ecajournal.haifa.ac.il

Enumerative Combinatorics and Applications

ECA **7:1** (2027) Article #S2R7
https://doi.org/10.54550/ECA2027V7S1R7

Note on Twisted Convolutions in Formal Power Series Calculus

Gilbert Labelle*

Département de mathématiques, Université du Québec à Montréal, Montréal (Québec) Canada
Email: labelle.gilbert@uqam.can

Received: June 13, 2026, **Accepted**: October 15, 2026, **Published**: November 6, 2026
The authors: Released under the CC BY-ND license (International 4.0)

Abstract: In this short note, we show that our approaches, in [4] and [7], concerning the composition, inversion, and continuous iterates of normalized formal power series making use of sequences of polynomials of binomial type and concerning the chain rule operation for cycle index series, can be unified as special instances of a general "twisted" algebraic variant of the convolution operation added on suitable large monoid algebras.

Keywords: Composition; Convolution; Cycle-index series; Formal power series; Twisted convolution
2020 Mathematics Subject Classification: 05A15; 13F25

1. Twisted convolutions added to large monoid algebras

The general algebraic variant of the convolution operation which we will use is set within the context of large monoid algebras. Let $\mathbf{M} = (M, \cdot, e)$ be a monoid with finite decompositions of order 2 (i.e., for every $n \in M$ there is a finite number of pairs $(i, j) \in M \times M$ such that $i \cdot j = n$) and let $\mathbf{A} = (A, +, \cdot, 0, 1)$ be a commutative ring. The classical *large* $\mathbf{A}$*-algebra of the monoid* $\mathbf{M}$, $\mathbf{A}[[\mathbf{M}]] = (A^M, +, \star, 0, \delta_e)$, is the set of all families

$$f = (f(n))_{n\in M} \in A^M, \text{ written as formal sums of the form } \sum_{n\in M} f(n)n,$$

equipped with term-wise addition "$+$" and multiplication "$\star$", called *convolution*, defined by†

$$h = f \star g \quad \Longleftrightarrow \quad h(n) = \sum_{i\cdot j=n} f(i) \cdot g(j), \tag{1}$$

with Kronecker $\delta_e = (\delta_e(n))_{n\in M}$ unit element, where $\delta_e(n) = 1$ if $n = e$ and $\delta_e(n) = 0$ otherwise.

The variant of this algebraic kind of convolution which we will consider is defined as follows.

Definition 1.1. *Let* $\tau : \mathbf{M} \to \hom(\mathbf{A}, \mathbf{A})$ *be a representation of the monoid* $\mathbf{M}$ *in the ring* $\mathbf{A}$*. That is,*

$$\tau = (\tau_i : \mathbf{A} \to \mathbf{A})_{i\in\mathbf{M}} \quad \text{such that} \quad \tau_e = id_{\mathbf{A}} \text{ and } \tau_{i\cdot j} = \tau_i \circ \tau_j. \tag{2}$$

The large $\mathbf{A}$-algebra of the monoid $\mathbf{M}$ twisted by τ, *denoted* $\mathbf{A}[[\mathbf{M}]]_\tau = (A^M, +, \star, \star_\tau, 0, \delta_e)$, *is the algebra* $\mathbf{A}[[\mathbf{M}]$ *together with an extra binary operation* "$\star_\tau$"*, called* twisted convolution by τ*, defined by*

$$h = f \star_\tau g \quad \Longleftrightarrow \quad h(n) = \sum_{i\cdot j=n} f(i) \cdot \tau_i(g(j)).$$

Remark 1.1. *In* $\mathbf{A}[[\mathbf{M}]]_\tau$*, the binary operation* "$\star_\tau$" *does not replace* "$\star$" *but is added as an extra binary operation. So that* $\mathbf{A}[[\mathbf{M}]]_\tau$ *is still an* $\mathbf{A}$*-algebra (under* $+, \star, 0, \delta_e$*). Twisted convolution is biadditive, associative (using* (2)*) and* δ_e *is its unit element. Also, if we embed* A *in* $\mathbf{A}[[\mathbf{M}]]_\tau$ *in the natural way (i.e.* $a = a\delta_e$*), then* $ab = a \star_\tau b$ *for* $a, b \in A$*. However, in general,* $a(f \star_\tau g) = (af) \star_\tau g \neq f \star_\tau (ag)$ *for* $a \in A$*. This implies that* "$\star_\tau$" *is only semi-bilinear. Technically speaking, this means that, contrarily to* $(A^M, +, \star, 0, \delta_e)$*,* $(A^M, +, \star_\tau, 0, \delta_e)$ *is not an* $\mathbf{A}$*-algebra but only an* $\mathbf{A}$*-module (or a "semi"* $\mathbf{A}$*-algebra). Finally, if* $\mathbf{M}$ *is a commutative monoid, then convolution* "$\star$" *is commutative but twisted convolution* "$\star_\tau$" *is not commutative in general.*

*No AI tools were used in the preparation of this work.

