G(m, k, g) = 0$. Furthermore, if $k = m + g - 1$, then*

$$G(m,k,g) = B(m-g, m-1) = \binom{2m-g-1}{m-1} - \binom{2m-g-1}{m}$$

where $B(x, y)$ counts the number of ballot paths from $(0, 0)$ to (x, y).

Proof. Suppose $k \geq m+g$, then Lemma 5.1 implies $y+b < 0$, an impossibility. Lemma 5.1 also implies $y+b = 0$ when $k = m+g-1$. Since this is an equation of nonnegative integers, $y = 0$ and $b = 0$. For any $p \in \mathbf{G_{m,m+g-1,g}}$, its merging path contains no bounces, reaches the point $(m - g, m - 1)$, and has only 1s beyond this point. Clearly, the merging paths have a one-to-one correspondence with ballot paths to $(m - g, m - 1)$. Substituting into the formula for ballot paths, $B(x, y) = \binom{x+y}{y} - \binom{x+y}{y+1}$, we obtain the result. □

The last of the obvious patterns in Table 1 occurs in the left half where $m > k$. We will establish the following theorem in two ways. The first is basic, using previous results and some convolution. The second gives a bijection between these numbers and odd-length ballot paths.

Theorem 5.2. *If $m > k \geq 0$, then*

$$G(m,k,g) = \binom{2m-1}{k-g+1} - \binom{2m-1}{k-g} = B(k-g+1, 2m-k+g-2).$$

Proof. The merging paths counted by $G(m, k, g)$ must reach the point $(m - g, m)$ with an up-step, or merging paths counted by $G(m, k - 1, g)$ reaching the point $(m - g, m - 1)$. Using Theorem 20 in [3], there would be $\binom{2m-g-1}{k-g-j+1} - \binom{2m-g-1}{k-g-j-1}$ merging paths when $m > k$. Suppose the arrival sequence beyond the block contains j 0s. There would be $\binom{g-1}{j}$ such subsequences. Adding over all possible j and using the Chu-Vandermonde identity, we obtain

$$\begin{aligned}\sum_{j=0}^{g-1}\binom{g-1}{j} M_{m-g,k-j-1}(m-1) &= \sum_{j=0}^{g-1}\binom{g-1}{j}\left[\binom{2m-g-1}{k-g-j+1} - \binom{2m-g-1}{k-g-j-1}\right] \\ &= \binom{2m-2}{k-g+1} - \binom{2m-2}{k-g-1} \\ &= \binom{2m-1}{k-g+1} - \binom{2m-1}{k-g}\end{aligned}$$

□

Next, we prove Theorem 5.2 by finding a bijection from $\mathbf{G_{m,k-1,g-1}}$ to $\mathbf{G_{m,k,g}}$ and then a bijection from $\mathbf{G_{m,k,1}}$ to the odd length ballot paths.

Theorem 5.3. *If $m > k > 0$ and $g > 2$, then $|\mathbf{G_{m,k-1,g-1}}| = |\mathbf{G_{m,k,g}}|$.*

Proof. Let $p \in \mathbf{G_{m,k-1,g-1}}$. Let L_p be the last point that the merging path for p reaches the diagonal, and let S_p be the ordered pair of steps just before and just after L_p. By Lemma 5.2, L_p is not the origin since the merging path contains at least two bounces.

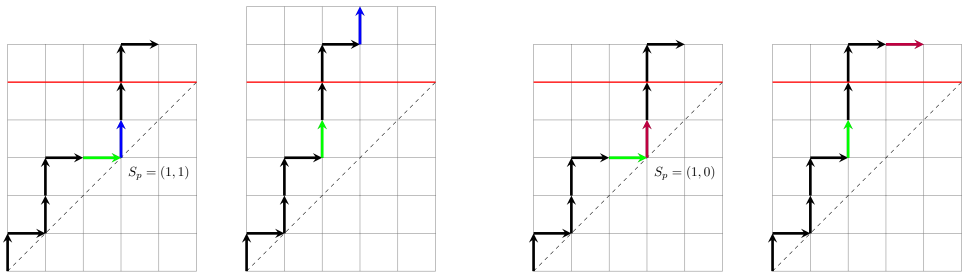

Figure 4: Two examples of the map ϕ, $S_p = (1,1)$ on left and $S_p = (1,0)$ on right. The steps above the red horizontal line represent the cars in the queue.

