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Enumerative Combinatorics and Applications

ECA **7:1** (2027) Article #S2R8
https://doi.org/10.54550/ECA2027V7S1R8

Normal Ordering of $(V^rU)^n$ and r-Sector Increasing Forests

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Received: June 8, 2026, **Accepted**: November 2, 2026, **Published**: November 6, 2026
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Abstract: For a fixed integer $r \geq 1$, we study the normal-ordering coefficients of $(V^rU)^n$ in the algebra defined by $UV = VU + V^2$. These coefficients are a specialization of generalized Stirling numbers; at $r = 1$ they give the unsigned meromorphic Stirling numbers, with the sign convention made explicit. We realize them as increasing forests in which every vertex has r distinguishable sectors, each containing a linearly ordered list of children. This places the coefficients in the increasing-tree framework with degree-weight function $\phi(u) = (1-u)^{-r}$. We give a unified derivation of the generating functions, component formula, and binomial-type identity. We also describe the finite-order lowering operator and the finite-band inverse transform, and establish fixed-column and row-sum asymptotics.

Keywords: Binomial-type polynomials; Increasing forests; Inverse transforms; Meromorphic Stirling coefficients; Normal ordering
2020 Mathematics Subject Classification: 05A15; 05A16; 05A40; 05C05

1. Introduction

Stirling-type arrays arise naturally from normal ordering in noncommutative algebras. The classical Weyl algebra gives the Stirling numbers through the normal ordering of powers of VU. More general normal-ordering problems have been studied extensively; see, for example, Mansour and Schork [7, 8], Blasiak, Penson, and Solomon [2], and the references therein.

The relation $UV = VU + V^2$ is a sign convention for the meromorphic Weyl relation considered in [4, 9, 10]. Mansour, Schork, and Shattuck studied the corresponding meromorphic Stirling numbers and their connection with Bessel numbers [9, 10]. Since both signs occur in this literature, we compare our convention with theirs in Section 1.2.

This paper studies the family obtained from $(V^rU)^n$, where $r \geq 1$. We first establish the algebraic recurrence and then explain its relation to prior work precisely. Throughout, $r \geq 1$ is fixed. If $\Phi(x) \in \mathbb{C}[[x]]$, then $[x^n]\Phi(x)$ denotes the coefficient of x^n. Binomial expressions with nonintegral exponents are interpreted formally until Section 5, where an analytic branch is specified.

Let $\mathcal{A}$ be the associative unital $\mathbb{C}$-algebra generated by U, V subject to

$$UV = VU + V^2. \tag{1}$$

A monomial is normally ordered if it has the form V^aU^b with $a, b \geq 0$.

Lemma 1.1. *The monomials V^aU^b, $a, b \geq 0$, form a $\mathbb{C}$-basis of $\mathcal{A}$.*

Proof. Let $R = \mathbb{C}[V]$ and define the derivation $\delta : R \to R$ by $\delta(V^m) = mV^{m+1}$ for $m \geq 0$. The Ore extension $R[U; \delta]$ is the algebra of polynomials in U with coefficients in R, with multiplication determined by $Up = pU + \delta(p)$ for $p \in R$; see [11, Chapter 1]. Taking $p = V$ gives (1), so $\mathcal{A} \simeq \mathbb{C}[V][U; \delta]$. Every element therefore has a unique finite expression $\sum_{b \geq 0} p_b(V)U^b$. Expanding each p_b proves the claim. □

Lemma 1.2. *For every $m \geq 0$,*

$$UV^m = V^mU + mV^{m+1}.$$

*OpenAI's ChatGPT was used to assist with language editing. The author takes responsibility for the mathematical content and the final version of the manuscript.

Proof. This is the Ore-extension rule applied to $p = V^m$. □

Definition 1.1. *For $n \geq 0$, define $H^{(r)}(n,k)$ by*

$$(V^r U)^n = \sum_{k=0}^{n} H^{(r)}(n,k) V^{n(r+1)-k} U^k. \tag{2}$$

We call these the r-meromorphic Stirling coefficients and define the row polynomial

$$P_n^{(r)}(y) = \sum_{k=0}^{n} H^{(r)}(n,k) y^k.$$

We use the convention $H^{(r)}(n,k) = 0$ for $k < 0$ or $k > n$.