†Of course, (1) is obtained by collecting similar terms in the formal expansion $(\sum_{i\in M} f(i)i)(\sum_{j\in M} g(j)j) = \sum_{i,j\in M} f(i)g(j)i\cdot j$.

2. Connections with the calculus of series

2.1 Twisted convolutions and composition of formal power series.

Consider the choices

$$\begin{aligned}\mathbf{M} &= \mathbb{N} = \{0,1,2,\dots\}, \text{the additive monoid of nonnegative integers,}\\ \mathbf{A} &= \mathbb{C}[t], \text{the ring of complex polynomials in a formal variable } t,\\ \tau &= (\tau_i : \mathbb{C}[t] \to \mathbb{C}[t])_{i\geq 0}, \text{the representation of } \mathbb{N} \text{ defined by } \tau_i P(t) = P(t+i).\end{aligned}$$

Then, since $\mathbb{N}$ is isomorphic to the multiplicative monoid $\langle z\rangle = \{1, z, z^2, \dots\}$ generated by a formal variable z, the elements of $\mathbf{A}[[\mathbf{M}]]_\tau$ are infinite sequences of polynomials $P = (P_n(t))_{n\geq 0}$, which can be encoded in the form

$$\sum_{n\geq 0} P_n(t) z^n \in \mathbb{C}[t][[z]].$$

In this case, for $P = (P_n(t))_{n\geq 0}, Q = (Q_n(t))_{n\geq 0}, R = (R_n(t))_{n\geq 0}$, the convolutions take the form

$$R = P \star Q \iff R_n(t) = \sum_{i+j=n} P_i(t) Q_j(t), \quad n = 0,1,2,\dots \quad \text{(standard)},$$

$$R = P \star_\tau Q \iff R_n(t) = \sum_{i+j=n} P_i(t) Q_j(t+i), \quad n = 0,1,2\dots \quad \text{(twisted)}.$$

Definition 2.1. *A sequence of polynomials $(P_n(t))_{n\geq 0} \in \mathbb{C}[t]$ is said to be of* binomial type *if it satisfies*[‡]

$$P_0(t) = 1, \quad \deg P_n(t) \leq n, \quad P_n(u+v) = \sum_{i+j=n} P_i(u) P_j(v),$$

for every $n \geq 0$ and formal variables u, v. Let

$$\mathbb{B} = \{P = (P_n(t))_{n\geq 0} \mid P \text{ is of binomial type}\} \subset A^M.$$

It turns out that (see [4])

$\mathbf{B} = (\mathbb{B}, \star, \delta_0)$ is an abelian group under convolution "$\star$",

the inverse of $P = (P_n(t))_{n\geq 0}$ is $P^{-1} = (P_n^{-1}(t))_{n\geq 0}$, where $P_n^{-1}(t) = P_n(-t)$,

$\mathbf{B}_\tau = (\mathbb{B}, \star_\tau, \delta_0)$ is a non abelian group under twisted convolution "$\star_\tau$",

the inverse of $P = (P_n(t))_{n\geq 0}$ is $P^{\langle -1\rangle} = (P_n^{\langle -1\rangle}(t))_{n\geq 0}$, where $P_n^{\langle -1\rangle}(t) = \dfrac{t}{t+n} P_n(-t-n)$.

Consider now the two sets of "normalized" formal power series,

$$\begin{aligned}\mathbb{U} &= \{\phi(z) \mid \phi(z) = 1 + c_1 z + c_2 z^2 + \cdots\} \subset \mathbb{C}[[z]],\\ \mathbb{S} &= \{f(z) \mid f(z) = z + a_2 z^2 + a_3 z^3 + \cdots\} \subset \mathbb{C}[[z]].\end{aligned}$$

Of course, $\mathbb{S} = z\mathbb{U}$, and

$$\begin{aligned}\mathbf{U} &= (\mathbb{U}, \cdot, 1) \text{ is an abelian group under product of series,}\\ \mathbf{S} &= (\mathbb{S}, \circ, z) \text{ is a non abelian group under composition of series.}\end{aligned}$$

The following result is the starting point of our analysis of continuous iteration of formal power series [4].