Define $\phi : \mathbf{G_{m,k-1,g-1}} \to \mathbf{G_{m,k,g}}$ as follows. If $p \in \mathbf{G_{m,k-1,g-1}}$, then $\phi(p)$ replaces S_p with a 0, and appends a 1 if $S_p = (1,1)$ or appends a 0 if $S_p = (1,0)$. See Figure 4 for an example. Notice that ϕ shifts the portion of the merging path of p past S_p to the left by 1, making it reach $(m-g, m)$ with an up-step. Clearly $\phi(p)$ contains k 0s, so $\phi(p) \in \mathbf{G_{m,k,g}}$.

Suppose for some $p, q \in \mathbf{G_{m,k-1,g-1}}$, $\phi(p) = \phi(q) = r \in \mathbf{G_{m,k,g}}$. Since ϕ shifts each point of the merging path of p past the last return to the diagonal to the left by one, the 0 that replaces S_p becomes that last time the merging path of p reaches the line $y = x+1$. Thus, there is a unique 0 in r where the merging path for r reaches $y = x+1$ for the last time, and must have replaced S_p by ϕ. This argument also applies to q, so $L_p = L_q$. The last bit of r uniquely determines what was replaced, so $S_p = S_q$. Finally, since ϕ doesn't change any other part of an arrival sequence, we have that $p = q$. Therefore, ϕ is one-to-one.

Finally, suppose $r \in \mathbf{G_{m,k,g}}$. We locate the last return of the merging path for r to the line $y = x+1$, which must be a 0. Denote this point by Q. Replace that 0 with a $(1,0)$ if r ends in a 0, and a $(1,1)$ if r ends in a 1. Finally, remove the last step and call the new sequence p. This has the effect of shifting every point past that 0 to the right by one and reduces the number of 0s by one, so $p \in \mathbf{G_{m,k-1,g-1}}$. This also makes the point to the right of Q the last return to the diagonal by p. By the definition of ϕ, $\phi(p) = r$, so ϕ is onto. □

Theorem 5.4. *There is a one-to-one correspondence between $\mathbf{G_{m,k,1}}$ and ballot paths reaching $(k, 2m-k-1)$.*

Proof. A ballot path reaching $(k, 2m-k-1)$ can be associated with a binary sequence with k 1's and $2m-k-1$ 0's. We create a merging path depending on whether the ballot path ends with a 0 or a 1.

Case I: If the ballot path ends with a 0. Remove this 0, then reverse and invert the remaining binary sequence. Finally, end this new sequence with a 1 and call it p. Use p to create a merging path with k red cars and $2m-k-1$ green cars. See Figure 5 for an example of this map. By Lemma 18 in [3], $b \geq m-k$. Inserting this into (3) gives $g \leq 1$. Of course, this implies $g = 1$ since 1 is the smallest possible gap size. Thus, the merging path obtained by p is in $\mathbf{G_{m,k,1}}$.

Case II: If the ballot path ends with a 1. Remove this 1, then reverse and invert the remaining binary sequence. Finally, end this new sequence with a 0 and call it p. Argument proceeds exactly as in Case I from this point. Now, if $p \in \mathbf{G_{m,k,1}}$, then the last step is an up-step. Removing this step gives a Dyck path. We can reverse this path to obtain a new Dyck path, and then change the bounces in p to up-steps. Lemma 5.1 implies that p contains $m-k$ bounces. Finally, we add the appropriate final step to this ballot path (a 0 if p ends in a bounce, and a 1 otherwise). We leave it to the reader to verify that the ballot reaches the point $(k, 2m-k-1)$.

□

6. Arrival sequences with exactly k red cars and b bounces

Returning to Table 1 and focusing on the section where $m = 6$, the portion not covered by the previous theorems are in bold in the Table 2.

6						1	6	15	20	15	6	1
5					1	10	**35**	**60**	**55**	**26**	5	
4				1	10	44	**96**	**109**	**62**	14		
3			1	10	44	110	**151**	**104**	28			
2		1	10	44	110	165	132	42				
1	1	10	44	110	165	132	42					
g/k	0	1	2	3	4	5	6	7	8	9	10	11

Table 2: Numbers not accounted for in the previous theorems where $m = 6$.