Lemma 1.2 proves existence of this expansion inductively. Lemma 1.1 proves uniqueness.

Proposition 1.1. *The coefficients are uniquely determined by $H^{(r)}(0,0) = 1$, the zero convention in Definition 1.1, and, for $n \geq 0$ and all integers k,*

$$H^{(r)}(n+1,k) = H^{(r)}(n,k-1) + \big(n(r+1) - k\big) H^{(r)}(n,k). \tag{3}$$

In particular, $H^{(r)}(n,0) = 0$ for $n \geq 1$, and $H^{(r)}(n,n) = 1$ for $n \geq 0$.

Proof. Multiplying (2) on the left by $V^r U$ and applying Lemma 1.2 gives

$$\begin{aligned} V^r U V^{n(r+1)-k} U^k &= V^{(n+1)(r+1)-(k+1)} U^{k+1} \\ &\quad + \big(n(r+1) - k\big) V^{(n+1)(r+1)-k} U^k. \end{aligned}$$

Comparison in the normal basis yields (3). The initial value follows from $n = 0$. Setting $k = 0$ and then $k = n+1$ in the recurrence proves the two stated boundary values by induction. □

1.1 Relation to prior work and scope of the contribution

The array itself belongs to an established generalized Stirling family. In the notation of Hsu and Shiue [6], their recurrence is

$$S(n+1,k;a,b,c) = S(n,k-1;a,b,c) + (bk - an + c) S(n,k;a,b,c).$$

Consequently, comparison with (3) and the initial conditions gives

$$H^{(r)}(n,k) = S(n,k;-(r+1),-1,0).$$

Thus we do not claim a new generalized Stirling array or a new general recurrence.

The contribution is the realization of the normal-ordering coefficients by increasing forests with r distinguishable sectors at every vertex (Theorem 2.1), together with a unified treatment of the associated operators, inverse coefficients, and asymptotic estimates. The insertion rule gives a direct combinatorial explanation of the coefficient $(r+1)n - k$ in (3). For the inverse transform, the polynomial substitution series of degree $r+1$ explains both its column support $k \leq n \leq (r+1)k$ and the finite recurrence in Theorem 4.3.

We distinguish these descriptions from the general tools used to derive them. The tree differential equation is an application of the increasing-tree theory of Bergeron, Flajolet, and Salvy [1]; the Bell-polynomial component formula is the standard decomposition of a labelled forest into components [3]; and the binomial-type identity, lowering operator construction, and inversion by series composition follow from finite operator calculus [12]. The asymptotic analysis applies standard singularity-transfer results [5]. Accordingly, the paper presents a concrete forest model and its consequences, rather than new general principles in these theories. The case $r = 1$ is already covered by [9,10] and is included to fix normalization and illustrate the extension.

1.2 The case r = 1: notation and normalization

At $r = 1$, (3) becomes

$$H^{(1)}(n+1,k) = H^{(1)}(n,k-1) + (2n-k) H^{(1)}(n,k).$$

In the notation $\mathfrak{S}_{s;h}(n,k)$ used in [10], this is $\mathfrak{S}_{2;1}(n,k)$. Their signed meromorphic convention uses $UV = VU - V^2$, giving $\mathfrak{S}_{2;-1}(n,k)$. Hence the precise identification is

$$H^{(1)}(n,k) = \mathfrak{S}_{2;1}(n,k) = (-1)^{n-k} \mathfrak{S}_{2;-1}(n,k).$$

Indeed, replacing U by $-U$ changes the sign of V^2 in the commutation relation and multiplies each normal-ordering coefficient by $(-1)^{n-k}$. Thus there is no additional scale factor or index shift. Equivalently, for $1 \le k \le n$, the unsigned Bessel-number expression is

$$H^{(1)}(n,k) = \frac{(2n-k-1)!}{2^{n-k}(n-k)!(k-1)!}.$$

It can also be verified directly from the recurrence and initial values. The column exponential generating function obtained below is

$$\sum_{n\ge k} H^{(1)}(n,k)\frac{x^n}{n!} = \frac{\left(1-\sqrt{1-2x}\right)^k}{k!}.$$

Table 1 displays the unsigned rows. Multiplying its (n,k) entry by $(-1)^{n-k}$ gives the corresponding signed meromorphic row; for example, the row at $n=4$ becomes $(0,-15,15,-6,1)$.