Proposition 2.1. *The correspondences*

$$\phi(z)^t = \sum_{n\geq 0} P_n(t) z^n \quad \text{and} \quad \left(\frac{f(z)}{z}\right)^t = \sum_{n\geq 0} P_n(t) z^n$$

respectively induce group isomorphisms

$$\mathbf{U} \xrightarrow{\sim} \mathbf{B} : \phi(z) \mapsto (P_n(t))_{n\geq 0} \quad \text{and} \quad \mathbf{S} \xrightarrow{\sim} \mathbf{B}_\tau : f(z) \mapsto (P_n(t))_{n\geq 0}.$$

In particular, we have the following version of Lagrange inversion,

Corollary 2.1. *Let $f(z), g(z) \in \mathbf{S}$ and $(f(z)/z)^t = \sum_{n\geq 0} P_n(t) z^n$, $(g(z)/z)^t = \sum_{n\geq 0} Q_n(t) z^n$, then*

$$f \circ g(z) = z \sum_{n\geq 0} (P \star_\tau Q)_n(1) z^n \quad \text{and} \quad f^{\langle -1\rangle}(z) = \sum_{n\geq 1} \frac{P_{n-1}(-n)}{n} z^n \quad (\text{inverse of } f(z) \text{ under "}\circ\text{"}).$$

[‡]This definition is slightly different from the original one in [8] in which they require the stronger condition $\deg P_n(t) = n$.

2.2 Twisted convolutions and chain rule for cycle index series.

Recall that the cycle index series of a combinatorial species F of structures is a formal power series, Z_F, in an infinity of variables $x_1, x_2, x_3, \dots$, which gives an inventory of the cyclic types of the automorphisms of the F-structures. More precisely, see Joyal [3] or [1], the cycle index series of F is defined in the following way:

$$Z_F = Z_F(x_1, x_2, x_3, \dots) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\sigma\in S_n} f_\sigma\, x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \cdots \tag{3}$$

where S_n denotes the symmetric group of degree n, f_σ is the number of the F-structures on the set $[n] = \{1, 2, \dots, n\}$ for which σ is an automorphism, and σ_i is the number of cycles of size i in the permutation σ. The n-tuple of integers $(\sigma_1, \sigma_2, \dots, \sigma_n)$ is called the cyclic type of σ.

Many terms in (3) are repeated. Since, for $n = i_1 + 2i_2 + 3i_3 + \cdots$, there are $\frac{n!}{1^{i_1} i_1! 2^{i_2} i_2! 3^{i_3} i_3! \cdots}$ permutations in S_n having cyclic type $(i_1, i_2, i_3, \dots)$, formula (3) can be rewritten in the more compact form

$$Z_F(x_1, x_2, x_3, \dots) = \sum_{n\geq 0} \sum_{i_1+2i_2+3i_3+\cdots=n} f_{i_1,i_2,i_3,\dots}\, \frac{x_1^{i_1} x_2^{i_2} x_3^{i_3} \cdots}{1^{i_1} i_1! 2^{i_2} i_2! 3^{i_3} i_3! \cdots} \tag{4}$$

where $f_{i_1,i_2,i_3,\dots}$ is the number of the F-structures on the set $[n]$ for which any given permutation $\sigma \in S_n$ of type $(i_1, i_2, i_3, \dots)$ is an automorphism. Cycle index series are important in enumerative combinatorics since they are compatible with operations between species, $F + G, F \cdot G, F \times G, F', F \circ G$, in the following way

$$Z_{F+G} = Z_F + Z_G, \quad Z_{F\cdot G} = Z_F \cdot Z_G, \quad Z_{F'} = \frac{\partial}{\partial x_1} Z_F, \quad Z_{F\circ G} = Z_F \circ Z_G \tag{5}$$

and reduce, by specialization of variables, to series enumerating labelled and unlabelled structures:

$$Z_F(x, 0, 0, \dots) = F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!} \quad \textit{and} \quad Z_F(x, x^2, x^3, \dots) = \widetilde{F}(x) = \sum_{n\geq 0} \tilde{f}_n\, x^n,$$

where f_n is the number of F-structures on $[n]$ and $\tilde{f}_n$ is the number of F-structures on n indistinguishable dots (i.e., number of isomorphism types of F-structures on n elements).