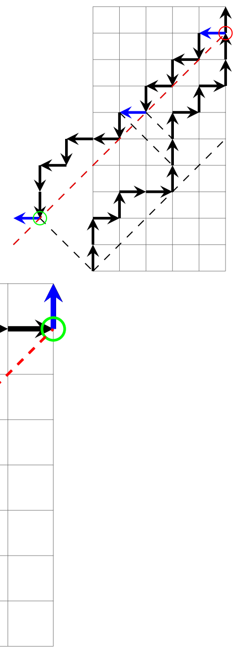

Figure 5: Example of changing a ballot path to a merging path in Case I. Top: Step-by-step visual of the reversed and inverted binary sequence with a new origin at the red circle. Bottom: ballot path and final merging path reflected and shifted from the top right.

To analyze these remaining numbers, we expand the definition of $\mathbf{G_{m,k,g}}$ by also accounting for the number of bounces.

Definition 6.1. *Let* $\mathbf{G_{m,k,g,b}}$ *be the set of arrival sequences of length* $2m-1$ *containing* k *0s, whose merging paths reach the point* $(m-g, m)$ *with an up-step, and* b *bounces. The number of such arrival sequences is* $G(m,k,g,b)$*.*

Tables 2 and 3 show the relationship between $G(6,k,g)$ and $G(6,k,g,b)$. Both have in bold the numbers where $k-g=2$. The sum of a row of bolded numbers in Table 4 is equal to the bolded number in Table 3 with the corresponding value of k.

g \ k	0	1	2	3	4	5	6	7	8	9	10	11
6						1	6	15	**20**	15	6	1
5					1	10	35	**60**	55	26	5	
4				1	10	44	**96**	109	62	14		
3			1	10	44	**110**	151	104	28			
2		1	10	44	**110**	165	132	42				
1	1	10	44	**110**	165	132	42					

Table 3: The values of $G(6,k,g)$ where the line $k-g=2$ are bold.

Theorem 6.1. *If* $g>1$ *and* $k \geq m$*, then*

$$G(m,k,g,b) = \binom{g-1}{k-m+b} B(m-g-b+1, m+b-1), \tag{4}$$

if $b \geq 1$ *and*

$$G(m,k,g,0) = \binom{g-1}{k-m} B(m-g, m-1). \tag{5}$$

Proof. If a merging path contains zero bounces, then the portion reaching $(m-g,m)$ with an up-step are the same as ballot paths reaching $(m-g, m-1)$. All of the cars in the right lane are red in this case, so there are $k-m$ out of the remaining $g-1$ cars that can appear in any order after the point $(m-g,m)$. This proves (5).

	$k-g=0$						$k-g=1$					$k-g=2$				$k-g=3$			$k-g=4$	
k/b	0	1	2	3	4	5	0	1	2	3	4	0	1	2	3	0	1	2	0	1
1						10														
2					9	1					44									
3				8	2					35	9				**110**					
4			7	3					27	16	1			**75**	**35**			165		
5		6	4					20	21	3			**48**	**54**	**8**		90	75		132
6	1	5					5	24	6			**14**	**60**	**21**	**1**	28	96	27	42	90
7							5	10				**20**	**36**	**4**		42	60	7	56	48
8												**10**	**10**			30	24	1	42	20
9																10	5		20	6
10																			5	1

Table 4: The values of $G(6, k, g, b)$ where $k-g=2$ are bold.

Now suppose $b \geq 1$. In this case, there are $m-b$ cars in the right lane, so there are $k-(m-b)$ red cars out of the $g-1$ cars in the queue that can appear in any order. Thus, what remains to prove is that the number of merging paths reaching $(m-g, m-1)$ with b bounces is the same as ballot paths to $(m-g-b+1, m+b-1)$.