Table 1: The coefficients $H^{(r)}(n,k)$ for $r=1,2$ and $0 \le n \le 5$. A dash indicates $k>n$, where the coefficient is zero.

	$r=1$							$r=2$					
$n\backslash k$	0	1	2	3	4	5	$n\backslash k$	0	1	2	3	4	5
0	1	–	–	–	–	–	0	1	–	–	–	–	–
1	0	1	–	–	–	–	1	0	1	–	–	–	–
2	0	1	1	–	–	–	2	0	2	1	–	–	–
3	0	3	3	1	–	–	3	0	10	6	1	–	–
4	0	15	15	6	1	–	4	0	80	52	12	1	–
5	0	105	105	45	10	1	5	0	880	600	160	20	1

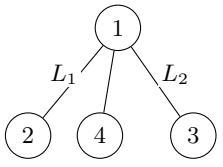
2. r-sector forests and generating functions

Definition 2.1. *An r-sector rooted tree is a rooted tree in which every vertex v carries an ordered r-tuple $(L_1(v),\dots,L_r(v))$ of finite lists of children. The sectors are distinguishable and indexed by $1,\dots,r$; each child occurs in exactly one list, and the order within each list is part of the structure. Empty lists are permitted. Swapping two sector lists generally changes the tree. An r-sector forest is an unordered finite collection of these trees. On $[n]=\{1,\dots,n\}$ it is increasing if labels increase along every path from a root to a descendant.*

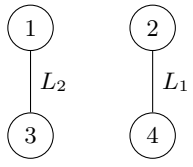
Let $\mathcal{F}_r(n,k)$ be the set of increasing r-sector forests on $[n]$ with k components, and let $N_r(n,k)=|\mathcal{F}_r(n,k)|$. The set $\mathcal{F}_r(0,0)$ consists of the empty forest. For $k<0$ or $k>n$ the set is empty and $N_r(n,k)=0$.

Examples. For $r=2$ and $n=2$, there are two trees with one component: the root is 1, and its child 2 lies either in sector 1 or in sector 2. Thus $H^{(2)}(2,1)=2$. The two singleton roots give the unique forest with two components.

For a larger example on [4], take a root 1 with sector lists $L_1(1)=(2,4)$ and $L_2(1)=(3)$, all other lists being empty. This is an increasing two-sector tree. Interchanging 2 and 4 within $L_1(1)$ gives a different tree. A two-component forest on [4] is obtained by taking $L_2(1)=(3)$ and $L_1(2)=(4)$ with roots $1,2$. Both examples appear in Figure 1. The left-to-right placement of components in the drawing carries no order.



(a) $L_1(1)=(2,4)$, $L_2(1)=(3)$



(b) Two unordered components

Figure 1: Increasing two-sector forests. Edge labels indicate sector membership; children within a sector are read from left to right. All unmentioned sector lists are empty.

Lemma 2.1. *If $F \in \mathcal{F}_r(n,k)$, the number of positions for inserting a new largest label $n+1$ as a non-root vertex is*

$$(r+1)n-k.$$

Proof. There are rn child lists. A list of length d has $d+1$ insertion gaps. The sum of the list lengths is the number of edges, $n-k$. Hence the total is $rn + \sum d = rn + (n-k)$. Inserting the largest label in any gap preserves the increasing condition. □

Theorem 2.1. *For all $n, k \geq 0$,*

$$H^{(r)}(n,k) = N_r(n,k) = |\mathcal{F}_r(n,k)|.$$

Proof. In an increasing forest, the largest label is a leaf. Deleting it reverses exactly one of two constructions: adjoining it as an isolated root, or inserting it in a child-list gap. Lemma 2.1 therefore gives

$$N_r(n+1,k) = N_r(n,k-1) + \big((r+1)n - k\big)N_r(n,k).$$

The initial and boundary conditions agree with those of Proposition 1.1, proving the result. □

Proposition 2.1. *The row polynomials satisfy*

$$P_{n+1}^{(r)}(y) = \big(n(r+1)+y\big)P_n^{(r)}(y) - y\frac{d}{dy}P_n^{(r)}(y), \tag{4}$$

with $P_0^{(r)}(y) = 1$.