By extension, any series, $a(x_1, x_2, x_3, \dots)$, of the form (3) or (4), with complex coefficients f_σ or $f_{i_1,i_2,\dots,i_n}$ will be called a cycle index series. Composition " $\circ$" in (5) is called *plethystic substitution* and defined by

$$a \circ b(x_1, x_2, x_3, \dots) = a(b(x_1, x_2, x_3, \dots), b(x_2, x_4, x_6, \dots), b(x_3, x_6, x_9, \dots), \dots).$$

Consider now the choices

$$\begin{aligned}
\mathbf{M} &= \mathbb{N}^* = \{1, 2, 3, \dots\}, \text{the multiplicative monoid of strictly positive integers,}\\
\mathbf{A} &= \mathbb{C}\{\{x_1, x_2, x_3, \dots\}\}, \text{the ring of cycle index series in the variables } x_1, x_2, x_3, \dots,\\
\tau &= (\tau_i : \mathbb{C}\{\{x_1, x_2, x_3, \dots\}\} \to \mathbb{C}\{\{x_1, x_2, x_3, \dots\}\})_{i\geq 1}, \text{the representation of } \mathbb{N}^* \text{ defined by}\\
\tau_i a(x_1, x_2, x_3, \dots) &= a(x_i, x_{2i}, x_{3i}, \dots) = a_i(x_1, x_2, x_3, \dots) \quad \text{(plethystic notation).}
\end{aligned}$$

In this context, plethystic substitution can, of course, be written in the equivalent forms

$$a \circ b = a(\tau_1 b, \tau_2 b, \tau_3 b, \dots) = a(b_1, b_2, b_3, \dots). \tag{6}$$

The elements of $\mathbf{A}[[\mathbf{M}]]_\tau$ are infinite sequences

$$\mathbf{a} = (a(n; x_1, x_2, x_3, \dots))_{n\geq 1}$$

of cycle index series, with convolutions of the form

$$\mathbf{c} = \mathbf{a} \star \mathbf{b} \iff c(n; x_1, x_2, \dots) = \sum_{i\cdot j = n} a(i; x_1, x_2, \dots) b(j; x_1, x_2, \dots), \quad \text{(standard)},$$

$$\mathbf{c} = \mathbf{a} \star_\tau \mathbf{b} \iff c(n; x_1, x_2, \dots) = \sum_{i\cdot j = n} a(i; x_1, x_2, \dots) b_i(j; x_1, x_2, \dots), \quad \text{(twisted)}.$$

Note that the gradient ∇a of a cycle index series a is an element of $\mathbf{A}[[\mathbf{M}]]_\tau$ since it is an infinite sequence

$$\nabla a = \left(\frac{\partial a}{\partial x_n}\right)_{n\geq 1}$$

of the cycle index series. The following proposition shows that the chain rule operation for plethystic substitution (introduced in [7]) becomes another instance of twisted convolution in $\mathbf{A}[[\mathbf{M}]]_\tau$.

Proposition 2.2. Chain rule for plethystic substitution. *Given two cycle index series a and b,*

$$\nabla(a \circ b) = ((\nabla a) \circ b) \star_\tau \nabla b.$$

Proof. By (6),

$$\nabla(a \circ b) = \left(\frac{\partial}{\partial x_n}(a \circ b)\right)_{n\geq 1} = \left(\frac{\partial}{\partial x_n}\, a(b_1, b_2, b_3, \dots)\right)_{n\geq 1} = \left(\sum_{i\geq 1} \frac{\partial a}{\partial x_i}(b_1, b_2, b_3, \dots)\, \frac{\partial b_i}{\partial x_n}\right)_{n\geq 1}, \tag{7}$$

$$\frac{\partial b_i}{\partial x_n} = \frac{\partial}{\partial x_n} b(x_i, x_{2i}, x_{3i}, \dots, x_{ki}, \dots) = \begin{cases} \frac{\partial b}{\partial x_k}(x_i, x_{2i}, x_{3i}, \dots, x_{ki}, \dots) & \text{if } n = ki \\ 0 & \text{if } n \neq ki, \end{cases}$$

$$= \begin{cases} \frac{\partial b}{\partial x_{n/i}}(x_i, x_{2i}, x_{3i}, \dots, x_{ki}, \dots) & \text{if } i \,|\, n \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} (\frac{\partial b}{\partial x_j})_i & \text{if } i \cdot j = n \\ 0 & \text{otherwise.} \end{cases}$$