Let $M := M_{m,g,b}$ be the set of merging paths reaching $(m-g, m-1)$ with b bounces and $B := B_{m,g,b}$ be the set of ballot paths to $(m-g-b+1, m+b-1)$. We define the map $f : M \to B$ by replacing the bounces with 00s. Since the first bounce may be the first step, it may not be preceded by a 1, although every other bounce has a preceding 1. So, the map f replaces the first bounce with a 00 and replaces all other bounces, along with their preceding 1 with a 00. For example, f(**1**01**11**01010**11**1000010) = **00**01**00**0101**00**000010 shown in Figure 6. Our map is well-defined since replacing bounces with 00s 'fixes' the merging path into a ballot path that will never cross the diagonal $y = x$. Moreover, changing the preceding 1s to 0s shifts the end of the path up $b-1$ times and left $b-1$ times.

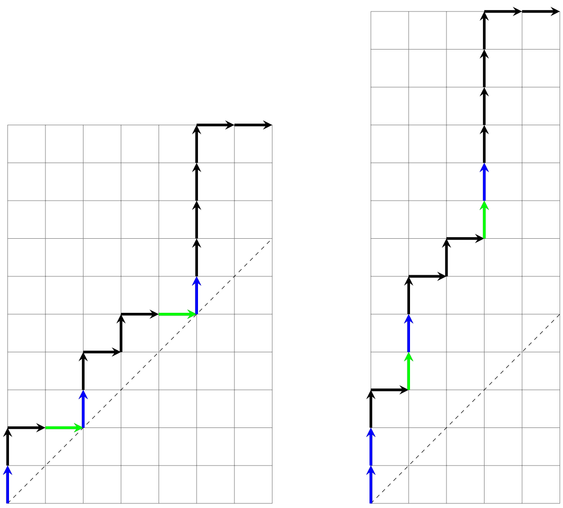

Figure 6: Example of the bijection f.

Next, we show that f is one-to-one. Suppose $x, y \in M$ where $x \neq y$, and let k be the first position where x and y are unequal. If k is not a bounce for either, or if it is a bounce (not the first) for one but not the other, then clearly $f(x) \neq f(y)$. The only interesting case is when k is the first bounce for one (say x), and the first bounce for y occurs after k. Suppose, for the sake of contradiction, $f(x)$ and $f(y)$ continue to agree until the first bounce for y. Then $f(x)$ reached the diagonal again beyond the first bounce for x. This implies that x reached a point under the diagonal.

Instead of directly showing f is onto, we define a map $g : B \to M$ and show that it is also one-to-one. This is a valid strategy since both sets are finite. The map g first changes the last occurrence of 00 from the diagonal to a 1, providing the first bounce. It then repeats the following $b-1$ times: Change the last occurrence of 00 from $y = x+1$ to a 11, providing a bounce each time. Note that g is well-defined and outputs a merging path with b bounces. We only need to show that the ballot path will contain enough valid occurrences of 00s to replace.

Suppose, for the sake of contradiction, that there are no more 00s from $y = x+1$ after $i < b$ replacements. If $b = 1$, then the ballot path contains no 00s, giving a simple staircase pattern. This is impossible since the ballot path reaches $(m-g, m)$, which is at least 2 above the diagonal. If $b \geq 2$, then after i replacements, the end of the path has been shifted down i times. At this point, the merging/ballot ('merlot', maybe?) path ends $(m+b-1)-(m-g-b+1)-i = 2b+g-2-i$ above the diagonal. Our assumption is that this distance is not more than 2, so $2b+g-2-i \leq 2$ or $2b+g-4 \leq i < b$. This implies that $b+g<4$, which is impossible since $g>1$.

Finally, we show that g is also one-to-one. Suppose $s,t \in B$ where $s \neq t$, and let k be the first position where s and t are unequal. Suppose after i replacements, k is the second 0 of the last 00 from $y=x+1$ in s but not in t. Then s will have $i+1$ bounces by k, and t will not. In all other cases, clearly $f(s) \neq f(t)$. □

7. Expected gap size with exactly k red cars and asymptotics

In Table 4 for fixed $k-g$, the columns sum to the same number when $b \geq 1$. We do not have a combinatorial interpretation for this, but we record it in the following corollary.