Proof. Multiplication of (3) by y^k and summation gives

$$\begin{aligned} P_{n+1}^{(r)}(y) &= \sum_{k=0}^{n+1} H^{(r)}(n,k-1)y^k + n(r+1)\sum_{k=0}^{n+1} H^{(r)}(n,k)y^k - \sum_{k=0}^{n+1} kH^{(r)}(n,k)y^k \\ &= yP_n^{(r)}(y) + n(r+1)P_n^{(r)}(y) - y\frac{d}{dy}P_n^{(r)}(y). \end{aligned}$$

The zero convention justifies the endpoint extensions. □

Theorem 2.2. *Let $G_r(y,x) = \sum_{n\geq 0} P_n^{(r)}(y)x^n/n!$, an exponential generating function in x whose coefficients are polynomials in the ordinary component marker y. Then*

$$G_r(y,x) = \exp\Big(y\left(1-(1-(r+1)x)^{1/(r+1)}\right)\Big). \tag{5}$$

Equivalently,

$$\sum_{n\geq k} H^{(r)}(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(1-(1-(r+1)x)^{1/(r+1)}\right)^k. \tag{6}$$

Proof. Set $f_r(x) = 1-(1-(r+1)x)^{1/(r+1)}$. Equation (4) implies

$$(1-(r+1)x)\frac{\partial G_r}{\partial x} + y\frac{\partial G_r}{\partial y} = yG_r, \qquad G_r(y,0) = 1.$$

The series $e^{yf_r(x)}$ satisfies this equation because $(1-(r+1)x)f_r'(x) + f_r(x) = 1$. Coefficient comparison in the differential equation recursively determines every row polynomial, so the solution is unique. Extracting $[y^k]$ yields (6). □

Corollary 2.1. *For $0 \leq k \leq n$,*

$$H^{(r)}(n,k) = \frac{n!}{k!}[x^n]\left(1-(1-(r+1)x)^{1/(r+1)}\right)^k.$$

Equivalently,

$$H^{(r)}(n,k) = \frac{n!(-r-1)^n}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\binom{j/(r+1)}{n}.$$

Proof. The first formula follows from (6); expanding the kth power by the binomial theorem gives the second. □

3. Increasing-tree and binomial-type structure

The forest model is an instance of the increasing-tree framework in [1], with degree-weight function

$$\phi(u) = (1-u)^{-r}.$$

Proposition 3.1. *Let $T_r(x)$ be the exponential generating function for increasing r-sector rooted trees. Then*

$$T_r'(x) = (1 - T_r(x))^{-r}, \qquad T_r(0) = 0,$$

and consequently

$$T_r(x) = 1 - (1-(r+1)x)^{1/(r+1)}. \tag{7}$$

Proof. At each vertex there are r distinguishable sectors, each containing a linearly ordered list of subtrees. The ordinary generating function in u for these child configurations is $(1+u+u^2+\cdots)^r = (1-u)^{-r}$. Here u counts subtrees, rather than vertex labels. The standard increasing-tree construction gives $T_r' = \phi(T_r)$: removing the smallest label at the root leaves precisely such an r-tuple of lists, and the derivative accounts for that root label. Solving the differential equation with the given initial value yields (7). □

Definition 3.1. *A sequence $(p_n(y))_{n\geq 0}$ is of binomial type if $p_0(y) = 1$, $\deg p_n = n$, and*

$$p_n(y+z) = \sum_{i=0}^{n} \binom{n}{i} p_i(y) p_{n-i}(z) \qquad (n \geq 0).$$

Theorem 3.1. *The row polynomials form a sequence of binomial type. In particular,*

$$P_n^{(r)}(y+z) = \sum_{i=0}^{n} \binom{n}{i} P_i^{(r)}(y) P_{n-i}^{(r)}(z). \tag{8}$$

Equivalently, for all $a, b, n \geq 0$,

$$\binom{a+b}{a} H^{(r)}(n, a+b) = \sum_{i=0}^{n} \binom{n}{i} H^{(r)}(i,a) H^{(r)}(n-i,b). \tag{9}$$