Finally, from (7),

$$\nabla(a \circ b) = \left(\sum_{i\cdot j = n} \frac{\partial a}{\partial x_i}(b_1, b_2, b_3, \dots) \cdot \left(\frac{\partial b}{\partial x_j}\right)_i\right)_{n\geq 1} = ((\nabla a) \circ b) \star_\tau \nabla b.$$

□

The above kind of chain rule has been used in [6] and [7] to compute the cycle index series of various families of combinatorial species. For example, let $T_R = XR(T_R)$ be the species of R-enriched rooted trees (introduced in [5]) and F be an arbitrary species. Then adapting Good's [2] multidimensional inversion formula to the present context, the coefficients of the cycle index series of the species $F(T_R)$ are given by

$$[x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots] Z_{F(T_R)} = [t_1^{n_1} t_2^{n_2} t_3^{n_3} \cdots] Z_F(t_1, t_2, t_3, \dots) \prod_{i\geq 1} \rho(t_i, t_{2i}, t_{3i}, \dots) Z_R(t_i, t_{2i}, t_{3i}, \dots)^{n_i}, \tag{8}$$

$$\text{where} \qquad \rho(t_1, t_2, t_3, \dots) = 1 - t_1 \frac{\partial Z_R(t_1, t_2, t_3, \dots)/\partial t_1}{Z_R(t_1, t_2, t_3, \dots)}. \tag{9}$$

In particular, taking $R = E$ (the species of finite sets) and $F = X$ (the species of singletons), we have $F(T_R) = T =$ the species of classical rooted trees. In this case, (8)-(9) give rise to the following extension [7] of Cayley's classical theorem stating that there are n^{n-1} rooted trees on n given points:

Let $\sigma \in S_n$ be a permutation of $[n]$ of cyclic type $(\sigma_1, \sigma_2, \dots, \sigma_n)$. Then the number of rooted trees on $[n]$ for which σ is an automorphism is given by 0 if $\sigma_1 = 0$ and otherwise by

$$\sigma_1^{\sigma_1 - 1} \prod_{k=2}^{n} \left(\phi_k^{\sigma_k} - k\sigma_k \phi_k^{\sigma_k - 1}\right) \quad \text{where} \quad \phi_k = \sum_{d|k} d\sigma_d.$$

3. Conclusion

The above framework provided a somewhat unexpected conceptual connection between two apparently different constructions in formal power series calculus. More instances of the above kind of twisted convolutions in combinatorics should be developed in the future.

Acknowledgement

The author thanks an anonymous reviewer for his constructive comments on an earlier draft of this paper.

References

[1] F. Bergeron, G. Labelle, and P. Leroux, *Combinatorial species and tree-like structures*, In: Encyclopedia Math. Appl., vol. 67, Cambridge U. Press, 480 p, 1998.

[2] I. J. Good, *Generalizations to several variables of Lagrange's expansion, with applications to stochastic processes*, Math. Proc. Cambridge Philos. Soc. 56 (1960), 367–380.

[3] A. Joyal, *Une théorie combinatoire des séries formelles*, Adv. Math. 42 (1981), 1–82.

[4] G. Labelle, *Sur l'inversion et l'itération continue des séries formelles*, European J. Combin. 1 (1980), 113–138.

[5] G. Labelle, *Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange*, Adv. Math. 42 (1981), 217–247.

[6] G. Labelle, *Eclosions combinatoires appliquées à l'inversion multidimensionnelle des séries formelles*, J. Combin. Theory Ser. A 39, (1985), 52–82.

[7] G. Labelle, *Some New Computational Methods in the Theory of Species*, In: Proceedings Combinatoire Enumérative, Lect. Notes in Math. 1234, Springer, New York, pp. 192–209, 1986.

[8] G.-C. Rota, D. Kahaner, and A. Odlyzko, *On the Foundations of Combinatorial Theory. VIII: Finite Operator Calculus*, J. Math. Anal. Appl. 42 (1973), 684–760.