Corollary 7.1. *If $k-g=c \geq 0$ and $b \geq 1$, then*

$$\sum_k G(m,k,g,b) = B(c+1, 2m-c-1).$$

Proof. Substituting $g=k+c$ into (4) in Theorem 6.1 gives

$$\sum_k G(m,k,g,b) = \sum_{g \geq m-b-c} \binom{g-1}{g-(m-b-c)} \left[\binom{2m-g}{m-g-b+1} - \binom{2m-g}{m-g-b} \right].$$

Using a variant of the Chu-Vandermonde identity,

$$\sum_{i \geq 0} \binom{x+i}{i}\binom{y-i}{s-i} = \binom{x+y+1}{r+s},$$

gives

$$\binom{2m}{c+1} - \binom{2m}{c} = B(c+1, 2m-c-1).$$

□

Definition 7.1. *Let $G(m,k)$ be the sum of the gaps for the merging sequences with capacity m, k red cars, and length $2m-1$. That is,*

$$G(m,k) = \sum_{g=1}^{m} g \cdot G(m,k,g).$$

Our results so far give formulas for $G(m,k)$. We record this with the following theorem.

Theorem 7.1. *If $m > k \geq 0$, then*

$$G(m,k) = \sum_{i=0}^{k} \binom{2m-1}{i}.$$

If $k \geq m$, then

$$G(m,k) = \sum_{i=0}^{2m-2-k} \binom{2m-1}{i} + (k-m+1)\binom{2m}{k+1}.$$

Proof. When $m > k$, Theorem 5.2 applies, and we obtain a sum that simplifies as follows.

$$G(m,k) = \sum_{g=1}^{m} g \left[\binom{2m-1}{k-g+1} - \binom{2m-1}{k-g} \right] = \sum_{i=0}^{k} \binom{2m-1}{i}.$$

When $k \geq m$, we substitute the results of Theorem 6.1 into the fact that $G(m,k,g) = \sum_b G(m,k,g,b)$ to obtain

$$G(m,k) = \sum_{g=1}^{m} g \left[\binom{g-1}{k-m} B(m-g, m-1) + \sum_{b \geq 1} \binom{g-1}{k-m+b} B(m-g-b+1, m+b-1) \right].$$

Interchanging the sums, using the formula for the ballot numbers, and a little algebra with the binomial coefficients gives

$$G(m,k) = \sum_{g=0}^{m}(k-m+1)\binom{g}{k-m+1}\left[\binom{2m-g-1}{m-1} - \binom{2m-g-1}{m}\right]$$

$$+\sum_{b\geq 1}\sum_{g=0}^{m}(k-m+b+1)\binom{g}{k-m+b+1}\left[\binom{2m-g-1}{m+b-1} - \binom{2m-g-1}{m+b}\right].$$

The inner sums are examples of a variant of the Chu-Vandermonde identity:

$$\sum_{i\geq 0}\binom{x-i}{r}\binom{y+i}{s} = \binom{x+y+1}{r+s}.$$

Applying it gives

$$G(m,k) = (k-m+1)\left[\binom{2m}{k+1} - \binom{2m}{k+2}\right] + \sum_{b\geq 1}(k-m+b+1)\left[\binom{2m+1}{k+2b+1} - \binom{2m+1}{k+2b+2}\right].$$

$$= (k-m+1)\left[\binom{2m}{k+1} - \binom{2m}{k+2}\right] + \sum_{b\geq 1}(k-m+b+1)\left[\binom{2m}{k+2b} - \binom{2m}{k+2b+2}\right].$$

This sum also simplifies in a similar way as the $m > k$ case, giving the result. □

Corollary 7.2. *The expected gap size $\mathbb{E}_{m,k}(g)$ for arrival sequences of length $2m-1$ with k red cars and capacity m is*

$$\mathbb{E}_{m,k}(g) = \left[\sum_{i=0}^{k}\binom{2m-1}{i}\right] \Big/ \binom{2m-1}{k} < \frac{1-\alpha}{1-2\alpha}.$$

where $\alpha = k/(2m-1)$ and $k < m$. When $k \geq m$,

$$\mathbb{E}_{m,k}(g) = \left[\sum_{i=0}^{2m-1-k}\binom{2m-1}{i} - \binom{2m-1}{k} + (k-m+1)\binom{2m}{k+1}\right] \Big/ \binom{2m-1}{k}$$

$$< \frac{1-\alpha}{2\alpha-1} + \frac{2m(k-m+1)}{k+1}.$$

Moreover, when $k = m$,

$$\mathbb{E}_{m,m}(g) = 2^{2m-1}\Big/\binom{2m}{m} + \frac{m-1}{m+1} \sim \frac{\sqrt{m\pi}}{2} + 1.$$