Proof. We have $P_0^{(r)} = 1$ and $\deg P_n^{(r)} = n$ because $H^{(r)}(n,n) = 1$. By (5), $G_r(y+z,x) = G_r(y,x)G_r(z,x)$. Comparison of coefficients of $x^n/n!$ gives (8); comparison of coefficients of $y^a z^b$ then gives (9). This is the standard generating-function argument of finite operator calculus [12]. □

Definition 3.2. *For $m \geq 1$, let*

$$\tau_r(m) = \prod_{j=1}^{m-1} \big((r+1)j - 1\big), \tag{10}$$

with the empty-product convention $\tau_r(1) = 1$.

Lemma 3.1. *The number of increasing r-sector rooted trees on $[m]$ is $\tau_r(m)$.*

Proof. Let $A_r(m)$ denote this number. We have $A_r(1) = 1$. By Lemma 2.1, insertion of m into a tree on $[m-1]$ gives $A_r(m) = ((r+1)(m-1)-1)A_r(m-1)$. Deletion of the largest leaf reverses this operation. Iteration proves (10). □

Definition 3.3. *The partial exponential Bell polynomials are defined by $B_{0,0} = 1$, $B_{n,0} = 0$ for $n \geq 1$, and*

$$B_{n,k}(x_1, x_2, \ldots) = \frac{n!}{k!} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i \geq 1}} \prod_{i=1}^{k} \frac{x_{n_i}}{n_i!} \qquad (1 \leq k \leq n). \tag{11}$$

We use the convention $B_{n,k} = 0$ outside $0 \leq k \leq n$. This is the notation of [3].

Theorem 3.2 (Component decomposition)**.** *For $0 \leq k \leq n$,*

$$H^{(r)}(n,k) = \frac{n!}{k!} \sum_{\substack{n_1+\cdots+n_k=n \\ n_i \geq 1}} \prod_{i=1}^{k} \frac{\tau_r(n_i)}{n_i!}. \tag{12}$$

The case $(n,k) = (0,0)$ is interpreted as the empty product, and the sum is zero if $k = 0 < n$. For $1 \leq k \leq n$, this is

$$H^{(r)}(n,k) = B_{n,k}\big(\tau_r(1), \tau_r(2), \ldots, \tau_r(n-k+1)\big). \tag{13}$$

Proof. A forest with k components is an unordered set of k trees. For an ordered list of component sizes $n_1, \dots, n_k$, the labels can be distributed among the components in $n!/(n_1! \cdots n_k!)$ ways. On any chosen block of size n_i, order-preserving relabelling gives $\tau_r(n_i)$ trees. Dividing by $k!$ to forget the component order and summing proves (12). Formula (13) follows from (11). □

4. Finite-order lowering and inverse transform

Let

$$f_r(x) = 1 - (1 - (r+1)x)^{1/(r+1)}.$$

Its compositional inverse has the polynomial expansion

$$\begin{aligned} g_r(x) &= \frac{1-(1-x)^{r+1}}{r+1} = \sum_{j=1}^{r+1} \frac{(-1)^{j+1}}{r+1} \binom{r+1}{j} x^j \\ &= x - \frac{r}{2}x^2 + \frac{r(r-1)}{6}x^3 - \cdots + \frac{(-1)^r}{r+1}x^{r+1}, \end{aligned}$$

where the finite sum is the defining expansion when $r = 1$ or 2. This polynomiality is the common source of the finite-order lowering operator and the inverse column support below.

Theorem 4.1. *Let $D = d/dy$ and*

$$Q_r = g_r(D) = \frac{1-(1-D)^{r+1}}{r+1}.$$

Equivalently,

$$Q_r = \sum_{j=1}^{r+1} \frac{(-1)^{j+1}}{r+1} \binom{r+1}{j} D^j.$$

Then

$$Q_r P_n^{(r)}(y) = n P_{n-1}^{(r)}(y) \qquad (n \geq 1),$$

and $Q_r P_0^{(r)} = 0$.

Proof. Since $De^{yf_r(x)} = f_r(x)e^{yf_r(x)}$ and $g_r(f_r(x)) = x$, applying Q_r to (5) gives

$$\sum_{n\geq 0} Q_r P_n^{(r)}(y) \frac{x^n}{n!} = x \sum_{n\geq 0} P_n^{(r)}(y) \frac{x^n}{n!} = \sum_{n\geq 1} n P_{n-1}^{(r)}(y) \frac{x^n}{n!}.$$

Coefficient comparison proves the claim. The construction is the usual inverse-series construction for a binomial-type sequence [12]; here the operator has order exactly $r+1$. □

Definition 4.1. *The matrix $\mathsf{H}_r = (H^{(r)}(n,k))_{n,k\geq 0}$ is unit lower triangular: the zero convention gives lower triangularity and $H^{(r)}(n,n) = 1$ gives its unit diagonal. Define its inverse by $\mathsf{H}_r^{-1} = (h^{(r)}(n,k))_{n,k\geq 0}$. Each entry of a matrix product involves a finite sum.*

Theorem 4.2. *For all $n, k \geq 0$,*

$$h^{(r)}(n,k) = \frac{n!}{k!}[x^n] g_r(x)^k,$$

or equivalently,

$$h^{(r)}(n,k) = \frac{n!}{k!(r+1)^k}[x^n]\big(1-(1-x)^{r+1}\big)^k. \tag{14}$$

In particular,

$$h^{(r)}(n,k) = 0 \quad \textit{unless} \quad k \leq n \leq (r+1)k. \tag{15}$$

Proof. By (6), if $A(x) = \sum_{k\geq 0} a_k x^k/k!$ and $b_n = \sum_{k=0}^n H^{(r)}(n,k) a_k$, then $\sum_{n\geq 0} b_n x^n/n! = A(f_r(x))$. Thus H_r represents substitution by f_r on exponential generating functions. Its inverse represents substitution by g_r, so

$$\sum_{n\geq 0} h^{(r)}(n,k) \frac{x^n}{n!} = \frac{g_r(x)^k}{k!}.$$

This proves the coefficient formulas. The lowest and highest degrees of g_r are 1 and $r+1$, respectively, giving (15). Thus each inverse column has finite support, not merely the lower-triangular support of a general substitution matrix. □

Corollary 4.1. *For $n, k \geq 0$,*

$$h^{(r)}(n,k) = \frac{(-1)^n n!}{k!(r+1)^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{(r+1)j}{n}.$$

Proof. Expand $(1-(1-x)^{r+1})^k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (1-x)^{(r+1)j}$ in (14). □

Theorem 4.3 (Finite recurrence for the inverse coefficients)**.** *The inverse coefficients satisfy $h^{(r)}(0,0) = 1$, $h^{(r)}(n,0) = 0$ for $n \geq 1$, and $h^{(r)}(n,k) = 0$ for $k < 0$ or $k > n$. For $n \geq 0$ and $k \geq 1$,*

$$h^{(r)}(n+1,k) = \sum_{j=0}^{\min(r,n)} (-1)^j \binom{r}{j} (n)_j h^{(r)}(n-j,k-1), \tag{16}$$

where $(n)_j = n(n-1)\cdots(n-j+1)$ and $(n)_0 = 1$.

Proof. Differentiating the inverse column exponential generating function and using $g_r'(x) = (1-x)^r$ gives

$$\sum_{n \geq 0} h^{(r)}(n+1,k) \frac{x^n}{n!} = (1-x)^r \frac{g_r(x)^{k-1}}{(k-1)!}.$$

The sum starts at $n = 0$; this includes the possibly nonzero term at $n = k-1$. To extract coefficients, if $A(x) = \sum_{m \geq 0} a_m x^m / m!$, then

$$n![x^n]\big(x^j A(x)\big) = \frac{n!}{(n-j)!} a_{n-j} = (n)_j a_{n-j} \qquad (0 \leq j \leq n),$$

and the coefficient is zero for $j > n$. Expanding $(1-x)^r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} x^j$ now proves (16). □

Corollary 4.2. *For every $n \geq 0$,*

$$y^n = \sum_{k=\lceil n/(r+1) \rceil}^{n} h^{(r)}(n,k) P_k^{(r)}(y).$$

Proof. Let $\mathbf{P} = (P_0^{(r)}(y), P_1^{(r)}(y), \ldots)^T$ and $\mathbf{y} = (1, y, y^2, \ldots)^T$. By definition $\mathbf{P} = \mathsf{H}_r \mathbf{y}$, so $\mathbf{y} = \mathsf{H}_r^{-1} \mathbf{P}$. The nth row gives the identity, with the restricted range supplied by (15). □

5. Asymptotics

We verify the fixed-column and row-sum estimates by singularity analysis [5, Chapter VI]. The integer k in the column estimate is fixed; no uniformity in growing k is asserted.