Proof. The proof follows from Theorem 7.1 and the following approximation

$$\sum_{i\leq \alpha n}\binom{n}{i}\Big/\binom{n}{\alpha n} < \frac{1-\alpha}{1-2\alpha}$$

found in exercise 42 on page 492 of [4]. □

This figure shows that the approximations in Corollary 7.2 are accurate when $|m-k|$ is sufficiently large, as well as when $k = m$.

m	1	14	92	378	1093	2380	4096	5383	5097	3381	1548	469	85	7	
6	1	12	67	232	562	1024	1354	1222	727	276	61	6			
5	1	10	46	130	256	340	286	145	41	5					
4	1	8	29	64	85	64	25	4							
3	1	6	16	21	13	3									
2	1	4	5	2											
1	1	1													
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	k

Table 5: The values of $G(m,k)$ for all arrival sequences of length $2m-1$ with k red cars, where the lane capacity is m.

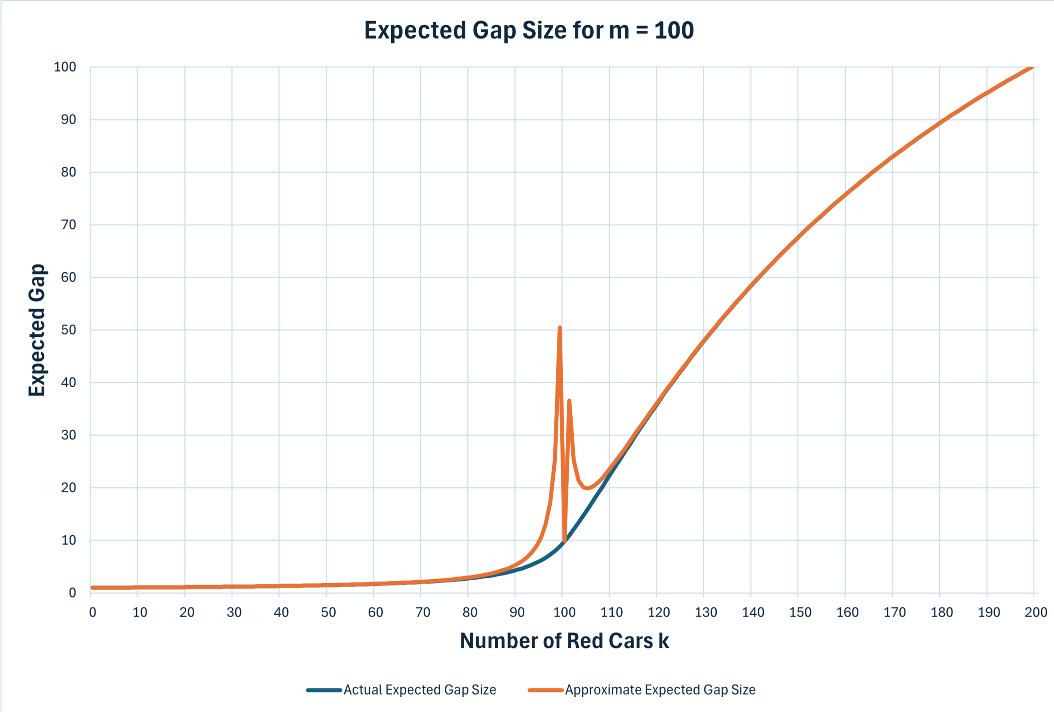

Figure 7: The blue curve shows the expected gap size for $m = 100$ and k red cars. The orange approximation is using the upper bounds involving α in Corollary 7.2 along with the approximation when $k = m$.

8. Open problems and conjectures

1. This paper considered the case where the right lane was preferable because the cars were forced to merge shortly after the light, but there are many instances of lights where a left-turn lane backs up and blocks the right lane. Create a combinatorial model for this scenario. Under what conditions should the turning-lane signals be given a green light before the through-traffic signals, and vice versa? Why are these phases not simultaneous, and how do their green-time durations differ?

2. Find a more accurate approximation for Corollary 7.2 in the cases where k is close to m.

3. Give a combinatorial interpretation of Corollary 7.1.

4. Develop a combinatorial model that incorporates merging after the light.

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