Theorem 5.1 (Fixed-column and row-sum asymptotics)**.** *Fix $r \geq 1$ and $k \geq 1$, and set $\alpha = 1/(r+1)$. As $n \to \infty$,*

$$H^{(r)}(n,k) \sim \frac{n!}{(k-1)!} \frac{\alpha}{\Gamma(1-\alpha)} (r+1)^n n^{-1-\alpha}. \tag{17}$$

If $B_n^{(r)} = P_n^{(r)}(1) = \sum_{k=0}^{n} H^{(r)}(n,k)$, then

$$B_n^{(r)} \sim e \frac{\alpha}{\Gamma(1-\alpha)} n! (r+1)^n n^{-1-\alpha}. \tag{18}$$

Proof. Let $\rho = 1/(r+1)$ and $X = 1 - x/\rho$. Use the principal branch of X^α, analytic in the slit plane $\mathbb{C} \setminus [\rho, \infty)$. Both generating functions below are analytic for $|x| < \rho$, extend analytically to a dented neighborhood of ρ, and have no other singularity on $|x| = \rho$. More explicitly, they are analytic in a domain

$$\{x : |x| < R,\ x \neq \rho,\ |\arg(x-\rho)| > \theta\} \quad \text{with } R > \rho,\ 0 < \theta < \pi/2.$$

This is a Δ-domain of the kind required by the transfer theorem.

For fixed $k \geq 1$, the column exponential generating function has expansion

$$\frac{(1-X^\alpha)^k}{k!} = \frac{1}{k!} - \frac{X^\alpha}{(k-1)!} + O(X^{2\alpha}).$$

Its first nonanalytic term is $-X^\alpha/(k-1)!$. Singularity transfer gives

$$[x^n]X^\alpha \sim \rho^{-n}\frac{n^{-1-\alpha}}{\Gamma(-\alpha)}, \qquad -\frac{1}{\Gamma(-\alpha)} = \frac{\alpha}{\Gamma(1-\alpha)}.$$

The constant contributes nothing for $n \geq 1$. The remainder contributes $O(\rho^{-n}n^{-1-2\alpha})$ by the O-transfer theorem, hence is of smaller order than the displayed leading term. If $r = 1$, then $2\alpha = 1$; any integer powers in the local expansion are analytic, and do not create an additional leading singular term. Multiplying the coefficient estimate by $n!$ proves (17).

For the row sums, use their exponential generating function directly:

$$\sum_{n\geq 0} B_n^{(r)}\frac{x^n}{n!} = G_r(1,x) = \exp(1-X^\alpha) = e - eX^\alpha + O(X^{2\alpha}).$$

The same transfer argument and multiplication by $n!$ prove (18). This does not require summing fixed-k asymptotic estimates over a growing range of columns. □

Remark 5.1. *When $r = 1$, $f_1(x) = 1-\sqrt{1-2x}$, $g_1(x) = x - x^2/2$, and $Q_1 = D - D^2/2$. The inverse support is $k \leq n \leq 2k$, consistent with the Bessel specialization. For example, $h^{(1)}(3,2) = -3$ and $h^{(1)}(3,3) = 1$, so $y^3 = P_3^{(1)}(y) - 3P_2^{(1)}(y)$. When $r = 2$, $g_2(x) = x - x^2 + x^3/3$, giving support $k \leq n \leq 3k$. In general, the width is determined by the degree $r+1$ of g_r.*

Acknowledgment

The author is grateful to the anonymous referee, who provided many helpful suggestions for improving the paper. The research of Mohsen Aliabadi is partially supported by an AMS–Simons Research Enhancement Grant (award no. SFI-MPS-T-Institutes-00021934).

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